Final Examination

Print your name: ________________ ________________

Print your ID #: ________________________________

You have 2 hours to solve the problems. Good luck!

1. Let $\alpha \geq 0$ and define $f_\alpha \in \mathrm{C}([0,1],\mathbb{R})$ by

$$f_\alpha(x) = \frac{1}{1+\alpha x}\,,\ x \in (0,1)\,.$$

For $A > 0$ consider the set $S_A = \big\{ f_\alpha \,\big|\, \alpha \in [0,A) \big\}$. Is it equicontinuous? What if $A = \infty$? Motivate your answers.

2. Show that

$$f : \mathbb{R} \to \mathbb{R}\,,\ x \mapsto \frac{\cos(x) - 1}{x^2}$$

is analytic at $x = 0$.

3. Consider

$$f_n(x) = e^{-nx}\,,\ x \in (0,\infty)\,,\ n \in \mathbb{N}\,.$$

What is $\sum_{n=0}^\infty f_n$? Is the convergence uniform? Motivate your answers.

4. Determine the function which is represented by the following power series

$$\sum_{n=1}^{\infty} n\, x^n$$

in the interval $(-1, 1)$. Justify your answer.

5. Let $(b_n)_{n\in\mathbb{N}}$ be a positive decreasing sequence. Show that $\sum_{n=0}^{\infty} a_n$ converges absolutely if

$$|a_n| \leq b_n - b_{n+1}\,,\ n \in \mathbb{N}\,.$$

6. Consider the following spaces of functions

$$\begin{aligned}\mathrm{BC}(\mathbb{R}, \mathbb{R}) &= \big\{ f \in \mathrm{C}(\mathbb{R}, \mathbb{R}) \,\big|\, \|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)| < \infty \big\}\,,\\ \mathrm{C}_0(\mathbb{R}, \mathbb{R}) &= \big\{ f \in \mathrm{C}(\mathbb{R}, \mathbb{R}) \,\big|\, \lim_{|x|\to\infty} f(x) = 0 \big\}\,.\end{aligned}$$

Show that $\mathrm{C}_0(\mathbb{R}, \mathbb{R})$ is a closed subset of $\big(\mathrm{BC}(\mathbb{R}, \mathbb{R}), \|\cdot\|_\infty\big)$.

7. Let $M_1 = (M, d_1)$ and $M_2 = (M, d_2)$ be metric spaces and assume that $d_2 \leq c\, d_1$ for some positive $c > 0$. Prove that

$$O \overset{o}{\subset} M_2 \Rightarrow O \overset{o}{\subset} M_1\,.$$

8. Fix $N \geq 1$ and show that the set

$$C_N = \Big\{ \sum_{n=1}^{N} a_n \cos(n\pi x) \,\Big|\, \sup_{1\leq n\leq N} |a_n| \leq 1 \Big\} \subset \mathrm{C}(\mathbb{R}, \mathbb{R})$$

is uniformly bounded and uniformly equicontinuous.

9. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in a metric space (M, d) and assume it possesses a convergent subsequence. Show that the whole sequence converges.