Assignment 16

1. Prove that a normed vector space $(V, |\cdot|_V)$ is an inner product space iff
$$|x+y|_V^2 + |x-y|_V^2 = 2\bigl(|x|_V^2 + |y|_V^2\bigr)\ ,\ x, y \in V\ .$$
[Hint: polarization identity.]

2. Give an example of a norm which cannot be derived from an inner product and an example of a metric which is not induced by a norm.

3. Define
$$l_\infty(\mathbb{K}) := \bigl\{x \in \mathbb{K}^{\mathbb{N}} \bigm| |x|_\infty := \sup_{n\in\mathbb{N}} |x_n| < \infty\bigr\}$$
and
$$c_0(\mathbb{K}) := \bigl\{x \in l_\infty(\mathbb{K}) \bigm| \lim_{n\to\infty} x_n = 0\bigr\}\ .$$
Show that $(l_\infty(\mathbb{K}), |\cdot|_\infty)$ and $(c_0(\mathbb{K}), |\cdot|_\infty)$ are complete normed vector spaces.

4. Two norms $|\cdot|_1$ and $|\cdot|_2$ on a vector space V are said to be *equivalent* iff there exists a constant $c \geq 1$ such that
$$\frac{1}{c}\,|x|_1 \leq |x|_2 \leq c\,|x|_1\ ,\ x \in V\ .$$
Prove that equivalent norms induce the same topology.

5. If you had to make a choice, what would you say is the single most important thing you learned in class?

The Homework is due on Friday, March 14.