Assignment 10

---

1. Let $\varphi \in \mathrm{C}(\mathbb{R},\mathbb{R})$ and $f \in \mathrm{C}([a,b],\mathbb{R})$ for $a < b \in \mathbb{R}$. Assume that φ is convex, that is, that it satisfies

   $\varphi\big((1-t)x+ty\big) \leq (1-t)\varphi(x)+t\varphi(y)\,,\ x,y\in\mathbb{R}\,,\ t\in[0,1]\,.$

   Prove the validity of the following *Jensen's inequality*

   $$\varphi\Big(\frac{1}{b-a}\int_a^b f(x)\,dx\Big) \leq \frac{1}{b-a}\int_a^b \varphi\big(f(x)\big)\,dx\,.$$

2. The Legendre ploynomials L_n can be defined by

   $$P_n(x)=\frac{1}{2^n\,n!}\frac{d^n}{dx^n}(x^2-1)^n\,,\ n=0,1,2,\dots$$

   Show that

   $$\int_{-1}^1 P_n(x)P_m(x)\,dx=0\,,\ m\neq n\,.$$

3. Show that the improper integral

   $$\Gamma(\alpha)=\int_0^\infty t^\alpha e^{-t}\,dt$$

   exists for every $\mathbb{R}\ni\alpha>-1$ and verify that

   $$\Gamma(n)=n!\,,\ n\in\mathbb{N}\,.$$

4. Assume that $f\in\mathrm{C}^1([-1,1])$ and prove that the following *principal value integral* exists (i.e. that the defining limit always exists)

   $$p.v.\int_{-1}^1\frac{f(x)}{x}\,dx=\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{f(x)}{x}\,dx\,.$$

   Also show that

   $$\int_{-1}^1\log(|x|)f'(x)\,dx=-p.v.\int_{-1}^1\frac{f(x)}{x}\,dx\,.$$

5. You ask a question.

The Homework is due Friday January 24