Math 205 Winter Term 2006

Assignment 12

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1. Let $f_n(x) := \frac{1}{nx}$, $x \in (0,\infty)$ and prove the following claims:
   (i) $(f_n)_{n\in\mathbb{N}}$ converges to zero pointwise.
   (ii) For any $r > 0$, $\big(f_n|_{[r,\infty)}\big)_{n\in\mathbb{N}}$ converges uniformly.
   (iii) $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly.

2. Determine which of the following sequences $(f_n)_{n\in\mathbb{N}}$ converge uniformly on $(0,1) \ni x$:
   (i) $f_n(x) := x^{\frac{1}{n}}$, $n \in \mathbb{N}$.
   (ii) $f_n(x) := \frac{1}{1+nx}$, $n \in \mathbb{N}$.
   (iii) $f_n(x) := \frac{x}{1+nx}$, $n \in \mathbb{N}$.

3. Find a sequence of functions $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{R}([0,1],\mathbb{R})$ which converges pointwise to zero but for which
$$\int_0^1 f_n(x)\,dx \not\to 0\,(n \to \infty)\,.$$
Also find a sequence which does not converge to zero pointwise but for which
$$\int_0^1 f_n(x)\,dx \to 0\,(n \to \infty)\,.$$

4. For $\alpha \in (0,1)$ let
$$\mathrm{C}^\alpha([0,1],\mathbb{K}) := \Big\{f \in \mathrm{C}([0,1],\mathbb{K}) \,\Big|\, [f]_\alpha = \sup_{x\neq y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty\Big\}$$
be the space of Hölder continuous real- or complex-valued functions defined on $[0,1]$. Show that it is complete w.r.t. $\|\cdot\|_\alpha = \|\cdot\|_\infty + [\cdot]_\alpha$. In other words, prove that any sequence satisfying
$$\forall\varepsilon > 0\ \exists M \in \mathbb{N} \text{ s.t. } \|f_n - f_m\|_\alpha \le \varepsilon,\ m,n \ge M\,.$$
converges to some limit $f \in \mathrm{C}^\alpha([0,1],\mathbb{K})$.

5. Let $f \in \mathrm{C}\big([0,1],\mathbb{R}\big)$ and define the Bernstein polynomials by
$$p_n(f,x) = \sum_{k=0}^n \binom{n}{k} f(\frac{k}{n}) x^k (1-x)^{n-k}\,,\ x \in [0,1]\,,\ n \in \mathbb{N}\,.$$

Show that

$$\|p_n(f,\cdot) - f\|_\infty \underset{n\to\infty}{\longrightarrow} 0\,.$$

This means that any continuous function defined on a compact interval can be uniformly approximated by a sequence of polynomials. We shall see a generalization of this fact in class.

[*Hint: Use the identity*

$$\sum_{k=0}^{n}(k-nx)^2\binom{n}{k}x^k(1-x)^{n-k} = nx(1-x)$$

and split the sum into two parts according to whether $|x-\frac{k}{n}| \le \delta$ *or* $|x-\frac{k}{n}| > \delta > 0$.]

Homework due by Thursday, February 9 2006.