Assignment 5

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1. Let $a, b \in \mathbb{R}$ with $a < b$ and a continuous function $f : (a, b) \to \mathbb{R}$ be given with
$$f\big((a,b)\big) = \{y_1 \, , \, y_2\} \, .$$
Prove that $y_1 = y_2$.

2. Give examples of a continuous function which is not Hölder continuous of any exponent and of a Hölder continuous function which is not Lipschitz.

3. Let $f \in \mathrm{C}(\mathbb{R})$. Then
$$[f = \alpha] = \{x \in \mathbb{R} \,|\, f(x) = \alpha\} \text{ is closed } \forall \alpha \in \mathbb{R}$$
$$[f > \alpha] = \{x \in \mathbb{R} \,|\, f(x) > \alpha\} \text{ is open } \forall \alpha \in \mathbb{R}$$
Show that
$$\partial[f > \alpha] \subset [f = \alpha] \, .$$

4. Let $g, h \in \mathrm{C}(\mathbb{R})$ and prove that $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) := \begin{cases} g(x) \, , & x \leq a \\ h(x) \, , & x > a \end{cases}$$
is continuous iff $g(a) = \lim_{x \to a+} h(x)$.

5. Assume that $f \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ is such that
$$\lim_{|x| \to \infty} f(x) = 0$$
meaning that
$$\forall \varepsilon > 0 \, \exists \, R > 0 \text{ s.t. } |f(x)| \leq \varepsilon \, \forall \, x \in \mathbb{R} \text{ with } |x| \geq R \, .$$
Prove that f attains at least its maximum or its minimum.

Homework due by Thursday, November 3 2005