Math 205 Spring Term 2006

Final Examination

Print your name: ________________ ________________

Print your ID #: ________________________________

You have 2 hours to solve the problems. Good luck!

1. Let $A \in \mathbb{R}^{n\times n}$ and show that the map

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\,,\ (x,y) \mapsto y^T A x$$

is differentiable and compute its derivative.

2. Let $f \in \mathrm{C}^2(\mathbb{R}^n,\mathbb{R})$ and assume that it is convex, that is, that

$$f\big((1-t)x + ty\big) \le (1-t)f(x) + tf(y)\ \forall x,y \in \mathbb{R}^n\ \forall t \in [0,1]\,.$$

Prove that $D^2 f(x) \ge 0$ (positive definite) for every $x \in \mathbb{R}^n$.

3. Show that the system

$$\begin{cases} e^{x+y+c} & = 1 \\ \frac{1}{1+(x-1)^2+y^2} & = d + \frac{1}{2} \end{cases}$$

has a unique small solution for every small $c, d \in \mathbb{R}$.

4. Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. Show that

$$M_{n-1} := \{x \in \mathbb{R}^n \,|\, x^T A x = 1\}$$

is a $(n-1)$-dimensional C^1-manifold in $\mathbb{R}^n$.

5. Compute the volume of the set

$$C := \{(x,y,z) \,|\, 0 \le x^2 + y^2 \le z\,,\ 0 \le z \le 1\}\,.$$

6. Find maxima and minima of the function $f(x,y,z) = 4y - 2z$ on the curve determined by

$$2x - y - z = 2\,,\ x^2 + y^2 = 1\,.$$

7. Compute the integral

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dxdy\,.$$

8. Find maxima and minima of the function

$$f(x,y) = x^2 e^{-x^2-y^2}\,,\ (x,y) \in \overline{\mathbb{B}}_{\mathbb{R}^2}(0,1) = \{x \in \mathbb{R}^n \,|\, |x|_2 \le 1\}\,.$$

Indicate which maxima and minima are strict and which are not.

9. Can the surface parametrized by

$$(s,t) \mapsto (s^3 + t^3, st, s^3 - t^3)\,,\ \mathbb{R}^2 \to \mathbb{R}^3$$

be represented as the graph of a function? If your answer is no, explain why. If it is yes, determine the function.