Assignment 17

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1. For $f \in \mathrm{B}([a,b],\mathbb{R})$ define its *oscillation* on the partition $P \in \mathcal{P}\big([a,b]\big)$ by
$$\operatorname{osc}(f,P) = S^{+}(f,P) - S^{-}(f,P)\,.$$
Prove that $f \in \mathcal{R}([a,b])$ iff
$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ s.t. } \operatorname{osc}(f,P) \le \varepsilon \text{ whenever } \triangle(P) \le \delta\,.$$

2. Consider two series of positive terms $\sum x_n$ and $\sum y_n$ and assume that $\sum y_n < \infty$. Show that the condition
$$\exists\, N \in \mathbb{N} \text{ s.t. } \frac{x_{n+1}}{x_n} \le \frac{y_{n+1}}{y_n}\ \forall n \ge N$$
implies $\sum x_n < \infty$.

3. Let $(x_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $[0,\infty)$ and prove that
$$\sum x_n < \infty \Longleftrightarrow \sum 2^k x_{2^k} < \infty\,.$$

4. Let (M,d) be a compact metric space and assume that a sequence $(A_n)_{n\in\mathbb{N}}$ of closed subsets of M be given with $A_{n+1} \subset A_n$ for $n \in \mathbb{N}$. Show that
$$\bigcap_{n\in\mathbb{N}} A_n \ne \emptyset\,.$$

5. Prove or disprove: The series $\sum \frac{(-1)^n}{nx}$ on $(0,1]$ converges pointwise or uniformly or absolutely.

Homework due by Thursday, March 16 2006.