Final Examination

Print your name: ________________ ________________

Print your ID #: ________________________________

You have 2 hours to solve the problems. Good luck!

1. Let A and B be compact subsets of $\mathbb{R}^n$. Prove that

$$A + B = \{a + b \,|\, a \in A\,,\ b \in B\}$$

is compact.

Solution:

Given an arbitrary sequence $(c_n)_{n\in\mathbb{N}}$ in $A+B$ we need to show that it has a convergent subsequence with limit in $A+B$. Since $c_n \in A+B$ we can find $a_n \in A$ and $b_n \in B$ such that

$$c_n = a_n + b_n \text{ for every } n \in \mathbb{N}\,.$$

We thus obtain two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in A and B, respectively.

Since A is compact we find an index sequence $(n_k)_{k\in\mathbb{N}}$ such that $(a_{n_k})_{k\in\mathbb{N}}$ has a limit $a_\infty \in A$, but then, since B is also compact, $(b_{n_k})_{k\in\mathbb{N}}$ contains a convergent subsequence $(b_{n_{k_j}})_{j\in\mathbb{N}}$ which has a limit $b_\infty \in B$.

Finally the subsequence $(c_{n_{k_j}})_{j\in\mathbb{N}}$ of $(c_n)_{n\in\mathbb{N}}$ will converge to $a_\infty + b_\infty \in A+B$ which concludes the proof.

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2. Let $K \subset \mathbb{R}^n$ be compact and $f \in \mathrm{C}(K,\mathbb{R})$ with $f > 0$. Show that $1/f$ is uniformly continuous.

Solution:

Since f is a continuous function defined on a compact set, it is uniformly continuous and $|f|$, which is also continuous, attains both a maximum $M < \infty$ and a minimum $m > 0$. The minimum is positive since the function is assumed to be positive. Next we observe that

$$|\frac{1}{f(x)} - \frac{1}{f(y)}| \leq \frac{1}{|f(x)f(y)|}|f(x)-f(y)| \leq \frac{1}{m^2}|f(x)-f(y)|\,, \forall x, y \in K\,.$$

Now, given $\varepsilon > 0$, it is possible to find $\delta >$ such that

$$|f(x) - f(y)| \leq m^2\varepsilon \text{ whenever } |x-y| \leq \delta \text{ for } x, y \in K\,.$$

since f is uniformly continuous. The claim follows combining the two inequalities.

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3. Let $A \subset \mathbb{R}^n$ be non compact. Show that there must exist an unbounded continuous function $f : A \to \mathbb{R}$.

Solution:

Two cases need to be considered. First the set A might be unbounded. In this case

$$f : A \to \mathbb{R}\,,\; x \mapsto \|x\|$$

is a continuous function (triangle inequality) which is obviously unbounded. On the other hand, if A is bounded it can not be closed. We therefore find $x_0 \in \partial A \setminus A$ and the function

$$f : A \to \mathbb{R}\,,\; x \mapsto \frac{1}{\|x - x_0\|}$$

is continuous ($\|x - x_0\|$ does not vanish on A) and unbounded (there are points in A which come arbitrarily close to x_0).

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4. Compute the limit:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{\sqrt{x^2 + y^2}}\,.$$

Justify your answer.

Solution:

Introducing polar coordinates $(x, y) = r\big(\cos(\theta), \sin(\theta)\big)$, we see that

$$(x, y) \to 0 \iff r \to 0\,.$$

The problem reduces to computing $\lim_{r\to 0} \frac{\sin(r^2)}{r}$. By L'Hôpital, or since $\sin(r^2) = O(r^2)$ (as $r \to 0$), we see that the limit is 0.

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5. For a given function $f : \mathbb{R}^n \to \mathbb{R}$ consider the following assertions:

(i) f is continuously differentiable .

(ii) f has directional derivatives in every direction at every point.

(iii) f has partial derivatives at every point.

Explain the implications between these assertions.

Solution:

(i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iii)

(ii) $\not\Rightarrow$ (i), (iii) $\not\Rightarrow$ (ii), (iii) $\not\Rightarrow$ (i).

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6. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable and assume that $Df(x)$ is invertible for every $x \in \mathbb{R}^n$. Prove that $\|f\|^2$ does not attain a maximum.
[Hint: Chain rule]
Solution:
By chain rule $\nabla\big(\|f\|^2\big)(x) = 2Df(x)^T f(x)$. At a point of maximum we would have $Df(x)^T f(x) = 0$. Since $Df(x)$ is invertible, this can only happen when $f(x) = 0$. Now, f cannot be constant, and thus, when $f(x) = 0$ we have a minimum of $\|f\|^2$.

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7. Let $f \in \mathrm{C}^2(\mathbb{R}^n, \mathbb{R})$. Assume that $\nabla f(x_0) = 0$ and that $D^2 f(x_0)$ is positive definite for some $x_0 \in \mathbb{R}^2$. Prove that there is $\delta > 0$ such that

$$f(x_0 + h) - f(x_0) \geq c\|h\|^2 \,,\ \forall h \text{ with } \|h\| \leq \delta \,.$$

Solution:
By a theorem in class

$$f(x_0 + h) = f(x_0) + h^T D^2 f(x_0) h + R_{f,x_0}(h)$$

for $R_{f,x_0}(h) = o(\|h\|^2)$ as $h \to 0$. Now

$$h^T D^2 f(x_0) h \geq \alpha \|h\|^2 \,,\ \forall h \in \mathbb{R}^n$$

since $D^2 f(x_0)$ is positive definite. We also can find $\delta > 0$ such that

$$R_{f,x_0}(h) \leq \frac{\alpha}{2}\|h\|^2 \text{ whenever } \|h\| \leq \delta \,.$$

Finally we see that

$$f(x_0 + h) - f(x_0) = h^T D^2 f(x_0) h + R_{f,x_0}(h) \geq \alpha\|h\|^2 - \frac{\alpha}{2}\|h\|^2 = \frac{\alpha}{2}\|h\|^2 \text{ whenever } \|h\| \leq \delta$$

which concludes the proof.

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8. Use the implicit function theorem to analyze solutions of

$$\begin{cases} a^3 + a^2 b + \sin(a+b+c) & = 0 \\ \log(1+a^2) + 2a + (bc)^4 & = 0 \end{cases}$$

about the point $(0,0,0)$ in $\mathbb{R}^3$.

Solution:

After computing

$$DF(0,0,0) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}$$

for

$$F(a,b,c) = (a^3 + a^2 b + \sin(a+b+c), \log(1+a^2) + 2a + (bc)^4)\,,$$

we use the implicit function thereom to conclude that the system can be solved for either (a,b) or (a,c) as functions of c or b, respectively, in a neighborhood of the origin.