# Midterm Examination

---

Print your name: ________________ ________________

Print your ID #: ________________________________

You have 50 minutes to solve the problems. Good luck!

**1.** Prove that the closure $\overline{O}$ of a convex set $O \subset \mathbb{R}^n$ is convex.
**Solution:**
Take any $x, y \in \overline{O}$. Then there are sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in the set $O$ such that

$$x_n \to x \text{ and } y_n \to y \text{ as } n \to \infty\,.$$

Since $O$ is convex, $(1-t)x_n + ty_n \in O$ for any $t \in [0,1]$. Therefore, for any $t \in [0,1]$, $(1-t)x + ty \in \overline{O}$ since

$$(1-t)x_n + ty_n \to (1-t)x + ty \text{ as } n \to \infty\,.$$

---

**2.** Prove that a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^n$ which possesses a convergent subsequence is already convergent. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if

$$\forall \varepsilon > 0\, \exists\, N \in \mathbb{N} \text{ s.t. } \|x_n - x_m\| \le \varepsilon \;\forall n, m \ge N\,.$$

**Solution:**
Fix $\varepsilon > 0$ and let $x_\infty$ be the limit of the convergent subsequence. By assumption there exists $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| \le \varepsilon/2\,,\; m, n \ge N$$

There also is $N \le M \in \mathbb{N}$ such that

$$\|x_M - x_\infty\| \le \varepsilon/2$$

since $x_\infty$ is the limit of a subsequence. We therefore conlcude that

$$\|x_n - x_\infty\| = \|x_n - x_M + x_M - x_\infty\| \le \|x_n - x_M\| + \|x_M - x_\infty\| \le \varepsilon\,, n \ge N$$

and the claimed convergence is established.

---

**3.** For a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^m$ define $X = \{x_n \,|\, n \in \mathbb{N}\}$.
Is every accumulation point of $(x_n)_{n\in\mathbb{N}}$ a limit point of $X$?
Is every limit point of $X$ an accumulation point of $(x_n)_{n\in\mathbb{N}}$?
If your answer is yes, prove it. If your answer is no, give a counterexample.
**Solution:**
The answer to the first question is **no**. Just take the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = x_\infty\,,\; n \in \mathbb{N}$ for some fixed $x_\infty \in \mathbb{R}^m$. In this case

$X = \{x_\infty\}$ is a set without limit points but $x_\infty$ is certainly an accumulation point of the sequence.
The converse is **true**. In fact, if $x_\infty$ is a limit point of $X$, then, by definition, there is a sequence $(x_{n_k})_{k\in\mathbb{N}}$ in $X \setminus \{x_\infty\}$ which converges to $x_\infty$ making it an accumulation point of $(x_n)_{n\in\mathbb{N}}$.

---

**4.** Let $f \in \mathrm{C}(\mathbb{R}^n, \mathbb{R})$. Is it possible that $f(\mathbb{R}^n) = \{0,1\}$?
Motivate your answer.
**Solution:**
The answer is **no**. In fact continuous functions map connected sets to connected sets. Since $\mathbb{R}^n$ is convex and therefore connected, the range of the continuous function $f$ cannot be the disconnected set $\{0,1\}$.

---

**5.** For $U \overset{o}{\subset} \mathbb{R}^n$ let $f : U \to \mathbb{R}$ be differentiable at $x \in U$ with $f(x) = 0$ and $g : U \to \mathbb{R}$ be continuous at $x$. Prove that $fg$ is differentiable at $x$ and compute $\nabla(fg)(x)$.

**Solution:**
Notice that

$$f(x+h) = \nabla f(x) \cdot h + o(\|h\|) \text{ as } h \to 0$$

since $f$ is differentiable at $x$ and $f(x) = 0$. Also

$$\lim_{h\to 0} \big(g(x+h) - g(x)\big) = 0$$

since $g$ is assumed to be continuous at $x$. Therefore

$$\begin{aligned} f(x+h)g(x+h) - g(x)\nabla f(x)\cdot h = \big[f(x+h) - \nabla f(x)\cdot h\big]g(x+h) + \\ \big[g(x+h) - g(x)\big]\nabla f(x)\cdot h = o(\|h\|) \text{ as } h \to 0 \end{aligned}$$

which gives differentiability and $\nabla(fg)(x) = g(x)\nabla f(x)$.

---