Chapter 7

Function Sequences and Series

7.1 Complex Numbers

This is not meant to be a thourogh and completely rigorous construction of complex numbers but rather a quick brush up of things assumed to be already known.

7.1.1 Basic Properties

The field of complex number can be viewed as $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ but is also convenient to have a geometric model like $\mathbb{C} = \mathbb{R}^2$. The next table summarizes the basic field operations in the two different model

$\mathbb{C} = \mathbb{R} + i\mathbb{R}$	$\mathbb{C} = \mathbb{R}^2$
$z = x + iy$	$z = (x, y)$
$x = \Re(z)\,,\ y = \Im(z)$	$x = \Re(z)\,,\ y = \Im(z)$
$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$	$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$
$z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$	$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$
$i^2 = -1$	$(0,1)(0,1) = (-1,0)$

$(\mathbb{C}\,,+\,,\cdot)$ is an algebraically closed field. We also recall the simple formula

$$\frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\,,\ x \neq 0 \neq y\,.$$

Further we define the *absolute value* (or *modulus*) of a complex number z by

$$|z| := \sqrt{x^2+y^2}$$

and mention the simple properties

$$|z_1\, z_2| = |z_1|\,|z_2|\,,\ |z_1 + z_2| \le |z_1| + |z_2|\,,\ z_1\,,\ z_2 \in \mathbb{C}\,.$$

What are their geometric meaning?

Definitions 7.1.1. (Convergence)
Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$.
(i) We say that it *converges* in $\mathbb{C}$ towards a limit $z_\infty \in \mathbb{C}$ iff

$$z_n \to z_\infty\ (n \to \infty)\ \forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N} \text{ s.t. } |z_n - z_\infty| \le \varepsilon\ \forall\, n \ge N\,.$$

(ii) We call it a *Cauchy sequence* iff

$$\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N} \text{ s.t. } |z_n - z_m| \le \varepsilon\ \forall\, m, n \ge N\,.$$

(iii) A function $f : \mathbb{R} \to \mathbb{C}$ is called *complex-valued*. In this case

$$f = g + ih \text{ for } g, h : \mathbb{R} \to \mathbb{R}\,.$$

Definition 7.1.2. A complex-valued function $f : D_f \overset{o}{\subset} \mathbb{R} \to \mathbb{C}$ is said to be *differentiable at* $x_0 \in D_f$ iff there exists $f'(x_0) \in \mathbb{C}$ such that

$$f(x_0 + h) = f(x_0) + f'(x_0)h + o(|h|)\,.$$

It is simply called *differentiable* if it differentiable everywhere and continuously differentiable if its derivative $f' : D_f \to \mathbb{C}$ is continuous. *Continuity* of a complex-valued function is defined similarly to that of real-valued function, where the absolute valued has to be substituted by the modulus. Finally f is *R-integrable* if any of the conditions of theorem 6.7.6 satisfied, where convergence is now in $\mathbb{C}$.

Remarks 7.1.3. (a) Observe that

$$z_n \to z_\infty\ (n \to \infty) \iff \begin{cases} x_n \to x_\infty \\ y_n \to y_\infty \end{cases} \quad (n \to \infty)$$

for $z_\infty = x_\infty + iy_\infty$. (b) Using the fact that $f = g + ih$, many properties of complex-valued functions can be reconducted to properties of g and h. In particular

$$f \text{ is continuous at } x_0 \in D_f \iff g, h \text{ are continuous at } x_0\,. \tag{7.1.1}$$

$$f \text{ is differentiable at } x_0 \in D_f \iff g, h \text{ are differentiable at } x_0\,. \tag{7.1.2}$$

$$T_m(f, x_0) = T_m(g, x_0) + iT_m(h, x_0)\,,\ f \in \mathrm{C}^m\big(D_f, \mathbb{C}\big)\,. \tag{7.1.3}$$

$$f \in \mathcal{R}(a, b; \mathbb{C}) \iff g, h \in \mathcal{R}(a, b)\,. \tag{7.1.4}$$

Theorem 7.1.4. *Let $f \in \mathcal{R}(a, b : \mathbb{C})$, then $|f| \in \mathcal{R}(a, b)$ and*

$$|\int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx\,.$$

Proof. The claim follows by the inequality

$$\big||z_1| - |z_2|\big| \leq |x_1 - x_2| + |y_1 - y_2|\,,$$

which follows from the triangular inequality and which implies that

$$Osc(|F|, \mathcal{P}) \leq \mathrm{osc}(f, \mathcal{P}) + \mathrm{osc}(g, \mathcal{P}) \text{ and } |S(F, \mathcal{P})| \leq S(|F|, \mathcal{P})\,.$$

$\surd$

7.2 Numerical series and Sequences

7.2.1 Convergence and Absolute Convergence

We begin by remarking that the convergence of a series $\sum_{k=1}^{\infty} x_k$ can simply be viewed as the convergence of the sequence $(s_n)_{n\in\mathbb{N}}$ of its partial sums $s_n = \sum_{k=1}^{n} x_k$. In this sense a series is nothing but a sequence itself.

Definition 7.2.1. A series $\sum_{k=1}^{\infty} x_k$ of complex numbers $x_k \in \mathbb{C}$, $k \in \mathbb{N}$ is called *absolutely convergent* iff $\sum_{k=1}^{\infty} |x_k|$ is convergent.

Example 7.2.2. (Geometric Series)
Let $r \in [0, \infty)$ and consider the geometric series $\sum_{k=1}^{\infty} r^k$. Considering its partial sums, we easily obtain that

$$\sum_{k=1}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

and conclude that convergence is given if $r < 1$. What is the limit (or value of the series)?

Proposition 7.2.3. *(Elementary Properties)*
Consider two convergent series $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ of complex numbers. Then, for $\lambda \in \mathbb{C}$,

$$\sum_{k=1}^{\infty}(x_k + y_k) \text{ converges and } \sum_{k=1}^{\infty}(x_k + y_k) = \sum_{k=1}^{\infty} x_k + \sum_{k=1}^{\infty} y_k\,, \tag{7.2.1}$$

$$\sum_{k=1}^{\infty} \lambda x_k \text{ converges and } \sum_{k=1}^{\infty} \lambda x_k = \lambda \sum_{k=1}^{\infty} x_k\,. \tag{7.2.2}$$

Theorem 7.2.4. $\sum_{k=1}^{\infty} x_k$ *converges iff*

$$\forall\, \varepsilon > 0\, \exists\, N(\varepsilon) \text{ such that } \Big|\sum_{k=m}^{n} x_k\Big| \le \varepsilon\, \forall\, m, n \ge N(\varepsilon)\,.$$

Proof. Since $\mathbb{C}$ is complete and the condition above implies that the sequence of partial sums is a Cauchy sequence, its converence gives the claim. $\surd$

Remarks 7.2.5. (a) If $\sum_{k=1}^{\infty} x_k$ converges, then $x_k \to 0\ (k \to \infty)$.
(b) Absolute convergence implies convergence since

$$\Big|\sum_{k=m}^{n} x_k\Big| \le \sum_{k=m}^{n} |x_k|\ \forall m, n \in \mathbb{N}\,.$$

Theorem 7.2.6. *(Comparison Test)*
Let $\sum_{k=1}^{\infty} x_k$ *and* $\sum_{k=1}^{\infty} y_k$ *be given such that*

$$|x_k| \le y_k\,,\ k \in \mathbb{N}\,.$$

Then, if $\sum_{k=1}^{\infty} y_k$ *converges, the series* $\sum_{k=1}^{\infty} x_k$ *is absolutely convergent.*

Proof. The inequality

$$\Big|\sum_{k=m}^{n} x_k\Big| \le \sum_{k=m}^{n} |x_k| \le \sum_{k=m}^{n} y_k\, \forall\, m \le n \in \mathbb{N}$$

implies that $\sum_{k=1}^{\infty} |x_k|$ satisfies Cauchy criterion if $\sum_{k=1}^{\infty} y_k$ does. If $\sum_{k=1}^{\infty} y_k$ converges, that is clearly the case. $\surd$

Corollary 7.2.7. *(Ratio and Root Tests)*

(i) $\exists\, N \in \mathbb{N}\,,\ r \in (0,1)$ *such that* $\Big|\dfrac{x_{k+1}}{x_k}\Big| \le r < 1\, \forall\, k \ge N$

$$\Longrightarrow \sum_{k=1}^{\infty} x_k \text{ converges absolutely.}$$

The series diverges, if $N \in \mathbb{N}$ *can be found such that* $|\frac{x_{k+1}}{x_k}| \ge 1$ *for all* $k \ge N$.

(ii) $\exists\, N \in \mathbb{N}\,,\ r \in (0,1)$ *such that* $|x_k|^{1/k} \le r < 1$

$$\Longrightarrow \sum_{k=1}^{\infty} x_k \text{ converges absolutely.}$$

Example 7.2.8. The series $\sum_{k=1}^{\infty} \frac{1}{k^a}$ converges for $a > 1$ and diverges for $a \leq 1$.

Definition 7.2.9. (Rearrangement)
Let $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation, then

$$\sum_{k=1}^{\infty} x_{\pi(k)}$$

is called *rearrangement* of $\sum_{k=1}^{\infty} x_k$.

It is natural to ask whether a convergent sequence preserves its convergence after a rearrangement. The next two theorems answer precisely this question.

Theorem 7.2.10. *(i) If $\sum_{k=1}^{\infty} x_k$ is absolutely convergent and π is any permutation of the naturals, then $\sum_{k=1}^{\infty} x_{\pi(k)}$ converges absolutely as well and has the same value.*
(ii) If $\sum_{k=1}^{\infty} x_{\pi(k)}$ converges regardless of the choice of permutation π, then $\sum_{k=1}^{\infty} x_k$ is absolutely convergent.

Proof. Let π be a rearrangement. Since $\sum_{k=1}^{\infty} x_k$ is absolutely convergent, for any $\varepsilon > 0$ we can find $\tilde{N} \in \mathbb{N}$ such that

$$\sum_{k=\tilde{N}+1}^{\infty} |x_k| \leq \varepsilon \,.$$

Then choose $N \in \mathbb{N}$ such that

$$\{x_1, \ldots, x_{\tilde{N}}\} \subset \{x_{\pi(1)}, \ldots, x_{\pi(N)}\} \,.$$

In that case

$$|\sum_{k=1}^{n} x_{\pi(k)} - \sum_{k=1}^{\tilde{N}} x_k| \leq \sum_{k=\tilde{N}+1}^{\infty} |x_k| \leq \varepsilon \,, \, \forall\, n \geq N$$

and consequently

$$|\sum_{k=1}^{\infty} x_k - \sum_{k=1}^{n} x_{\pi(k)}| \leq |\sum_{k=1}^{\infty} x_k - \sum_{k=1}^{\tilde{N}} x_n| + |\sum_{k=1}^{\tilde{N}} x_n - \sum_{k=1}^{n} x_{\pi(k)}|$$
$$\leq 2\varepsilon \,, \, \forall\, n \geq N$$

which gives the desired convergence.
(ii) We prove the converse. Assume that $\sum_{k=1}^{\infty} |x_k|$ is divergent. The we need to find at least a permutation π such that $\sum_{k=1}^{\infty} x_{\pi(k)}$ diverges, too. Without loss of generality we can assume that

$$x_k \in \mathbb{R} \,\forall\, k \in \mathbb{N} \text{ and } \exists\, (k_n)_{n\in\mathbb{N}} \text{ with } x_{k_n} > 0 \text{ and } \sum_{n=1}^{\infty} x_{k_n} = \infty\,.$$

To get a proper rearragement we still need to include the terms which are still missing. Let $(\tilde{k}_n)_{n\in\mathbb{N}}$ be the sequence of the indeces missed by $(k_n)_{n\in\mathbb{N}}$. Notice that it might be finite. Then define N_1 such that

$$s_1 + x_{\tilde{k}_1} := \sum_{n=1}^{N_1} x_{k_n} + x_{\tilde{k}_1} \geq 1$$

and squeeze $x_{\tilde{k}_1}$ into $(x_{kn})_{n\in\mathbb{N}}$ right after $x_{k_{N_1}}$. Then proceed inductively to choose $N_m \in \mathbb{N}$ for $m \geq 2$ such that

$$\sum_{l=1}^{m-1} \bigl(s_l + x_{\tilde{k}_l}\bigr) + \sum_{n=N_{m-1}}^{N_m} x_{k_n} + x_{\tilde{k}_m} \geq m$$

and squeeze $x_{\tilde{k}_m}$ right after $x_{k_{N_m}}$. In this fashion we obtain a rearrangement of the original series which diverges as well. $\surd$

Theorem 7.2.11. *Let $\sum_{k=1}^{\infty} x_k$ be a conditionally convergent series and $x_\infty \in \mathbb{R}$. Then there exists a permutation π of the naturals such that*

$$\sum_{k=1}^{\infty} x_{\pi(k)} = x_\infty\,.$$

Thus a conditionally convergent series can be made to converge to any limit by rearranging it.

Proof. (i) Consider the two subsequences $(x_k^+)_{k\in\mathbb{N}}$ and $(x_k^-)_{k\in\mathbb{N}}$ comprising the positive and negative terms of $(x_k)_{k\in\mathbb{N}}$, respectively. Then necessarily

$$\text{either } \sum_{k=1}^{\infty} x_k^+ = \infty \text{ or } \sum_{k=1}^{\infty} x_k^- = -\infty$$

since, otherwise, the series would converge absolutely.
(ii) Next we put together a new series recusively as follows. First find $N_1^\pm$

such that

$$\begin{cases} N_1^+ := \inf\left\{N \,\middle|\, s_1^+ := \sum_{k=1}^N x_k^+ > x_\infty\right\}, \\ N_1^- := \inf\left\{N \,\middle|\, \underbrace{s_1^+ + \sum_{k=1}^N x_k^-}_{=:s_1^-} < x_\infty\right\}, \end{cases}$$

and then, for $m \geq 2$,

$$\begin{cases} N_m^+ := \inf\left\{N \,\middle|\, \underbrace{s_{m-1}^- + \sum_{k=N_{m-1}^+ +1}^{N_m} x_k^+}_{s_m^+} > x_\infty\right\}, \\ N_m^- := \inf\left\{N \,\middle|\, \underbrace{s_m^+ + \sum_{k=N_{m-1}^-}^{N_m^-} x_k^-}_{s_m^-} < x_\infty\right\}, \end{cases}$$

(iii) Finally we observe that

$$s_m^- \to x_\infty \text{ and } s_m^+ \to x_\infty \ (m \to \infty)$$

follows from

$$|x_\infty - s_m^+| \leq |x_{N_m^+}| \text{ and } |x_\infty - s_m^-| \leq |x_{N_m^-}| \ \forall m \in \mathbb{N}$$

combined with remark 7.2.5(a). $\surd$

7.2.2 Summation by Parts

Motivation. As the name says, summation by parts will have something to do with integration by parts. Since an integral is a limit of sums and a series can be viewed as an integration, the two can be viewed as one and the same thing. This can actually be made precise but we won't do it. Instead we shall see that integration by parts provides a simple trick to investigate the convergence of conditionally convergent series.

Let

$$B_n = \sum_{k=1}^n b_k\,,\ a_n = A_{n+1} - A_n \text{ for } (b_n)_{n\in\mathbb{N}}\,,\ (A_n)_{n\in\mathbb{N}}\mathbb{C}^{\mathbb{N}}\,,$$

then the idea lies in the simple observation that

$$\begin{aligned}
&\sum_{n=1}^{m} a_n B_n \\
&\quad = a_1 B_1 + \cdots + a_m B_m = (A_2 - A_1) b_1 + (A_3 - A_2)(b_1 + b_2) + \ldots \\
&\qquad + (A_{m+1} - A_m)(b_1 + b_2 + \cdots + b_m) \\
&\quad = b_1 (A_2 - A_1 + A_3 - A_2 + \cdots + A_{m+1} - A_m) + \\
&\qquad b_2 (A_3 - A_2 + \cdots + A_{m+1} - A_m) + \cdots + b_m (A_{m+1} - A_m) \\
&\quad = b_1 (A_{m+1} - A_1) + b_2 (A_{m+1} - A_2) + \cdots + b_m (A_{m+1} - A_m) \\
&\quad = -\sum_{n=1}^{m} b_n A_n + B_m A_{m+1} \,. \qquad (7.2.3)
\end{aligned}$$

Example 7.2.12. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. Defining $A_n = \frac{1}{n}$ and $b_n = (-1)^n$ for $n \in \mathbb{N}$ we obtain

$$a_n = \frac{1}{n+1} - \frac{1}{n} = -\frac{1}{n(n+1)} \,, \; B_n = \begin{cases} 0 \,, \; n \text{ is even} \,, \\ -1 \,, \; n \text{ is odd} \end{cases} .$$

Since $|A_{n+1} B_n| \leq \frac{1}{n} \to 0 \, (n \to \infty)$ we finally get that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges since

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)2n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \,.$$

Theorem 7.2.13. *Let $(A_n)_{n \in \mathbb{N}}$ be a decreasing sequence of positive numbers converging to zero.*
(i) Then

$$\sum_{n=1}^{\infty} (-1)^n A_n \text{ converges.}$$

(ii) Let $(b_n)_{n \in \mathbb{N}}$ be given and let $B_n := \sum_{k=1}^{n} b_k$ with $|B_n| \leq M$ for all $n \in \mathbb{N}$ and some $M > 0$. Then

$$\sum_{n=1}^{\infty} A_n b_n \text{ converges.}$$

Proof. (ii) Defining $a_n = A_{n+1} - A_n$ we obtain that $-a_n \geq 0$ and that

$$\begin{aligned}
-\sum_{n=1}^{m} a_n &= -(A_2 - A_1) - (A_3 - A_2) - \cdots - (A_{m+1} - A_m) \\
&= A_1 - A_{m+1} \to A_1 \; (m \to \infty) \,,
\end{aligned}$$

which makes $\sum_{n=1}^m a_n$ absolutely convergent. Now, multiplying by B_n term by term, we arrive at

$$\sum_{n=k}^{l} |a_n B_n| \leq M \sum_{n=k}^{l} |a_n| \to 0 \; (k, l \to \infty)$$

which, by Cauchy's criterion, implies absolute convergence of $\sum_{n=k}^l a_n B_n$. Next

$$|A_{n+1} B_n| \leq M |A_{n+1}| \to 0 \; (n \to \infty)$$

together with (7.2.3) implies the convergence of $\sum_{n=k}^l A_n b_n$ as desired.
(i) Just take $b_n = (-1)^n$ in (ii). $\surd$

7.3 Uniform Convergence

A function can be thought of as a "very long" vector with uncountably many components, aka the values it assumes. If we are to introduce a concept of convergence for function sequences we will have to bear that in mind. In particular we should specify if we would like all components to converge at the same rate, or if we would rather allow each component to get there at its own pace. These two requirements lead to two different concepts of convergence for function sequences. They both have a *raison d'être* and are both useful in a variety of applications.

7.3.1 Uniform Limits and Continuity

Definition 7.3.1. (Pointwise and Uniform Convergence)
Given a sequence of real- or complex-values functions $(f_n)_{n\in\mathbb{N}}$ defined on some domain $D \subset \mathbb{R}$ we say that it
(i) converges *pointwise* to $f_\infty : D \to \mathbb{R}$ iff

$$\forall\, x \in D\, \forall \varepsilon > 0\, \exists\, N \in \mathbb{N} \text{ such that } |f_\infty(x) - f_n(x)| \leq \varepsilon\, \forall\, n \geq N\,.$$

(ii) converges *uniformly* to $f_\infty : D \to \mathbb{R}$ iff

$$\forall\, \varepsilon > 0\, \exists\, N \in \mathbb{N} \text{ such that } |f_\infty(x) - f_n(x)| \leq \varepsilon\,,\ \forall\, n \geq N\,,\ \forall\, x \in D\,.$$

(iii) converges *locally uniformly* to $f_\infty : D \to \mathbb{R}$ iff there is a neighborhood about each point in which the convergence is uniform.

What does N depend on in the first case? In the second? And in the last?

Example 7.3.2. Consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$ given by

$$f_n(x) := x^n\,,\; x \in [0,1)\,,\; n \in \mathbb{N}\,.$$

It converges to the zero function pointwise, but not uniformly! Explain why.

Remarks 7.3.3. (a) A more suggestive way to visualize uniform continuity is by the use of the so-called supremum norm

$$f_n \to f_\infty \; (n \to \infty) \text{ uniformly iff } \forall\, \varepsilon > 0\; \exists\, N \in \mathbb{N} \text{ with}$$
$$\|f_\infty - f_n\|_\infty := \sup_{x\in D} |f_\infty(x) - f_n(x)| \leq \varepsilon\,,\; \forall\, n \geq N\,.$$

In the example $\sup_{x\in[0,1)} |x^n - 0| = 1\; \forall\, n \in \mathbb{N}$ and therefore no uniform convergence is possible.
(b) Pointwise convergence is very weak. In fact many properties shared by a whole sequence often go lost in the limit. Consider the sequence $(f_n)_{n\in\mathbb{N}}$ given by

$$f_n(x) := \begin{cases} 0\,, & x \in [-1,0] \\ nx\,, & x \in (0,\frac{1}{n}) \\ 1\,, & x \in [\frac{1}{n},1] \end{cases}\,,\; n \in \mathbb{N}$$

Then, f_n converges to f_∞, where

$$f_\infty(x) := \begin{cases} 0\,, & x \in [-1,0]\,, \\ 1 & x \in (0,1]\,. \end{cases}$$

Notice that $[-1,1]$ is compact and that

$$\|f_\infty - f_n\|_\infty = 1\; \forall\, n \in \mathbb{N}\,.$$

Theorem 7.3.4. *(Cauchy Criterion)*
A sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly iff

$$\forall\, \varepsilon > 0\; \exists\, N \in \mathbb{N} \textit{ such that } \|f_n - f_m\|_\infty \leq \varepsilon\; \forall\, m,n \geq N\,.$$

Proof. "$\Longrightarrow$": By uniform convergence, there is a limit function f_∞ such that, for any given $\varepsilon > 0$,

$$\|f_\infty - f_n\|_\infty \leq \frac{\varepsilon}{2}\; \forall\, n \geq N$$

for some $N \in \mathbb{N}$ and therefore

$$\|f_m - f_n\|_\infty \leq \|f_m - f_\infty\|_\infty + \|f_\infty - f_n\|_\infty \leq \varepsilon\; \forall\, m,n \geq N\,.$$

"$\Longleftarrow$": For $\varepsilon > 0$ choose $N \in \mathbb{N}$ such that

$$\|f_n - f_m\|_\infty \leq \varepsilon \forall\, m, n \geq N\,.$$

Then

$$|f_n(x) - f_m(x)| \leq \varepsilon\, \forall\, x \in D\,,\ \forall\, m, n \geq N\,.$$

Thus $\big(f_n(x)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for each $x \in D$. Since $\mathbb{C}$ is complete, it therefore has a limit for each $x \in D$, call it $f_\infty(x)$. This defines a function f_∞ on D for which

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \leq \varepsilon\, \forall\, x \in D\,,\ \forall\, m, n \geq N\,.$$

Letting m take off to infinity we therefore finally obtain

$$|f_n(x) - f_\infty(x)| \leq \varepsilon\, \forall\, x \in D\,,\ \forall\, n \geq N$$

which is the desired uniform convergence. $\surd$

Theorem 7.3.5.

$$\left.\begin{array}{l} f_n \in \mathrm{C}(D, \mathbb{C})\,,\ n \in \mathbb{N} \\ \|f_n - f_\infty\|_\infty \to 0\ (n \to \infty) \end{array}\right\} \Longrightarrow\ f_\infty \in \mathrm{C}(D, \mathbb{C})\,.$$

Proof. Fix $x_0 \in D$. For $\varepsilon > 0$ find $N \in \mathbb{N}$ such that

$$\|f_n - f_\infty\|_\infty \leq \frac{\varepsilon}{3}\, \forall\, n \geq N\,,$$

and $\delta > 0$ such that

$$|f_N(x) - f_N(x_0)| \leq \frac{\varepsilon}{3}\, \forall\, x \in D \text{ with } |x - x_0| \leq \delta\,.$$

Both are possible by assumption. Then

$$\begin{aligned} |f_\infty(x) - f_\infty(x_0)| \leq{}& \\ & |f_\infty(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f_\infty(x_0)| \\ & \qquad \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\, \forall\, x \in D \text{ with } |x - x_0| \leq \delta \end{aligned}$$

and continuity at x_0 is proven. The claim follows since x_0 was arbitrary. $\surd$

Remarks 7.3.6. (a) Example 7.3.2 shows that pointwise convergence is not enough to preserve continuity in the limit.
(b) Uniform continuity is also preserved. In fact

$$\left.\begin{array}{l} f_n \in \mathrm{BUC}(D, \mathbb{C})\,,\ n \in \mathbb{N} \\ \|f_n - f_\infty\|_\infty \to 0\ (n \to \infty) \end{array}\right\} \Longrightarrow\ f_\infty \in \mathrm{BUC}(D, \mathbb{C})\,.$$

7.3.2 Integration and Differentiation of Limits

Remark 7.3.7. We observe here that the Riemann integrability of a function $f \in \mathrm{B}([a,b],\mathbb{K})$ is equivalent to the validity of one of the following conditions:

$$\begin{aligned}
&\text{(i) } \forall\, \varepsilon > 0\, \exists\, \delta > 0 \text{ s.t. } |S_f(P) - S_f(\tilde{P})| \le \varepsilon \\
&\qquad \forall\, P, \tilde{P} \in \mathcal{P}\big([a,b]\big) \text{ with } \triangle(P), \triangle(\tilde{P}) \le \delta \\
&\text{(ii) } \exists I \in \mathbb{K} \text{ s.t. } \forall\, \varepsilon > 0\, \exists\, \delta > 0 \text{ s.t. } |S_f(P) - I| \le \varepsilon \\
&\qquad \forall\, P \in \mathcal{P}\big([a,b]\big) \text{ with } \triangle(P) \le \delta
\end{aligned}$$

We leave the proof as an exercise. It can be based on theorem 6.7.6.

Theorem 7.3.8.

$$\left.\begin{array}{l} f_n \in \mathcal{R}(a,b;\mathbb{C})\,,\ n \in \mathbb{N} \\ \|f_n - f_\infty\|_\infty \to 0\ (n \to \infty) \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} f_\infty \in \mathcal{R}(a,b;\mathbb{C})\,, \\ \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f_\infty(x)\,dx\,. \end{array}\right.$$

Proof. (i) Given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$\|f_n - f_\infty\|_\infty \le \frac{\varepsilon}{3(b-a)}\,.$$

It follows that

$$\begin{aligned}
|S(f_n,\mathcal{P}) - S(f_\infty,\mathcal{P})| &= \big|\sum_{k=1}^{n}[f_n(y_k) - f_\infty(y_k)](x_k - x_{k-1})\big| \\
&\le \sum_{k=1}^{n}|f_n(y_k) - f_\infty(y_k)|(x_k - x_{k-1}) \le \frac{\varepsilon}{3(b-a)}\sum_{k=1}^{n}(x_k - x_{k-1}) \le \frac{\varepsilon}{3}
\end{aligned}$$

for any partition $\mathcal{P}$ of $[a,b]$. By assumption we can find $\delta > 0$ with

$$|S(f_N,\mathcal{P}) - S(f_N,\tilde{\mathcal{P}})| \le \frac{\varepsilon}{3}\,,\ \text{whenever } \triangle(\mathcal{P})\,,\ \triangle(\tilde{\mathcal{P}}) \le \delta\,.$$

Finally we obtain that

$$\begin{aligned}
|S(f_\infty,\mathcal{P}) - S(f_\infty,\tilde{\mathcal{P}})| &\le |S(f_\infty,\mathcal{P}) - S(f_N,\mathcal{P})| \\
&\quad + |S(f_N,\mathcal{P}) - S(f_N,\tilde{\mathcal{P}})| + |S(f_N,\tilde{\mathcal{P}}) - S(f_\infty,\tilde{\mathcal{P}})| \le \varepsilon\,.
\end{aligned}$$

(ii) Now that we know that f_∞ is integrable we can find $\delta > 0$ with

$$|S(f_\infty,\mathcal{P}) - \int_a^b f_\infty(x)\,dx| \le \frac{\varepsilon}{3} \text{ when } \triangle(\mathcal{P}) \le \delta\,.$$

Then

$$|S(f_n,\mathcal{P}) - \int_a^b f_\infty(x)\,dx| \le \frac{2\varepsilon}{3}\ \forall\, n \ge N$$

and there is $\delta_n > 0$ such that

$$|S(f_n,\mathcal{P}) - \int_a^b f_n(x)\,dx| \le \frac{\varepsilon}{3} \text{ if } \triangle(\mathcal{P}) \le \delta_n\,.$$

Putting everything together we obtain

$$\begin{aligned}|\int_a^b f_\infty(x)\,dx - \int_a^b f_n(x)\,dx| &\le |\int_a^b f_\infty(x)\,dx - S(f_n,\mathcal{P})| \\ &\quad + |S(f_n,\mathcal{P}) - \int_a^b f_n(x)\,dx| \le \varepsilon\ \forall\, n \ge N\end{aligned}$$

and the proof is finished. $\surd$

We can also ask the question as to what conditions need to be satisfied for a sequence of differentiable functions to preserve that property in the limit.

Example 7.3.9. Uniform convergence is <u>not</u> enough! In fact, consider the sequence $(f_n)_{n\in\mathbb{N}}$ given through

$$f_n(x) := \sqrt{x^2 + \frac{1}{n^2}}\,,\ x \in [-1,1]\,.$$

Then f_n converges to $|\cdot|$ uniformly, but the absolute value function is not differentiable in the origin.

Theorem 7.3.10. *Let a sequence $(f_n)_{n\in\mathbb{N}}$ be given such that*

$$f_n \in \mathrm{C}^1\big((a,b),\mathbb{R}\big)$$

for $n \in \mathbb{N}$. Assume that

$$\left.\begin{array}{l} f_n \to f\ (n\to\infty)\ \textit{pointwise} \\ f_n' \to g\ (n\to\infty)\ \textit{uniformly}\end{array}\right\} \implies f \in \mathrm{C}^1\big((a,b),\mathbb{R}\big)\ \textit{and}\ f' = g\,.$$

Proof. Let $x_0 \in (a,b)$. Then by theorem 6.8.7 we have that

$$f_n(x) = f_n(x_0) + \int_{x_0}^x f_n'(x)\,dx\,,\ x \in (a,b)\,.$$

Letting $n \to \infty$, we obtain

$$f(x) = f(x_0) + \int_{x_0}^{x} g(x)\, dx\,,\ x \in (a,b)$$

by the assumptions and theorem 7.3.8. Moreover g is continuous by theorem 7.3.5 and therefore $f \in \mathrm{C}^1\big((a,b),\mathbb{R}\big)$ and $f'(x) = g(x)\,,\ x \in (a,b)$. $\surd$

Remarks 7.3.11. (a) Convergence at one single point for f_n together with uniform convergence of the sequence of derivatives would suffice.
(b) Local uniform convergence for f_n' would be enough as well.
(c) Uniform convergence of f_n' is not enough.

7.3.3 Unrestricted Convergence

Next we give another characterization of uniform convergence (on compact subsets).

Theorem 7.3.12. *Let $D \subset \mathbb{R}$ be compact and $(f_n)_{n\in\mathbb{N}} \in \mathrm{C}(D,\mathbb{R})^{\mathbb{N}}$. Then*

$$\begin{aligned}&f_n(x_n) \to f(x)\ \forall\, (x_n)_{n\in\mathbb{N}} \in D^{\mathbb{N}}\ \textit{s.t.}\ x_n \to x\ (n \to \infty)\\ &\qquad\qquad\iff \|f_n - f\|_\infty \to 0\ (n \to \infty)\,.\end{aligned}$$

Proof. "$\Longleftarrow$": Let a sequence $(x_n)_{n\in\mathbb{N}}$ be given which converges to $x \in D$. Then, given $\varepsilon > 0$ we find $\delta > 0$ and then $N \in \mathbb{N}$ such that

$$\begin{aligned}&|f(y) - f(x)| \le \frac{\varepsilon}{2} \text{ if } |y - x| \le \delta \text{ and}\\ &\qquad\qquad |x_n - x| \le \delta\,,\ \|f_n - f\|_\infty \le \frac{\varepsilon}{2}\ \forall\, n \ge N\,.\end{aligned}$$

Then

$$|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ \forall\, n \ge N\,.$$

"$\Longrightarrow$": Define the set

$$A_{m,N} := \Big\{x \in D \,\Big|\, |f_n(x) - f(x)| \le \frac{1}{m}\ \forall\, n \ge N\Big\}\,.$$

Then uniform convergence amounts to

$$\forall\, m \in \mathbb{N}\, \exists\, N(m) \text{ s.t. } A_{m,N(m)} = D\,.$$

We now claim that

$$\forall\, x \in D\, \forall\, m \in \mathbb{N}\, \exists\, N \in \mathbb{N} \text{ s.t. } A_{m,N} \in \mathcal{U}(x)\,.$$

Assume that this is not the case. Then we can find $x \in D$ and $m \in \mathbb{N}$ as well as a sequence $(x_N)_{N\in\mathbb{N}}$ in D with $x_N \to x$ as $N \to \infty$, but $x_N \notin A_{m,N}$. It then follows that

$$\exists\, k(N) \geq N \text{ s.t. } |f_{k(N)}(x_N) - f(x)| > \frac{1}{m}\,.$$

Next define the sequence $(y_n)_{n\in\mathbb{N}}$ by

$$y_k := \begin{cases} x_N\,, & k = k(N)\,,\\ x\,, & \text{otherwise}\,. \end{cases}$$

Now $y_k \to x$ as $k \to \infty$, but infinitely many $y_k's$ satisfy satisfy

$$|f_k(y_k) - f(x)| > \frac{1}{m}\,.$$

contradicting the assumed convergence. Finally, for any $m \in \mathbb{N}$ given,

$$\forall\, x \in D\, \exists\, N_x \in \mathbb{N} \text{ s.t. } x \in \overset{\circ}{A}_{m,N_x}\,.$$

Now since $D \subset \cup_{x\in D}\overset{\circ}{A}_{m,N_x}$, compactness of D implies the existence of a finite subcover, or that

$$D \subset \cup_{j=1,\dots,n}\overset{\circ}{A}_{m,N_{x_j}}$$

which gives $D \subset \overset{\circ}{A}_{m,\max_j(N_{x_j})} = \cup_{j=1,\dots,n}\overset{\circ}{A}_{m,N_{x_j}}$. $\checkmark$

7.4 Power Series

We have already talked about local approximation of functions by polynomials when we introduced the Taylor polynomial of differentiable functions. If a function turns out to possess infinitely many derivatives we can increase the degree of the Taylor polynomial more and more. In the limit we would get a series like

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n\,,\ (a_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}\,,$$

which is called *power series* centered at x_0 for obvious reasons. Here we used the notation $\mathbb{K} := \mathbb{R}, \mathbb{C}$. When Taylor developing a smooth function we would of course have

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

Eventhough it is easy to write down a power series or the Taylor expansion of a smooth functions, what really matters is their convergence! So the central question are: How can we assess the convergence of a power series? What kind of convergence can be expected?

7.4.1 Radius of Convergence

We first observe that by simple translation we can assume, without loss of generality, that the power series is centered at $x_0 = 0$. This corresponds to the intuitive observation the convergence will have something to do with the distance from the center but not with its location.

Remark 7.4.1. Convergence of a power series can be limited to its center as the example

$$\sum_{n=1}^{\infty}(nx)^n$$

shows.

Definition 7.4.2. (Radius of Convergence)
The *radius of convergence* $\rho \geq 0$ of a given power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is defined by

$$\rho = \sup\left\{|x-x_0| \in [0,\infty) \,\middle|\, \sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges}\right\}.$$

Remarks 7.4.3. (a) A power series converges uniformly and absolutely strictly within its radius of convergence. In other words, convergence is uniform and absolut in $[x_0-\rho+\varepsilon, x_0+\rho-\varepsilon]$ for any $\rho > \varepsilon > 0$. It therefore represents a contiunuous function on $(x_0-\rho, x_0+\rho)$. Why? To prove the claimed convergence, observe that for $|x-x_0| \leq \rho-\varepsilon$ the following estimate holds

$$\begin{aligned}\Big\|\sum_{n=0}^{N} a_n\Big(\frac{x-x_0}{\rho-\varepsilon/2}\Big)^n(\rho-\varepsilon/2)^n\Big\| &\leq \sum_{n=0}^{N} M\Big(\frac{|x-x_0|}{\rho-\varepsilon/2}\Big)^n \\ &\leq \sum_{n=0}^{N} M\Big(\frac{\rho-\varepsilon}{\rho-\varepsilon/2}\Big)^n < \infty.\end{aligned}$$

Notice that we have used the fact that $a_n(\rho-\varepsilon/2)^n \le M \;\forall\, n\in\mathbb{N}$ for some $M>0$ since the associated series $\sum_{n=1}^{\infty} a_n(\rho-\varepsilon/2)^n$ converges.
(b) At $x=x_0\pm\rho$ anything can happen as the following examples show

$$\sum_{n=1}^{\infty} x^n \text{ diverges at } x=\pm 1\,,$$

$$\sum_{n=1}^{\infty} nx^n \text{ diverges at } x=\pm 1\,,$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ diverges at } x=1 \text{ and converges at } x=-1\,,$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} \text{ converges at } x=\pm 1\,.$$

Lemma 7.4.4.

$$\lim_{n\to\infty} M^{1/n}=1\,,\ M>0\,.$$

Proof. It is clear that $\frac{1}{n}\log M\to 0$ $(n\to\infty)$, then continuity of the exponential function implies that

$$e^{\frac{1}{n}\log M}=M^{1/n}\to 1\ (n\to\infty)$$

and we are done. $\surd$

Theorem 7.4.5. *The radius of convergence of a power series $\sum_{n=1}^{\infty} a_n x^n$ is given by*

$$\rho=\frac{1}{\limsup_{n\to\infty}|a_n|^{1/n}}$$

Proof. The idea of the proof is to compare the series

$$\sum_{n=1}^{\infty}\left(|a_n|^{1/n}x\right)^n$$

to the geometric series.
(i) First we prove that

$$\limsup_{n\to\infty}|a_n|^{1/n}\le\frac{1}{\rho}\,.$$

This is obviously true if $\rho=0$. Assume then that $\rho>0$ and take $r\in(0,\rho)$. It follows that $|a_n x^n|\le M \;\forall\, n\in\mathbb{N}$ for some $M>0$ and $|x|\le r$. Thus we get

$$|a_n|^{1/n}\le\frac{M^{1/n}}{r}\;\forall\, n\in\mathbb{N}\,.$$

It follows that

$$\limsup_{n\to\infty} |a_n|^{1/n} \le \frac{1}{r} \limsup_{n\to\infty} M^{1/n} = \frac{1}{r}\,.$$

(ii) Let now $\frac{1}{\rho_0} := \limsup_{n\to\infty} |a_n|^{1/n}$. If we can show convergence of the series for $|x| < \rho_0$, then $\rho \ge \rho_0$, which would conclude the proof. Let $r < \rho_0$ and pick $\rho_1 \in (r, \rho_0)$, then

$$\limsup_{n\to\infty} |a_n|^{1/n} < \frac{1}{\rho_1}$$

and, consequently, we can find $k \in \mathbb{N}$ s.t.

$$|a_n|^{1/n} \le \frac{1}{\rho_1}\, \forall\, n \ge k\,.$$

Finally this gives

$$|a_n x^n| \le \left(\frac{r}{\rho_1}\right)^n \text{ for } |x| \le r \text{ and } n \ge k$$

and, finally,

$$\sum_{n\ge k} |a_n x^n| \le \sum_{n\ge k} \left(\frac{r}{\rho_1}\right)^n < \infty$$

implies the desired convergence. $\surd$

Examples 7.4.6. (a) Let p, q be polynomial functions such that $q(n) \neq 0\, \forall\, n \in \mathbb{N}$ and define

$$a_n := \frac{p(n)}{q(n)}\,.$$

Then $\rho = 1$.

Proof. Itis enough to show that

$$\lim_{n\to\infty} |p(n)|^{1/n} = 1$$

for any nonzero polynomial p. For n large we find constants $\underline{M}\,, \overline{M} > 0$ such that

$$\underline{M} n^k \le |p(n)| \le \overline{M} n^k$$

and it is therefore enough to show

$$\lim_{n\to\infty} \left(M n^k\right)^{1/n}$$

for $M > 0$, which follows from

$$\frac{1}{n}\log(Mn^k) = \frac{1}{n}\log M + \frac{k}{n}\log n \to 0 \; (n \to \infty)$$

and the continuity of the exponential function. $\surd$

(b) The series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence $\rho = \infty$. To see this we need to show $\lim_{n\to\infty}\big(\frac{1}{n!}\big)^{1/n} = 0$. For any $m \in \mathbb{N}$ we have that

$$n! \geq m^{n-m} \text{ if } n \geq m$$

and, consequently, $(n!)^{1/n} \geq m\, m^{-m/n}$ if $n \geq m$. Finally it follows that

$$\liminf_{n\to\infty}(n!)^{1/n} \geq m \;\forall\, m \in \mathbb{N}$$

and the proof is finished.

Theorem 7.4.7. *Let $\rho > 0$ be the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$. Define*

$$f(x) := \sum_{n=0}^{\infty} a_n x^n \,,\; x \in (-\rho, \rho)\,.$$

Then

$$f \in \mathrm{C}^1\big((-\rho,\rho), \mathbb{K}\big) \text{ and } f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}\,.$$

Then, it follows that

$$f \in \mathrm{C}^\infty\big((-\rho,\rho), \mathbb{K}\big) \text{ and } f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1) a_n x^{n-k}\,.$$

Proof. Define $f_n(x) = a_n x^n$, then, obviously

$$f_n \in \mathrm{C}^1\big((-\rho,\rho), \mathbb{K}\big)$$

and $\sum_{n=0}^{\infty} f_n$ converges uniformly in $(-r, r)$ for any $r \in (0, \rho)$. What about $\sum_{n=0}^{\infty} f_n'$? Since $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ we see that its radius of convergence is given by

$$\frac{1}{\rho'} = \limsup_{n\to\infty}\Big((n+1)|a_{n+1}|\Big)^{1/n} = \limsup_{n\to\infty} |a_{n+1}|^{1/n} = \frac{1}{\rho}$$

whrere the last equality follows from example 7.4.6(a). Thus $\sum_{n=0}^{\infty} f_n'$ converges uniformly in $(-r,r)$ as well and therefore

$$f \in \mathrm{C}^1\big((-r,r),\mathbb{K}\big)\;\forall\, r \in (0,\rho)$$

which amount to the claim. The argument can then be iterated to cover all derivatives. $\surd$

Remark 7.4.8. The above proof shows that a power series can be differentiated term-by-term within its domain of convergence and that the resulting power series for the derivative has the same radius of convergence as the original series.

Warning. It has to be pointed out that NOT ALL C^∞-functions can be represented by power series! In fact the non-zero function

$$f(x) = \begin{cases} e^{-1/x^2}\,, & x \in \mathbb{R}\setminus\{0\}\,, \\ 0\,, & x = 0\,, \end{cases}$$

has a vanishing power series expansion in $x_0 = 0$.

Definition 7.4.9. (Analytic Functions)
Any function $f : D \subset \mathbb{R} \to \mathbb{K}$ is called *analytic* if it has a power series expansion about every point in its domain D. We also define

$$\mathrm{C}^\omega(D,\mathbb{K}) := \big\{f : D \to \mathbb{K} \,\big|\, f \text{ is analytic}\big\}\,.$$

Proposition 7.4.10. *(Identity)*
Let

$$\sum_{n=0}^{\infty} a_n x^n \equiv \sum_{n=0}^{\infty} b_n x^n$$

and let both series have the same radius of convergence $\rho > 0$. Then

$$a_n = b_n \;\forall\, n \in \mathbb{N}\,.$$

Proof. Let $|x| < r < \rho$, then the both series define the same function there

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n\,.$$

It follows that $a_n = \frac{f^{(n)}(0)}{n!} = b_n$ for any $n \in \mathbb{N}$. $\surd$

7.4.2 Analytic Continuation

Consider a power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $\rho > 0$. Take x_0 with $|x_0| \in (0, \rho)$. What is the power series $\sum_{n=0}^{\infty} b_n (x - x_0)^n$ representation about x_0 of $\sum_{n=0}^{\infty} a_n x^n$ and what is its radius of convergence?

Theorem 7.4.11. *If $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is convergent in $[|x - x_0| < \rho]$ and defines a function f there, then f also has a power series expansion about each $x_1 \in (x_0 - \rho, x_0 + \rho)$ which converges in the largest interval $(x_1 - r, x_1 + r)$ still contained in $(x_0 - \rho, x_0 + \rho)$.*

Proof. Without loss of generality take $x_0 = 0$ and $x_1 = x_0$. Write $x = (x - x_0) + x_0$ which then gives

$$x^n = \big[(x - x_0) + x_0\big]^n = \sum_{k=0}^{n} \binom{n}{k} (x - x_0)^k x_0^{n-k} .$$

Moreover

$$\sum_{n=0}^{m} a_n x^n = \sum_{n=0}^{m} a_n \sum_{k=0}^{n} \binom{n}{k} (x - x_0)^k x_0^{n-k} = \sum_{k=0}^{m} \Big(\sum_{n=k}^{m} a_n \binom{n}{k} x_0^{n-k} \Big) (x - x_0)^k .$$

Letting $m \to \infty$ we expect to have

$$b_k = \sum_{n=k}^{\infty} a_n \binom{n}{k} x_0^{n-k} = \sum_{n=0}^{\infty} a_{n+k} \binom{n+k}{k} x_0^n$$

and

$$f(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n .$$

Now the b_k's are given by a power series with convergence radius ρ since

$$\binom{n+k}{k} = \frac{(n+k)!}{n!k!} = \frac{1}{k!}(n+k)(n+k-1)\cdots(n+1)$$

and therefore $\lim_{n\to\infty} \binom{n+k}{k}^{1/n} = 1$. It follows that

$$b_k = \sum_{n=0}^{\infty} a_{n+k} \binom{n+k}{k} x_0^n$$

is well-defined for any $x_0 \in (-\rho, \rho)$. We still have to show that

$$f(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n \text{ for } |x - x_0| < R - |x_0|\,.$$

To that end observe that

$$\begin{aligned} |\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_n \binom{n}{k} (x-x_0)^k x_0^{n-k}| \leq & \sum_{n=0}^{\infty}\sum_{k=0}^{n} |a_n| \binom{n}{k} |x-x_0|^k |x_0|^{n-k} \\ & = \\ \sum_{n=0}^{\infty} |a_n| \big(|x-x_0| + |x_0|\big)^n & = \sum_{n=0}^{\infty} |a_n| r^n < \infty \end{aligned}$$

which gives absolute convergence. Furthermore

$$\begin{aligned} f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_n \binom{n}{k} (x-x_0)^k x_0^{n-k} \\ = \sum_{k=0}^{\infty}\Big(\sum_{n=k}^{\infty} a_n \binom{n}{k} x_0^{n-k}\Big)(x-x_0)^k \end{aligned}$$

which converges in $|x - x_0| < \rho - |x_0|$. $\surd$

Remarks 7.4.12. (a) The previous theorem shows that power series define functions which are analytic within the domain of covergence of the series.
(b) Convergence could hold in a larger interval than we proved in the theorem. Consider

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

where the power series has radius convergence $\rho = 1$. For $x_0 \in (-1, 1)$ we compute

$$b_n = \frac{f^{(n)}(x_0)}{n!}$$

or

$$\begin{aligned} \frac{1}{1-x} = \frac{1}{1 - x_0 + (x - x_0)} & = \frac{1}{1-x_0}\frac{1}{1 - \frac{x-x_0}{1-x_0}} \\ & = \frac{1}{1-x_0}\sum_{n=0}^{\infty}\big(\frac{x-x_0}{1-x_0}\big)^n = \sum_{n=0}^{\infty}(1-x_0)^{-n-1}(x-x_0)^n \end{aligned}$$

where to ensure convergence of the series we need to assume that $\left|\frac{x-x_0}{1-x_0}\right| < 1$. Now, this is what the theorem predicts for $x_0 \in (0,1)$, but, if $x_0 \in (-1,0)$, we get the larger interval $[|x-x_0| < |1-x_0|] = (2x_0-1,1)$.

7.4.3 Analytic Functions on Complex Domains

Motivation. Next we would like to extend real functions to the complex plane. Since $\mathbb{C}$ is itself a field, we can certainly easily do so for polynomials. In general, however, given $f : \mathbb{R} \to \mathbb{K}$, is is not clear how to define an extension $\tilde{f} : \mathbb{C} \to \mathbb{K}$ in natural way. The zero function could be extended by $f(x+iy) := y\,,\ x,y \in \mathbb{R}$, for instance. For power series it seems natural to extend

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

by simply allowing x to become complex

$$f(z) = \sum_{n=0}^{\infty} a_n z^n\,.$$

This is justified since

$$\Big|\sum_{n=m}^{M} a_n z^n\Big| \le \sum_{n=m}^{M} |a_n||z|^n \le \varepsilon\, \forall\, m, M \ge N_\varepsilon$$

for $|z| < \rho$ and some N_ε by absolute convergence of the real series with the radius of convergence $\rho > 0$ implies convegence in $\mathbb{C}$ by Cauchy's criterion.

Remarks 7.4.13. (a) It follows that analytic functions can locally be extended to the complex plane.
(b) The function $f(x) = \frac{1}{1-x}$ can be extended to $\frac{1}{1-z}$ to $[|z| < 1]$ by observing that the series $\sum_{n=0}^{\infty} z^n$ actually converges for all $|z| < 1$.
(c) The function $f(x) = \frac{1}{1+x^2}$ is analytic and can be locally extended to by its power series expansion. Take for instance the origin as center, then

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}\,,\ |z| < 1\,.$$

Observe that this function has no singularities along the real line but has poles at $z = \pm i$. They clearly seem to have something to do with the size of the radius of convergence.

Theorem 7.4.14. *(Elementary Properties)*
Let

$$\begin{cases} \sum_{n=0}^\infty a_n(x-x_0)^n =: f(x)\,, \\ \sum_{n=0}^\infty b_n(x-x_0)^n =: g(x)\,, \end{cases}$$

both converge for $|x-x_0|<\rho$. *Then*

$$f \pm g\,,\ f\,g \text{ and } \frac{f}{g}$$

all possess a power series expansion about x_0 *as well provided* $g(x_0)\neq 0$ *for the quotient. In the first two cases the radius of convergence remains unchanged, whereas in the third it might shrink.*

Theorem 7.4.15. *(Composition)*
Let

$$\begin{cases} \sum_{n=0}^\infty a_n(x-x_0)^n =: f(x)\,, \\ \sum_{n=0}^\infty b_n(x-x_1)^n =: g(x)\,, \end{cases}$$

converge in $[|x-x_j|<\rho_j]$, $j=0,1$ *and assume that* $x_1=f(x_0)$. *Then* $g\circ f$ *has a convergent power series expansion in a neighborhood of* x_0 *obtained by rearranging*

$$\sum_{n=0}^\infty b_n\Big(\sum_{k=0}^\infty a_k(x-x_0)^k-x_1\Big)^n\,.$$

Why does rerranging not create troubles?

7.5 Approximation by Polynomials

Motivation. Functions are many and can show wildly different behaviors. Among them we find functions which are simpler to describe and/or understand than others. Functions which are arguably quite simple are polynomial functions. It is therefore natural to ask how well and in which sense a general function can be approximated by polynomials. We have seen that analytic functions can be locally represented by power series, which in turn, implies that they can be approximated locally uniformly by polynomials. But what about more general, less regular functions?

7.5.1 Lagrange Interpolation

Definition 7.5.1. Given n arguments

$$x_1,\dots,x_n\in\mathbb{R} \text{ and } n \text{ values } y_1,\dots,y_n\in\mathbb{K}\,,$$

the associated *Lagrange polynomial* is given by

$$\sum_{k=1}^{n} y_k Q_k$$

where Q_k is defined by

$$Q_k(x) = \frac{q_k(x)}{q_k(x_k)}\,,\ q_k(x) := \Pi_{j\neq k}(x - x_j)\,,\ k = 1, \ldots, n\,.$$

Notice that $Q_k(x_j) = \delta_{jk}$.

Even though Lagrange interpolation represents a viable way of approximating functions by interpolating between their values as described in the definition above, it provides no direct control on the behavior of the approximating polynomial in betweeb interpolation points. We therefore use another method based on convolutions which has a much wider range of applications.

7.5.2 Convolutions and Mollifiers

Definitions 7.5.2. (i) Given $f \in \mathrm{C}(\mathbb{R}, \mathbb{K})$, its *support* $\operatorname{supp} f$ is defined by

$$\operatorname{supp} f := \overline{\{x \in \mathbb{R} \,|\, f(x) \neq 0\}}\,.$$

We also define

$$\mathrm{C}_c(\mathbb{R}, \mathbb{K}) := \big\{ f \in \mathrm{C}(\mathbb{R}, \mathbb{K}) \,\big|\, f \text{ has compact support} \big\}\,.$$

(ii) Given $f, g \in \mathrm{C}_c(\mathbb{R}, \mathbb{K})$ we defined their *convolution* $f * g$ by

$$\big(f * g\big)(x) := \int_{-\infty}^{\infty} f(x - y) g(y)\, dy\,,\ x \in \mathbb{R}\,.$$

Remarks 7.5.3. (a) $\big(f * g\big)(x) = \big(g * f\big)(x)\,,\ x \in \mathbb{R}\,,\ \forall\, f, g \in \mathrm{C}_c(\mathbb{R}, \mathbb{K})$.
(b) $\big((f * g) * h\big)(x) = \big(f * (g * h)\big)(x)\,,\ x \in \mathbb{R}\,,\ \forall\, f, g, h \in \mathrm{C}_c(\mathbb{R}, \mathbb{K})$.
(c) $f \in \mathrm{C}_c(\mathbb{R}, \mathbb{K})\,,\ p \in \mathbb{R}[x] \implies p * f \in \mathbb{R}[x]$.

Proof. Let $p(x) = \sum_{k=0}^{n} a_k x^k$, $x \in \mathbb{R}$, and consider

$$\big(p * f\big)(x) = \int_{-\infty}^{\infty} p(x-y) f(y)\, dy = \int_{-\infty}^{\infty} f(y) \sum_{k=0}^{n} a_k (x-y)^k =$$

$$\int_{-\infty}^{\infty} \sum_{k=0}^{n} a_k \sum_{j=0}^{k} \binom{k}{j} x^j (-y)^{k-j} f(y)\, dy$$

$$= \sum_{j=0}^{n} \Big[\sum_{k=j}^{n} (-1)^{k-j} a_k \binom{k}{j} \int_{-\infty}^{\infty} y^{k-j} f(y)\, dy \Big] x^j$$

which shows that the convolution is indeed a polynomial. $\surd$

Definition 7.5.4. (Approximation of the identity)
A sequence of functions $(g_n)_{n\in\mathbb{N}}$ is called *approximation of the identity* if the following conditions are satisfied
(i) $0 \leq g_n \in \mathrm{C}(\mathbb{R}, \mathbb{R})$.
(ii) $\int_{-\infty}^{\infty} g_n(x)\, dx = 1$.
(iii) $\lim_{n\to\infty} \int_{|x|\geq \frac{1}{m}} g_n(x)\, dx = 0$ for every $m \in \mathbb{N}$.

Lemma 7.5.5. *Let $(g_n)_{n\in\mathbb{N}}$ be an approximation of the identity and $f \in \mathrm{C}_c(\mathbb{R}, \mathbb{K})$. Then*

$$\|f * g_n - f\|_\infty = \sup_{x\in\mathbb{R}} \big|\big(f * g_n\big)(x) - f(x)\big| \to 0\ (n \to \infty)\,.$$

Proof. First observe that

$$\big(f * g_n\big)(x) - f(x) = \int_{\mathbb{R}} f(y) g_n(x-y)\, dy - f(x) =$$

$$\int_{\mathbb{R}} g_n(y) \big[f(x-y) - f(x)\big]\, dy = \int_{[|y|\geq 1/n]} g_n(y) \big[f(x-y) - f(x)\big]\, dy$$

$$+ \int_{[|y|\leq 1/n]} g_n(y) \big[f(x-y) - f(x)\big]\, dy\,.$$

Then, given $\varepsilon > 0$, N_ε and M_ε can be found such that

$$\int_{[|y|\geq 1/n]} g_n(y)\, dy \leq \frac{\varepsilon}{4\|f\|_\infty}\ \forall n \geq N_\varepsilon$$

$$\text{and } |f(x-y) - f(x)| \leq \frac{\varepsilon}{2}\ \forall\, y \in \mathbb{R} \text{ s.t. } |y| \leq \frac{1}{M_\varepsilon}\,.$$

We observe that M_ε does not depend on x since f is uniformly continuous (why?). Combining the following two inequalities

$$\begin{aligned} &\Big| \int_{[|y|\geq 1/n]} g_n(y)\big[f(x-y)-f(x)\big]\,dy\Big| \\ &\qquad\qquad \leq 2\|f\|_\infty \int_{[|y|\geq 1/n]} g_n(y)\,dy \leq \frac{\varepsilon}{2}\,\forall\, n\geq N_\varepsilon\,, \\ &\qquad\qquad \Big| \int_{[|y|\leq 1/n]} g_n(y)\big[f(x-y)-f(x)\big]\,dy\Big| \\ &\qquad\qquad\qquad\qquad \leq \frac{\varepsilon}{2}\int_{[|y|\leq 1/n]} g_n(y)\,dy \leq \frac{\varepsilon}{2}\,\forall\, n\geq M_\varepsilon\,, \end{aligned}$$

we obtain

$$\big|\big(f*g_n\big)(x)-f(x)\big| \leq \varepsilon\,\forall\, x\in\mathbb{R}\,\forall\, n\geq \max(M_\varepsilon, N_\varepsilon)$$

and the proof is finished. $\surd$

7.5.3 Weierstrass Approximation Theorem

Next we are going to use an approximation of the identity to prove that any continuous map on a compact interval can approximated arbitrarily well by a polynomial.

Theorem 7.5.6. *Let $-\infty < a < b < \infty$ and $f\in \mathrm{C}\big([a,b],\mathbb{K}\big)$. Then there exists a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ such that*

$$\|f-p_n\|_\infty \to 0\,(n\to\infty)\,.$$

Proof. We want to use an approximation of the identity but, to do so, we need to resolve two issues:
1. f is not defined on the whole real line.
2. We need an approximation of the identity by polynomials to exploit remark 7.5.3(c).
Let us resolve **1.** first. By substituting f by $\tilde f = f-(Ax+B)$, we can assume w.l.o.g. that $f(a)=f(b)=0$. In fact, if we can approximate $\tilde f$ by a polynomial p, then $p+Ax+B$ approximates f. We can therefore extend the function f by zero to the whole real line, if needed.
For **2.** remember that we only need an approximation on $[a,b]$. To compute

$$f*g(x) = \int_a^b f(y)g(x-y)\,dy$$

for $x \in [a,b]$ we only need values of g on $[x-b, x-a]$. Thus we only need values of g on $[a-b, b-a]$ to get the convolution $f * g$ on $[a,b]$.
So let us construct an approximation of the identity as follows. First define

$$\tilde{g}_n(x) := \begin{cases} (1-x^2)^n\,,\ |x| < 1\,, \\ 0\,,\ |x| \geq 1\,, \end{cases}$$

and then compute the normalizing constant

$$c_n := \int_{-1}^{1} \tilde{g}_n(x)\,dx$$

in order to set $g = \frac{\tilde{g}_n}{c_n}$ and obtain $\int_{\mathbb{R}} g_n(x)\,dx = 1$ for $n \in \mathbb{N}$. Let us next get a bound for c_n. Observe that

$$\begin{cases} p_n(0) = q_n(0)\,, \\ p_n'(x) = -2nx < -2xn(1-x^2)^{n-1} = q_n'(x)\,,\ |x| \leq \frac{1}{\sqrt{n}}\,, \end{cases}$$

implies that

$$p_n(x) = 1 - nx^2 \leq (1-x^2)^n = q_n(x)\,,\ |x| \leq \frac{1}{\sqrt{n}}\,.$$

It follows that

$$\int_{-1}^{1} (1-x^2)^n\,dx \geq \int_{-1/2\sqrt{n}}^{1/2\sqrt{n}} (1-nx^2)\,dx \geq \frac{3}{4\sqrt{n}}$$

and therefore $c_n \leq \frac{4}{3}\sqrt{n}$. Now we have

$$g_n \geq 0\,,\ \int_{\mathbb{R}} g_n(x)\,dx = 1\,.$$

To obtain an approximation of the identity we only need make sure that

$$\int_{|x| \leq \frac{1}{m}} g_n(x)\,dx \to 0\ (n \to \infty)\,,$$

which would follow from the uniform convergence of $g_n \to 0$ on $|x| \geq \frac{1}{m}$. To prove the latter, observe that

$$0 \leq g_n(x) \leq c(1 - \frac{1}{m^2})^n \sqrt{n}\,\forall\,|x| \geq \frac{1}{m}$$

and that

$$(1 - \frac{1}{m^2})^n \sqrt{n} \to 0 \, (n \to \infty) \, .$$

Finally

$$f * g_n \to f \text{ uniformly on } [a, b]$$

since g_n is an approximation of the identity and $f * g_n$ is a polynomial on $[a, b]$. $\surd$

Corollary 7.5.7. *Let $f \in \mathrm{C}\big([0,1]\big)$ and assume that $\int_0^1 x^n f(x) \, dx = 0$ for every $n \in \mathbb{N}$. Then $f \equiv 0$.*

Proof. Take a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ such that $p_n \to \overline{f} \; (n \to \infty)$ uniformly on $[0, 1]$, then

$$0 = \int_0^1 p_n(x) f(x) \, dx \longrightarrow \int_0^1 |f(x)|^2 \, dx \; (n \to \infty)$$

and f must vanish identically. $\surd$

7.6 Equicontinuity

Motivation. We are already familiar with the concept of compactness. We know for instance that sequences in compact sets have to have convergent subsequences. If we are given a sequence of continuous functions we might be interested in establishing its uniform convergence or that of a subsequence. The general concept of compactness can be used within the space of continuous functions but it would be really handy to have a more concrete characterization of compactness than the one given by the definition or the one that states that every open cover has a finite subcover. Such a characterization is precisely the main goal of this section.

Example 7.6.1. The function sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{C}\big([0,1]\big)$ defined by $f_n(x) := n$ shows that we need some boundedness, that is

$$|f_n(x)| \le M \; \forall \, x \in [0,1] \; \forall \, n \in \mathbb{N} \quad (\text{ or, equivalently, } \sup_{n\in\mathbb{N}} \|f\|_\infty \le M) \, .$$

But, in contrast to sequences of reals, this is not enough as the example

$$f_n(x) := \sin(n\pi x) \, , \; x \in [0,1] \, ,$$

shows.

Let us try to figure out what possible additional condition needs to be satisfied by looking at a convergent sequence. Assume that $(f_n)_{n\in\mathbb{N}}$ is a uniformly convergent sequence of continuous functions defined on $[0,1]$. Then

$$\forall\,\varepsilon > 0\,\exists\,N(\varepsilon) \text{ s.t. } \|f_n - f_\infty\|_\infty \le \varepsilon/3\,\forall\,n \ge N(\varepsilon)\,,$$

and, by continuity of the limiting function, we find $\tilde{\delta}(\varepsilon)$ such that

$$|f_\infty(x) - f_\infty(y)| \le \varepsilon/3\,\forall\,x,y \in [0,1] \text{ with } |x-y| \le \tilde{\delta}(\varepsilon)\,.$$

Making combined use of these two inequalities we arrive at

$$\begin{aligned} |f_n(x) - f_n(y)| \le |f_n(x) - f_\infty(x)| + |f_\infty(x) - f_\infty(y)| + |f_\infty(y) - f_n(y)| \le \varepsilon \\ \forall\,x,y \in [0,1] \text{ with } |x-y| \le \tilde{\delta}(\varepsilon)\,\forall\,n \ge N(\varepsilon)\,. \end{aligned}$$

On the other hand, for $k = 1,\ldots,N(\varepsilon)-1$, we can find $\delta_k(\varepsilon) > 0$ such that

$$|f_k(x) - f_k(y)| \le \varepsilon\,\forall\,x,y \in [0,1] \text{ with } |x-y| \le \tilde{\delta}_k(\varepsilon)\,.$$

Finally this gives that

$$\begin{aligned} \forall\,\varepsilon > 0\,\exists\delta(\varepsilon)\big(= \min(\tilde{\delta}(\varepsilon), \delta_1(\varepsilon), \ldots, \delta_{N(\varepsilon)-1}(\varepsilon))\big) > 0 \text{ s.t.} \\ |f_n(x) - f_n(y)| \le \varepsilon\,\forall\,x,y \in [0,1] \text{ with } |x-y| \le \delta(\varepsilon)\,\forall\,n \in \mathbb{N}\,. \end{aligned}$$

In other words all functions are continuous “in the same way”: They are *equicontinuous.*

Definition 7.6.2. (Equicontinuity)
A set of functions $A \subset \mathrm{C}(D,\mathbb{K})$ is called *uniformly equicontinuous* iff

$$\begin{aligned} \forall\,\varepsilon > 0\,\exists\delta > 0 \text{ s.t. } |f(x) - f(y)| \le \varepsilon \\ \forall\,x,y \in D \text{ with } |x-y| \le \delta\,,\ \forall\,f \in A\,. \end{aligned}$$

Theorem 7.6.3. *(Arzéla-Ascoli)*
Let $(f_n)_{n\in\mathbb{N}} \in \mathrm{C}\big([a,b]\big)^{\mathbb{N}}$ for $a < b \in \mathbb{R}$ be uniformly bounded and uniformly equicontinuous. Then $(f_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence, that is, there exists $f_\infty \in \mathrm{C}\big([a,b]\big)$ and a subsequence of indices $(n_k)_{k\in\mathbb{N}}$ with

$$f_{n_k} \to f_\infty\ (k \to \infty) \text{ uniformly.}$$

Proof. The proof uses two main ingredients. Boundedness allows to construct a subsequence which converges on a dense countable subset. Equicontinuity makes it possible to show that the convergence occurs everywhere and is uniform.

Step 1: Let $\{x_k \,|\, k \in \mathbb{N}\}$ be a dense subset of $[a,b]$. For fixed $k \in \mathbb{N}$ the sequence $\big(f_n(x_k)\big)_{n\in\mathbb{N}}$ is bounded. In particular, by Problem 10 of Chapter 2, there exists a subsequence of indices $(n_j^1)_{j\in\mathbb{N}}$ such that

$$f_{n_j^1}(x_1) \longrightarrow f_\infty(x_1)\ (j \to \infty)$$

for some $f_\infty(x_1) \in \mathbb{R}$. Then there exists a further subsequece $(n_j^2)_{j\in\mathbb{N}}$ of $(n_j^1)_{j\in\mathbb{N}}$ such that

$$f_{n_j^i}(x_i) \longrightarrow f_\infty(x_i)\ (j \to \infty)\,,\ i = 1, 2\,,$$

for some $f_\infty(x_2) \in \mathbb{R}$. Step by step we construct a subsequence $(n_j^k)_{j\in\mathbb{N}}$ of $(n_j^{k-1})_{j\in\mathbb{N}}$ such that

$$f_{n_j^i}(x_i) \longrightarrow f_\infty(x_i)\ (j \to \infty)\,,\ i = 1, 2, \ldots, k\,,$$

for some $f_\infty(x_k) \in \mathbb{R}$. Eventually taking the diagonal $(n_k^k)_{k\in\mathbb{N}}$ we obtain that

$$f_{n_k^k}(x_i) \longrightarrow f_\infty(x_i)\ (k \to \infty)\,,\ i \in \mathbb{N}\,.$$

Step 2: By theorem 7.3.4 uniform convergence would follow if we had

$$\forall\, \varepsilon > 0\, \exists\, M \text{ s.t. } \|f_{n_k^k} - f_{n_l^l}\|_\infty \le \varepsilon\, \forall\, k, l \ge M\,.$$

Let $\varepsilon > 0$, then we can find $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\begin{aligned}|f_j(x) - f_j(y)| \le \varepsilon/3\, \forall\, x, y \in [a,b] \text{ with } |x-y| \le \delta\,,\ \forall\, j \in \mathbb{N}\\ \text{and } \forall\, x \in [a,b]\, \exists\, k = k(x) \le N \text{ with } |x - x_{k(x)}| \le \delta\,.\end{aligned}$$

Then

$$\begin{aligned}|f_{n_j^j}(x) - f_{n_l^l}(x)| &\le |f_{n_j^j}(x) - f_{n_j^j}(x_{k(x)})| + |f_{n_j^j}(x_{k(x)}) - f_{n_l^l}(x_{k(x)})|\\ &\qquad + |f_{n_l^l}(x_{k(x)}) - f_{n_l^l}(x)|\\ &\le \frac{2}{3}\varepsilon + |f_{n_j^j}(x_{k(x)}) - f_{n_l^l}(x_{k(x)})| \le \varepsilon\, \forall\, n_j^j, n_l^l \ge M\end{aligned}$$

for an $M \in \mathbb{N}$ which exists by Step 1 and the fact that $\{k(x) \,|\, x \in [a,b]\}$ contains less than N indices, thus only finitely many. $\surd$

Corollary 7.6.4. *Let $(f_n)_{n\in\mathbb{N}} \in \mathrm{C}\big([a,b]\big)^{\mathbb{N}}$ for $a < b \in \mathbb{R}$ be such that*

$$|f_k(x)| \leq M\,,\ |f_k'(x)| \leq M \,\forall\, x \in [a,b] \,\forall\, k \in \mathbb{N}\,,$$

for some $M > 0$. Then $(f_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence, that is, there exist $f_\infty \in \mathrm{C}\big([a,b]\big)$ and $(n_k)_{k\in\mathbb{N}}$ such that

$$\|f_{n_k} - f_\infty\|_\infty \longrightarrow 0\ (n \to \infty)\,.$$

Proof. Since boundedness is assumed, we need only prove equicontinuity. The estimate

$$|f_n(x) - f_n(y)| \leq |\int_y^x f_n'(\xi)\,d\xi| \leq M|x-y| \,\forall\, n \in \mathbb{N}$$

readily implies it. $\surd$

7.7 Problems