Final Examination–Solutions

1. Let $A \in \mathbb{R}^{n\times n}$ and show that the map

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\,,\ (x,y) \mapsto y^T A x$$

is differentiable and compute its derivative.
Solution: Let $h = (h_1, h_2) \in \mathbb{R}^{2n}$ be given and consider

$$(y + h_2)^T A(x + h_1) = y^T Ax + [h_2^T Ax + y^T Ah_1] + h_2^T Ah_1\,.$$

Since $|h_2^T Ah_1| \leq c\,|h|_2^2$, it follows that

$$\Phi(x + h_1, y + h_2) - \Phi(x, y) - [h_2^T Ax + y^T Ah_1] = o(|h|_2)$$

and therefore the derivative is given by

$$D\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\,,\ (x, y) \mapsto D\Phi(x, y)$$

for $D\Phi(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\,,\ h \mapsto h_2^T Ax + y^T Ah_1\,.$

2. Let $f \in \mathrm{C}^2(\mathbb{R}^n, \mathbb{R})$ and assume that it is convex, that is, that

$$f\big((1-t)x + ty\big) \leq (1-t)f(x) + tf(y)\ \forall x, y \in \mathbb{R}^n\ \forall t \in [0,1]\,.$$

Prove that $D^2 f(x) \geq 0$ (positive definite) for every $x \in \mathbb{R}^n$.
Solution: The assumption implies that the real real-valued function

$$\phi_{x,y} : [0,1] \to \mathbb{R}\,,\ t \mapsto f\big((1-t)x + ty\big)$$

is convex and therefore satisfies $\phi''_{x,y} \geq 0$. Since

$$\phi''_{x,y}(t) = (y - x)^T D^2 f\big((1-t)x + ty\big)(y - x)$$

is valid for any choice of $x, y \in \mathbb{R}^n$ it follows that

$$h^T D^2 f\big((1-t)x + ty\big)h \geq 0\ \forall\, h \in \mathbb{R}^n$$

and therefore the claim.

3. Show that the system

$$\begin{cases} e^{x+y+c} & = 1 \\ \frac{1}{1+(x-1)^2+y^2} & = d + \frac{1}{2} \end{cases}$$

has a unique small solution for every small $c, d \in \mathbb{R}$.
Solution: Rewrite the system as

$$\begin{cases} e^{x+y} & = e^{-c} \\ \frac{1}{1+(x-1)^2+y^2} & = d + \frac{1}{2} \end{cases}$$

and observe that the function f given by left-hand-side evaluated at $(0,0)$ gives $(1, \frac{1}{2})$ for $c = d = 0$. Since

$$Df(0,0) = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix}$$

is invertible, the claim is a direct consequence of the inverse function theorem.

4. Let $A \in \mathbb{R}^{n \times n}$ be symmetric and positive definite. Show that

$$M_{n-1} := \{x \in \mathbb{R}^n \mid x^T A x = 1\}$$

is a $(n-1)$-dimensional C^1-manifold in $\mathbb{R}^n$.
Solution: The first derivative of the map $\phi = [x \mapsto x^T A x]$ is given by

$$D\phi(x) = 2x^T A$$

and, since A is invertible, it can only vanish for $x = 0 \notin M_{n-1}$. Hence the claim follows from the regular value theorem.

5. Compute the volume of the set

$$C := \{(x, y, z) \mid 0 \leq x^2 + y^2 \leq z\,,\ 0 \leq z \leq 1\}\,.$$

Solution: The volume is given by

$$\int_C 1\, d(x, y, z)\,.$$

By using cylindrical coordinates, the change of variables theorem gives

$$\int_C 1\, d(x, y, z) = \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{z}} r\, dr d\phi dz = \frac{\pi}{2}\,.$$

6. Find maxima and minima of the function $f(x,y,z) = 4y - 2z$ on the curve determined by

$$2x - y - z = 2\,,\; x^2 + y^2 = 1\,.$$

Solution: Introduce the function

$$H(x,y,z,\lambda,\mu) := 4y - 2z + \lambda(2x - y - z - 2) + \mu(x^2 + y^2 - 1)\,.$$

Its critical points satisfy

$$\begin{cases} 2\lambda + 2\mu x & = 0 \\ 4 - \lambda + 2\mu y & = 0 \\ -2 - \lambda & = 0 \\ 2x - y - z - 2 & = 0 \\ x^2 + y^2 & = 1 \end{cases}$$

Since $\lambda = -2$ by the third equation, the first and the second imply that x, y, μ are all different from zero. It follows that $x = \frac{2}{\mu}$ and that $y = -\frac{3}{\mu}$. The last equation then gives $\mu = \pm\sqrt{13}$. Finally the solutions are

$$(x,y,z) = (\pm\frac{2}{\sqrt{13}}, \mp\frac{3}{\sqrt{13}}, -2 \pm \frac{7}{\sqrt{13}})\,.$$

Since the curve determined by the two equations is closed and bounded, it is also compact. The function f therefore takes on both its maximum and its minimum. Evaluation at the two points just computed reveals that $(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}, -2 + \frac{7}{\sqrt{13}})$ is the point of minimum and the other is the point of maximum.

7. Compute the integral

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dxdy\,.$$

Solution: Interchanging the order of integration we obtain

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dxdy = \int_0^3 \int_0^{x/3} e^{x^2}\, dydx\,.$$

Then

$$\int_0^3 \int_0^{x/3} e^{x^2}\, dydx = \frac{1}{3}\int_0^3 xe^{x^2}\, dx = \frac{1}{6}\int_0^9 e^z\, dz = \frac{1}{6}(e^9 - 1)\,.$$

8. Find maxima and minima of the function

$$f(x,y) = x^2 e^{-x^2-y^2}\,,\ (x,y) \in \overline{\mathbb{B}}_{\mathbb{R}^2}(0,1) = \{x \in \mathbb{R}^n \mid |x|_2 \leq 1\}\,.$$

Indicate which maxima and minima are strict and which are not.
Solution: Computing the gradient of f and looking for critical points gives

$$\begin{cases} 2x(1-x^2) &= 0 \\ -2yx^2 &= 0 \end{cases}$$

from which follows that

$$(x,y) = (0,y) \text{ for } y \in [-1,1] \text{ or } (x,y) = (\pm 1, 0)\,.$$

At the points $(0,y)$ the nonnegative function f vanishes. These points are therefore minima and are clearly non strict. At the other two points the function value is $\frac{1}{e}$. Since the function must assume its maximum, this has to occur there. Both maxima are strict.

9. Can the surface parametrized by

$$(s,t) \mapsto (s^3+t^3, st, s^3-t^3)\,,\ \mathbb{R}^2 \to \mathbb{R}^3$$

be represented as the graph of a function? If your answer is no, explain why. If it is yes, determine the function.
Solution: Consider the system

$$\begin{cases} s^3+t^3 &= u \\ s^3-t^3 &= v \end{cases}$$

and observe that it is always uniquely solvable with solution

$$(s,t) = \big((\frac{u+v}{2})^{\frac{1}{3}}, (\frac{u-v}{2})^{\frac{1}{3}}\big)\,.$$

The surface is therefore the graph of the function

$$(u,v) \mapsto (\frac{u^2-v^2}{4})^{\frac{1}{3}}\,.$$