Assignment 16

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1. Prove that a normed vector space $(V, |\cdot|_V)$ is an inner product space iff
$$|x+y|_V^2 + |x-y|_V^2 = 2\big(|x|_V^2 + |y|_V^2\big)\ ,\ x, y \in V\ .$$
[Hint: Use the polarization identity.]

2. Give an example of a norm which cannot be derived from an inner product and an example of a metric which is not induced by a norm.

3. Define
$$l_\infty(\mathbb{K}) := \big\{x \in \mathbb{K}^{\mathbb{N}} \,\big|\, |x|_\infty := \sup_{n\in\mathbb{N}} |x_n| < \infty\big\}$$
and
$$c_0(\mathbb{K}) := \big\{x \in l_\infty(\mathbb{K}) \,\big|\, \lim_{n\to\infty} x_n = 0\big\}\ .$$
Show that $(l_\infty(\mathbb{K}), |\cdot|_\infty)$ and $(c_0(\mathbb{K}), |\cdot|_\infty)$ are complete normed vector spaces.

4. Two norms $|\cdot|_1$ and $|\cdot|_2$ on a vector space V are said to be *equivalent* iff there exists a constant $c \geq 1$ such that
$$\frac{1}{c}\,|x|_1 \leq |x|_2 \leq c\,|x|_1\ ,\ x \in V\ .$$
Prove that equivalent norms induce the same topology.

5. Let (M, d) be a metric space and show that
$$\tau := \{O \subset M \,|\, O \text{ is open}\}$$
defines a topology on M. Compute τ for $M = \mathbb{R}$ and d defined by
$$d(x,y) = \begin{cases} 0\,, & x = y \\ 1\,, & x \neq y\,. \end{cases}$$
For $M = \mathbb{R}$ define
$$d(x,y) = \frac{2}{\pi}\int_{\min(x,y)}^{\max(x,y)} \frac{d\xi}{1+\xi^2}\ ,\ x, y \in \mathbb{R}\,.$$
Show that (M, d) is a metric space and compute $\mathbb{B}(0,1)$, $\mathbb{B}(1, \frac{\pi}{4})$.

Homework due by Thursday, March 9 2006