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A Note on Günther's Analysis of Couple Stress

By

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The following considerations are concerned with GÜNTHER's form of couple stress theory [1]. While the observations which follow were made without knowledge of the earlier work[1], they are offered here as a supplement to it.

We assume an orthogonal coordinate system x_i with position vector $\boldsymbol{x}$ and with coordinate tangent unit vectors $\boldsymbol{t}_i = \boldsymbol{x}_{,i}/\alpha_i$ where $\boldsymbol{x}_{,i} \cdot \boldsymbol{x}_{,j} = \alpha_i \alpha_j \delta_{ij}$. We designate force stress vectors by $\boldsymbol{\sigma}_i$, moment stress vectors by $\boldsymbol{\tau}_i$ and body force and moment intensity vectors by $\boldsymbol{p}$ and $\boldsymbol{q}$. We take as basic relations the two equations of force and moment equilibrium

$$\sum (S_i \boldsymbol{\sigma}_i)_{,i} + V \boldsymbol{p} = 0, \qquad \sum (S_i \boldsymbol{\tau}_i)_{,i} + \boldsymbol{x}_{,i} \times (S_i \boldsymbol{\sigma}_i) + V \boldsymbol{q} = 0 \qquad (1)$$

where $V = \alpha_1 \alpha_2 \alpha_3$, $S_1 = \alpha_2 \alpha_3$, etc.

We next introduce force and moment strain vectors $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{\varkappa}_i$, and translational and rotational displacement vectors $\boldsymbol{u}$ and $\boldsymbol{\phi}$. We obtain expressions for $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{\varkappa}_i$ in terms of $\boldsymbol{u}$ and $\boldsymbol{\phi}$ through use of the principle of virtual work, written in the form

$$\begin{aligned} &\int \left(\sum \boldsymbol{\sigma}_i \cdot \delta \boldsymbol{\varepsilon}_i + \boldsymbol{\tau}_i \cdot \delta \boldsymbol{\varkappa}_i\right) dV \\ &\qquad = \int (\boldsymbol{p} \cdot \delta \boldsymbol{u} + \boldsymbol{q} \cdot \delta \boldsymbol{\phi})\, dV + \oint (\boldsymbol{\sigma}_n \cdot \delta \boldsymbol{u} + \boldsymbol{\tau}_n \cdot \delta \boldsymbol{\phi})\, dS. \qquad (2) \end{aligned}$$

In this p and q are taken from (1) and $\boldsymbol{\sigma}_n$ and $\boldsymbol{\tau}_n$ are surface traction vectors. Integration by parts to eliminate derivatives of $\boldsymbol{\sigma}_i$ and $\boldsymbol{\tau}_i$ in (2) and observation that $\boldsymbol{\sigma}_i$ and $\boldsymbol{\tau}_i$ are arbitrary functions in the remaining volume integral leads to the vectorial strain displacement relations

$$\alpha_i \boldsymbol{\varepsilon}_i = \boldsymbol{u}_{,i} + \boldsymbol{x}_{,i} \times \boldsymbol{\phi}, \qquad \alpha_i \boldsymbol{\varkappa}_i = \boldsymbol{\phi}_{,i} \qquad (3)$$

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[1] We are indebted to E. KRÖNER for bringing GÜNTHER's paper to our attention.

which are equivalent to relations stated by GÜNTHER [1]. The present approach and Günther's approach differ from each other inasmuch as Günther departs from (3) and obtains (1) by use of the principle of virtual work. It seems to us easier to depart from (1), which is in accordance with elementary principles of dynamics, and avoid the less elementary geometrical considerations which are involved in stipulating (3).

From (3) follows by inspection, as noted previously in [1], a system of six vectorial compatibility equations

$$(\alpha_1 \boldsymbol{\varkappa}_1)_{,2} - (\alpha_2 \boldsymbol{\varkappa}_2)_{,1} = (\alpha_2 \boldsymbol{\varkappa}_2)_{,3} - (\alpha_3 \boldsymbol{\varkappa}_3)_{,2} = (\alpha_3 \boldsymbol{\varkappa}_3)_{,1} - (\alpha_1 \boldsymbol{\varkappa}_1)_{,3} = 0, \quad (4)$$

$$(\alpha_1 \boldsymbol{\varepsilon}_1)_{,2} - (\alpha_2 \boldsymbol{\varepsilon}_2)_{,1} = \boldsymbol{x}_{,1} \times (\alpha_2 \boldsymbol{\varkappa}_2) - \boldsymbol{x}_{,2} \times (\alpha_1 \boldsymbol{\varkappa}_1), \quad (5\text{a})$$

$$(\alpha_2 \boldsymbol{\varepsilon}_2)_{,3} - (\alpha_3 \boldsymbol{\varepsilon}_3)_{,2} = \boldsymbol{x}_{,2} \times (\alpha_3 \boldsymbol{\varkappa}_3) - \boldsymbol{x}_{,3} \times (\alpha_2 \boldsymbol{\varkappa}_2), \quad (5\text{b})$$

$$(\alpha_3 \boldsymbol{\varepsilon}_3)_{,1} - (\alpha_1 \boldsymbol{\varepsilon}_1)_{,3} = \boldsymbol{x}_{,3} \times (\alpha_1 \boldsymbol{\varkappa}_1) - \boldsymbol{x}_{,1} \times (\alpha_3 \boldsymbol{\varkappa}_3). \quad (5\text{c})$$

We complement Günther's formulation by phenomenological stress strain relations as follows. Writing

$$(\boldsymbol{\sigma}_i, \boldsymbol{\tau}_i, \boldsymbol{\varepsilon}_i, \boldsymbol{\varkappa}_i) = \sum (\sigma_{ij}, \tau_{ij}, \varepsilon_{ij}, \varkappa_{ij})\, \boldsymbol{t}_j \quad (6)$$

we stipulate the existence of functions $A(\varepsilon, \varkappa)$ and $B(\sigma, \tau)$ such that

$$\sigma_{ij} = \frac{\partial A}{\partial \varepsilon_{ij}}, \qquad \tau_{ij} = \frac{\partial A}{\partial \varkappa_{ij}}, \qquad \varepsilon_{ij} = \frac{\partial B}{\partial \sigma_{ij}}, \qquad \varkappa_{ij} = \frac{\partial B}{\partial \tau_{ij}}. \quad (7)$$

An appropriate definition of vectorial derivatives allows us to write (7) in the form

$$\boldsymbol{\sigma}_i = \frac{\partial A}{\partial \boldsymbol{\varepsilon}_i}, \qquad \boldsymbol{\tau}_i = \frac{\partial A}{\partial \boldsymbol{\varkappa}_i}, \qquad \boldsymbol{\varepsilon}_i = \frac{\partial B}{\partial \boldsymbol{\sigma}_i}, \qquad \boldsymbol{\varkappa}_i = \frac{\partial B}{\partial \boldsymbol{\tau}_i}. \quad (7')$$

Observing (1), (3) and (7) we readily have as variational principles, generalizing corresponding principles of elasticity without couple stresses, a variational principle for stresses and displacements, $\delta I_{SD} = 0$, and a variational principle for stresses, displacements and strains, $\delta I_{SDS} = 0$, where

$$I_{SD} = \int \left[\sum \boldsymbol{\sigma}_i \cdot \frac{\boldsymbol{u}_{,i} + \boldsymbol{x}_{,i} \times \boldsymbol{\phi}}{\alpha_i} + \boldsymbol{\tau}_i \cdot \frac{\boldsymbol{\phi}_{,i}}{\alpha_i} - \boldsymbol{p} \cdot \boldsymbol{u} - \boldsymbol{q} \cdot \boldsymbol{\phi} - B \right] dV -$$

$$- \int_{S_s} [\overline{\boldsymbol{\sigma}}_n \cdot \boldsymbol{u} + \overline{\boldsymbol{\tau}}_n \cdot \boldsymbol{\phi}]\, dS - \int_{S_d} [(\boldsymbol{u} - \overline{\boldsymbol{u}}) \cdot \boldsymbol{\sigma}_n + (\boldsymbol{\phi} - \overline{\boldsymbol{\phi}}) \cdot \boldsymbol{\tau}_n]\, dS \quad (8)$$

and

$$I_{SDS} = \int \left[\sum \boldsymbol{\sigma}_i \cdot \left(\frac{\boldsymbol{u}_{,i} + \boldsymbol{x}_{,i} \times \boldsymbol{\phi}}{\alpha_i} - \boldsymbol{\varepsilon}_i \right) + \boldsymbol{\tau}_i \cdot \left(\frac{\boldsymbol{\phi}_{,i}}{\alpha_i} - \boldsymbol{\varkappa}_i \right) - \right.$$

$$\left. - \boldsymbol{p} \cdot \boldsymbol{u} - \boldsymbol{q} \cdot \boldsymbol{\phi} + A \right] dV -$$

$$- \int_{S_s} [\overline{\boldsymbol{\sigma}}_n \cdot \boldsymbol{u} + \overline{\boldsymbol{\tau}}_n \cdot \boldsymbol{\phi}]\, dS - \int_{S_d} [(\boldsymbol{u} - \overline{\boldsymbol{u}}) \cdot \boldsymbol{\sigma}_n + (\boldsymbol{\phi} - \overline{\boldsymbol{\phi}}) \cdot \boldsymbol{\tau}_n]\, dS. \quad (9)$$

In δI_{SD} stresses and displacements are varied independently and the Euler differential equations consist of the equations of equilibrium and the stress strain relations. In δI_{SDS} stresses, displacements and strains are varied independently and the Euler differential equations are equilibrium equations, strain displacement relations and stress strain relations[1].

A recent result [4] on a static-geometric analogue of the two-dimensional elastic-shell theory version of the variational principle for stresses and displacements [3] suggests the formulation of a variational principle for strains and stress functions in couple stress elasticity, such that the compatibility Eqs. (4) and (5), together with the stress strain relations, are Euler differential equations.

A small amount of mathematical experimentation indicates that a suitable stress function representation of the solutions of the homogeneous Eqs. (1) is

$$S_1\,\boldsymbol{\sigma}_1 = \boldsymbol{F}_{3,2} - \boldsymbol{F}_{2,3}, \quad S_2\,\boldsymbol{\sigma}_2 = \boldsymbol{F}_{1,3} - \boldsymbol{F}_{3,1}, \quad S_3\,\boldsymbol{\sigma}_3 = \boldsymbol{F}_{2,1} - \boldsymbol{F}_{1,2}, \tag{10}$$

$$S_1\,\boldsymbol{\tau}_1 = \boldsymbol{H}_{3,2} - \boldsymbol{H}_{2,3} + \boldsymbol{x}_{,2} \times \boldsymbol{F}_3 - \boldsymbol{x}_{,3} \times \boldsymbol{F}_2, \tag{11a}$$

$$S_2\,\boldsymbol{\tau}_2 = \boldsymbol{H}_{1,3} - \boldsymbol{H}_{3,1} + \boldsymbol{x}_{,3} \times \boldsymbol{F}_1 - \boldsymbol{x}_{,1} \times \boldsymbol{F}_3, \tag{11b}$$

$$S_3\,\boldsymbol{\tau}_3 = \boldsymbol{H}_{2,1} - \boldsymbol{H}_{1,2} + \boldsymbol{x}_{,1} \times \boldsymbol{F}_2 - \boldsymbol{x}_{,2} \times \boldsymbol{F}_1. \tag{11c}$$

Taking Eqs. (10) and (11) as equations of definition, we find that the variational equation

$$\delta \int \left[\sum \boldsymbol{\sigma}_i \cdot \boldsymbol{\varepsilon}_i + \boldsymbol{\tau}_i \cdot \boldsymbol{\varkappa}_i - A\right] d V = 0 \tag{12}$$

where the $\boldsymbol{F}_i$, $\boldsymbol{H}_i$, $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{\varkappa}_i$ are varied independently does in fact have the compatibility Eqs. (4) and (5) and the stress strain relations (7) as Euler equations. In addition, it is found that the associated Euler boundary conditions are displacement boundary conditions expressed in terms of tangential strain vectors. These conditions are homogeneous conditions which may be made non-homogeneous conditions upon adding in (12) an appropriate surface integral.

We state two further variational principles which we think will eventually be found useful. The first of these is a *mixed* principle, in the sense that it has parts of the equilibrium equations and compatibility equations as Euler equations, while the complementary parts are equations of definition.

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[1] A restricted form of the variational principle $\delta I_{SDS} = 0$ has previously been stated by NAGHDI [2]. The restriction consists in assuming symmetry conditions $\varepsilon_{ij} = \varepsilon_{ji}$ as equations of definition. As a consequence of this, an additional Euler differential equation $\boldsymbol{\phi} = \frac{1}{2} \operatorname{curl} \boldsymbol{u}$ is obtained, the force stress strain relations become $\frac{1}{2}(\sigma_{ij} + \sigma_{ji}) = \partial A / \partial \varepsilon_{ij} = \partial A / \partial \varepsilon_{ji}$ and the number of associated Euler boundary conditions is five instead of six.

Restricting attention to the case of absent body forces and moments, we take as equations of definition the conditions of force equilibrium in (1), via the stress function Eqs. (10), and the equations for bending strains in (3), these being equivalent to the compatibility Eqs. (4). Additionally, we now take the relations between stresses and strains in the mixed form

$$\boldsymbol{\varepsilon}_i = \frac{\partial B_f}{\partial \boldsymbol{\sigma}_i}, \qquad \boldsymbol{\tau}_i = \frac{\partial A_c}{\partial \boldsymbol{\varkappa}_i} \tag{13}$$

as equations of definition.

We then have that the equation $\delta I_M = 0$ where

$$I_M = \int \left[\frac{\boldsymbol{F}_1}{\alpha_1} \cdot \left(\frac{\boldsymbol{x}_{,2}}{\alpha_2} \times \frac{\boldsymbol{\phi}_{,3}}{\alpha_3} - \frac{\boldsymbol{x}_{,3}}{\alpha_3} \times \frac{\boldsymbol{\phi}_{,2}}{\alpha_2} \right) + \cdots + B_f - A_c \right] dV \tag{14}$$

and where $\boldsymbol{\phi}$ and the $\boldsymbol{F}_i$ are varied independently has as Euler differential equations the conditions of moment equilibrium in (1) together with the force strain compatibility Eqs. (5).

The second additional variational principle is for boundary values. It is a generalization of a principle previously stated for elasticity without couple stresses [5]. Define a functional

$$I_B = \int_{S_s} [\tfrac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{\sigma}_n + \boldsymbol{\phi} \cdot \boldsymbol{\tau}_n) + \Psi_s(\boldsymbol{u}, \boldsymbol{\phi})] \, dS -$$
$$- \int_{S_d} [\tfrac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{\sigma}_n + \boldsymbol{\phi} \cdot \boldsymbol{\tau}_n) + \Psi_d(\boldsymbol{\sigma}_n, \boldsymbol{\tau}_n)] \, dS \tag{15}$$

where $\boldsymbol{u}$, $\boldsymbol{\phi}$, $\boldsymbol{\sigma}_n$ and $\boldsymbol{\tau}_n$ are the surface values of states of displacement and stress which in the interior satisfy equilibrium and stress displacement relations, subject to the limitation of absent body forces and moments and subject to the limitation that the stress energy function B in (7) is homogeneous of the second degree in the sense that $\sum \boldsymbol{\sigma}_i \cdot \partial B / \partial \boldsymbol{\sigma}_i + \boldsymbol{\tau}_i \cdot \partial B / \partial \boldsymbol{\tau}_i = 2B$. It can then be shown, in extension of what is done in [5] for elasticity without couple stresses, that the Euler equations of $\delta I_B = 0$ are the boundary conditions.

$$\boldsymbol{\sigma}_n + \frac{\partial \Psi_s}{\partial \boldsymbol{u}} = 0, \qquad \boldsymbol{\tau}_n + \frac{\partial \Psi_s}{\partial \boldsymbol{\phi}} = 0 \quad \text{on } S_s, \tag{16}$$

$$\boldsymbol{u} + \frac{\partial \Psi_d}{\partial \boldsymbol{\sigma}_n} = 0, \qquad \boldsymbol{\phi} + \frac{\partial \Psi_d}{\partial \boldsymbol{\tau}_n} = 0 \quad \text{on } S_d. \tag{17}$$

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