Math205b WINTER TERM 2003

Assignment 9

Let $a < b \in \mathbb{R}$ and denote by P the generic partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

of the interval $[a,b]$. Let y_k denote any point in $[x_{k-1}, x_k]$. Finally define

$$\triangle(P) = \sup_{k=1,\ldots,n} (x_k - x_{k-1})$$

the maximum interval length of the partition P.

1. Let

$$\mathrm{S}([a,b],\mathbb{R}) = \Big\{\varphi : [a,b] \to \mathbb{R} \;\Big|\; \varphi\big|_{[x_{k-1},x_k]} \equiv \varphi_k \in \mathbb{R}\,,\ k = 1,\ldots,n \text{ for some } P\Big\}$$

   be the vector space of step functions defined on the interval $[a,b]$. Show that any $f \in \mathrm{C}([a,b],\mathbb{R})$ can be approximated by a sequence of step functions, that is, $\forall \varepsilon > 0$ there is $\varphi \in \mathrm{S}([a,b],\mathbb{R})$ s.t

$$\|f - \varphi\|_\infty = \sup_{x \in [a,b]} |f(x) - \varphi(x)| \le \varepsilon\,.$$

2. Let $g \in \mathrm{C}^1([a,b],\mathbb{R})$ be increasing. Prove that

$$\int_a^b f(x)\,dg(x) = \lim_{\triangle(P)\to 0} \sum_{k=1}^n f(y_k)\big(g(x_k) - g(x_{k-1})\big)$$

   is well-defined for $f \in \mathrm{C}([a,b],\mathbb{R})$. Also show that

$$\int_a^b f(x)\,dg(x) = \int_a^b f(x)g'(x)\,dx\,.$$

3. Let $f \in \mathrm{C}^1([a,b],\mathbb{R})$ and $x, y \in [a,b]$. Prove the mean value theorem in integral form

$$f(y) = f(x) + (y - x)\int_0^1 f'((1-t)x + ty)dt\,.$$

4. Let $x_0 \in U_{x_0} \overset{o}{\subset} \mathbb{R}$ and $f, g \in \mathrm{C}^3(U_{x_0})$ with

$$f(x_0) = f'(x_0) = 0 = g'(x_0) = g(x_0) \text{ and } g''(x_0) \neq 0\,.$$

   Show that f/g is differentiable at x_0 and compute the derivative there.

5. You ask a question.

The Homework is due Friday January 17 2002