Final Examination – Solutions

1. Let $g \in \mathrm{C}^1\big([c,d],[a,b]\big)$ and $f \in \mathrm{C}\big([a,b]\big)$. Define

$$F(x) := \int_a^{g(x)} f(y)\,dy\,,\ x \in (c,d)\,.$$

Show that F is differentiable and compute its derivative.
Solution:
Let $G(x) := \int_0^y f(\xi)\,d\xi\,,\ y \in (a,b)$. Then, since f is continuous, G is differentiable and the claim follows from the chain rule thanks to

$$F(x) = G(g(x))\,,\ x \in (c,d)\,.$$

The chain rule and the fundamental theorem of calculus also give

$$F'(x) = f\big(g(x)\big)g'(x)\,.$$

2. Let $(x_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $[0,\infty)$ and prove that

$$\sum x_n < \infty \Longleftrightarrow \sum 2^k x_{2^k} < \infty\,.$$

Solution:
"$\Longleftarrow$": Taking into account that the sequence is decreasing one sees that

$$x_1 + \underbrace{x_2 + x_3} + \underbrace{x_4 + x_5 + \cdots + x_8} + \cdots \leq x_1 + \underbrace{x_2 + x_2} + \underbrace{4x_4} + \cdots < \infty\,.$$

"$\Longrightarrow$": As for the converse the proof goes similarly since

$$x_1 + \underbrace{x_2 + x_2} + \underbrace{4x_4} \leq \ldots x_1 + \underbrace{x_1 + x_2} + \underbrace{x_2 + x_3 + x_3 + x_4} + \ldots$$
$$= 2\sum_{n=0}^{\infty} x_n < \infty\,.$$

3. Let a sequence of functions $(f_n)_{n\in\mathbb{N}}$ be defined by

$$f_n(x) = \cos(x)^n\,,\ x \in [0, \frac{\pi}{2}]\,,\ n \in \mathbb{N}\,.$$

Let $g \in \mathrm{C}\big([0, \frac{\pi}{2}]\big)$ be such that $g(0) = 0$. What is the limit of $(gf_n)_{n\in\mathbb{N}}$? Is the convergence pointwise? Is it uniform? Justify your answer.
Solution:
The convergence is uniform to the limit $f_\infty \equiv 0$ as the the following argument shows. For any given $\varepsilon > 0$ we can find $\delta > 0$ such that

$$|g(x)| \leq \varepsilon \text{ whenever } x \in [0, \delta]$$

since g is assumed to be continuous. Also since

$$\cos(x) \leq \cos(\delta)\ \forall\, x \in [\delta, \frac{\pi}{2}]\,,$$

we can find $N \in \mathbb{N}$ such that

$$\cos(x)^n \leq \frac{\varepsilon}{\|g\|_\infty}\ \forall\, n \geq N\,.$$

Combining the two inequalities it is obtained that

$$g(x)\cos(x)^n \leq \begin{cases} \varepsilon\,, & x \in [0, \delta]\,, \\ |g(x)|\frac{\varepsilon}{\|g\|_\infty} \leq \varepsilon\,, & x \in [\delta, \frac{\pi}{2}]\,. \end{cases}\,,\ \forall\, n \geq N\,.$$

4. Show that f defined through

$$f(x) = \log^2(1 + x)$$

is analytic in a neighborhood of the origin. Compute the coefficients of its power series expansion about $x = 0$.
Solution:
The function f satisfies $f'(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n\,,\ x \in (-1, 1)$. It follows that

$$\log(1 + x) = \sum_{k=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}\,,\ x \in (-1, 1)\,.$$

For the product it therefore follows that

$$[\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}]\,[\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}] = x^2 + \sum_{n=2}^{\infty}[\sum_{k=1}^{n}\frac{(-1)^{n+1}}{k(n+1-k)}]x^{n+1}\,.$$

5. Let (M, d) be a metric space. For a subset $A \subset M$ define

$$\bar{A} = A \cup \mathrm{LP}(A)$$

and show that

$$\bar{A} = \bigcap \{B \subset M \,|\, A \subset B \text{ and } B \text{ is closed}\}\,.$$

Solution:
Since $A \cup \mathrm{LP}(A)$ is closed, it readily follows that

$$A \cup \mathrm{LP}(A) \supset \bigcap \{B \subset M \,|\, A \subset B \text{ and } B \text{ is closed}\}\,.$$

As for the converse, we show that $A \subset B$ implies that $\bar{A} \subset \bar{B} = B$ where latter equality follows if B is closed. In fact, if $x \in \bar{A}$, then we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $x_n \to x$ as $n \to \infty$. If B is assumed to be closed, then $x \in \bar{B} = B$, since the sequence is clearly also on B. Thus any closed set B which contains A also contains $\bar{A}$ and the claim follows.

6. Let (M, d) be a metric space. For a subset $A \subset M$ define

$$\mathring{A} := \{x \in A \,|\, \exists\, r > 0 \text{ s.t. } \mathbb{B}(x, r) \subset A\}\,.$$

Prove or disprove: $(A \cup B)^\circ = \mathring{A} \cup \mathring{B}$, $(A \cap B)^\circ = \mathring{A} \cap \mathring{B}$
Solution:
The first equality does not hold since

$$A = [0, \frac{1}{2}]\,,\ B = [\frac{1}{2}, 1]$$

gives a counter-example in $\mathbb{R}$ with the standard metric. The second equality holds. In fact, if $x \in (A \cap B)^\circ$, then we find $r > 0$ such that $\mathrm{B}(x, r) \subset A \cap B$ which implies

$$\mathrm{B}(x, r) \subset A\,,\ \mathrm{B}(x, r) \subset B$$

and therefore $x \in \mathring{A}$ as well as $x \in \mathring{B}$. Also, if $x \in \mathring{A} \cap \mathring{B}$, we find $r_1, r_2 > 0$ with

$$\mathrm{B}(x, r_1) \subset A \text{ and } \mathrm{B}(x, r_2) \subset B$$

which gives

$$\mathrm{B}(x, r) \subset A \cap B$$

for $r := \min(r_1, r_2)$.

7. Prove or disprove:

$$\{\frac{1}{\sqrt{n}}\tanh(nx) : \mathbb{R} \to \mathbb{R} \,|\, n \in \mathbb{N}\}$$

is uniformly equicontinuous.

<u>Solution:</u>

The sequence it is uniformly convergent to 0 since

$$\frac{1}{\sqrt{n}}\tanh(nx) \leq \frac{1}{\sqrt{n}} \,\forall\, x \in \mathbb{R}\,.$$

Thus it equicontinuous by the Arzéla-Ascoli Theorem. A more hands-on approach would be to observe that

$$|\frac{1}{\sqrt{n}}\tanh(nx) - \frac{1}{\sqrt{n}}\tanh(ny)| \leq \frac{1}{\sqrt{n}}\,,$$

and that

$$\|\frac{d}{dx}\frac{1}{\sqrt{n}}\tanh(nx)\|_\infty = \|\sqrt{n}\big[1 - \tanh^2(nx)\big]\|_\infty \leq \sqrt{n}\,.$$

Latter implies that

$$|\frac{1}{\sqrt{n}}\tanh(nx) - \frac{1}{\sqrt{n}}\tanh(ny)| \leq \sqrt{n}\,|x-y|$$

and thus

$$\begin{aligned} &|\frac{1}{\sqrt{n}}\tanh(nx) - \frac{1}{\sqrt{n}}\tanh(ny)| = \\ &|\frac{1}{\sqrt{n}}\tanh(nx) - \frac{1}{\sqrt{n}}\tanh(ny)|^{1/2}|\frac{1}{\sqrt{n}}\tanh(nx) - \frac{1}{\sqrt{n}}\tanh(ny)|^{1/2} \\ &\qquad\qquad \leq \frac{1}{n^{1/4}}\,n^{1/4}\,|x-y|^{1/2} = |x-y|^{1/2} \end{aligned}$$

which readily implies uniform equicontinuity.

8. Assume that the improper integral $\int_0^\infty \frac{f(x)}{x}\,dx$ exists and show that

$$\int_0^\infty \frac{f(xy)}{x}\,dx = \int_0^\infty \frac{f(x)}{x}\,dx \,\forall\, y \in (0,\infty)\,.$$

Solution:
The integration domain is invariant with respect to rescaling. Thus simple substitution gives

$$\int_0^\infty \frac{f(xy)}{x}\,dx = \int_0^\infty \frac{f(xy)}{xy}\,d(xy) = \int_0^\infty \frac{f(z)}{z}\,dz\,.$$

To be more detailed, first observe that

$$\int_0^\infty f(z)\,dz = \lim_{r\to 0}\int_r^1 \frac{f(z)}{z}\,dz + \lim_{R\to\infty}\int_1^R \frac{f(z)}{z}\,dz$$

and, then by change of variable, that

$$\begin{aligned}\int_0^\infty f(z)\,dz &= \lim_{r\to 0}\int_r^1 \frac{f(xy)}{xy}\,d(xy) + \lim_{R\to\infty}\int_1^R \frac{f(xy)}{xy}\,d(xy)\\ &= \lim_{r\to 0}\int_{r/y}^{1/y} \frac{f(xy)}{x}\,dx + \lim_{R\to\infty}\int_{1/y}^{R/y} \frac{f(xy)}{x}\,dx = \int_0^\infty \frac{f(xy)}{x}\,dx\,.\end{aligned}$$

9. Let $(f_n)_{n\in\mathbb{N}}$ be a decreasing sequence of real-valued functions on $[a,b]$ which converges uniformly to $f_\infty \equiv 0$. Show that

$$\sum_{n=1}^\infty (-1)^n f_n$$

converges uniformly.
Solution:
Arguing just like in the case of numeric sequence we obtain that

$$\|\sum_{j=n}^m (-1)^j f_j\|_\infty \leq \|f_n\|_\infty$$

by virtue of the fact that the sequence is decreasing. Now the claim follows since the right-hand-side converges to 0 by assumption and the Cauchy criterion for series (that is, the sequence of partial sums is Cauchy).