Assignment 13

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1. Let $f \in \mathrm{C}\big([0,1],\mathbb{R}\big)$ and define the Bernstein polynomials by
$$p_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} f(\frac{k}{n}) x^k (1-x)^{n-k}\ ,\ x \in [0,1]\ ,\ n \in \mathbb{N}\,.$$
Show that
$$\|p_n(f,\cdot) - f\|_\infty \underset{n\to\infty}{\longrightarrow} 0\,.$$
[*Hint: Use the identity*
$$\sum_{k=0}^{n} (k-nx)^2 \binom{n}{k} x^k (1-x)^{n-k} = nx(1-x)$$
and split the sum into two parts according to whether $|x-\frac{k}{n}| \leq \delta$ *or* $|x-\frac{k}{n}| > \delta > 0$.]

2. Determine the radius of convergence of the following power series
$$\sum_{n=0}^{\infty} \frac{\sqrt{n\,2^n}}{(n+1)^5} x^n\ ,\ \sum_{n=0}^{\infty} (-1)^n \frac{n!}{n^n} x^n\,.$$

3. Assume that the power series
$$\sum_{n=0}^{\infty} a_n x^n \text{ and } \sum_{n=0}^{\infty} b_n x^n$$
have positive radii of convergence. Suppose that there exists a sequence $(y_j)_{j\in\mathbb{N}}$ with $y_j \to 0\ (j\to\infty)$ and $y_j \neq 0$ such that
$$\sum_{n=0}^{\infty} a_n y_j^n = \sum_{n=0}^{\infty} b_n y_j^n\,.$$
Prove that $a_n = b_n$, $n \in \mathbb{N}$.

4. Assume $D \overset{o}{\subset} \mathbb{R}$ and let $f: D \to \mathbb{R}$ be analytic. Show that, for every $x_0 \in D$, constants $M, r, \delta > 0$ can be found such that
$$|f^{(k)}(x)| \leq M k!\, r^k\ ,\ x \in (x_0-\delta, x_0+\delta)\,.$$

5. You ask a question.

The Homework is due on Friday, February 21.