Assignment 10

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1. Let $\varphi \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ and $f \in \mathrm{C}([a,b], \mathbb{R})$ for $a < b \in \mathbb{R}$. Assume that φ is convex, that is, that it satisfies

   $\varphi\big((1-t)x + ty\big) \leq (1-t)\varphi(x) + t\varphi(y)\,,\ x, y \in \mathbb{R}\,,\ t \in [0,1]\,.$

   Prove the validity of the following *Jensen's inequality*

   $$\varphi\big(\frac{1}{b-a}\int_a^b f(x)\,dx\big) \leq \frac{1}{b-a}\int_a^b \varphi\big(f(x)\big)\,dx\,.$$

2. The Legendre ploynomials L_n can be defined by

   $$P_n(x) = \frac{1}{2^n\,n!}\frac{d^n}{dx^n}(x^2-1)^n\,,\ n = 0, 1, 2, \dots$$

   Show that

   $$\int_{-1}^{1} P_n(x) P_m(x)\,dx = 0\,,\ m \neq n\,.$$

3. Show that the improper integral

   $$\Gamma(\alpha) = \int_0^\infty t^\alpha e^{-t}\,dt$$

   exists for every $\mathbb{R} \ni \alpha > -1$ and verify that

   $$\Gamma(n) = n!\,,\ n \in \mathbb{N}\,.$$

4. Assume that $f \in \mathrm{C}^1([-1,1])$ and prove that the following *principal value integral* exists (i.e. that the defining limit always exists)

   $$p.v.\int_{-1}^{1} \frac{f(x)}{x}\,dx = \lim_{\varepsilon \to 0}\int_{|x|\geq\varepsilon} \frac{f(x)}{x}\,dx\,.$$

   Also show that

   $$\int_{-1}^{1} \log(|x|) f'(x)\,dx = -p.v.\int_{-1}^{1}\frac{f(x)}{x}\,dx\,.$$

5. Let $f \in \mathrm{C}^1([a,b], \mathbb{R})$ and $x, y \in [a,b]$. Prove the mean value theorem in integral form

   $$f(y) = f(x) + (y-x)\int_0^1 f'((1-t)x + ty)dt\,.$$

Homework due by Thursday, January 26 2006