Semiinfinite Orthotropic Cantilevered Strips and the Foundations of Plate Theories

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Elastostatic problems of semiinfinite orthotropic cantilevered strips with traction-free edges and loading at infinity are reduced to the solution of a single scalar Fredholm integral equation of the first kind with a generalized Cauchy kernel. The known complex variable method for equations with a Cauchy type kernel is extended to handle the singularities in the solution for the generalized Cauchy kernel. The reduced problem lends itself to a more efficient numerical solution scheme than all existing methods. Moments of stresses at the root of the cantilever are accurately evaluated and used for the correct formulation of displacement boundary conditions for a plate theory solution (or the actual interior solution) of the elastostatics of thin flat bodies.

1. Introduction

We are concerned here with some canonical problems for an orthotropic semiinfinite cantilevered strip within the framework of the linear theory of elastostatics, as well as the applications of their solution to theories of thin elastic plates.

The extension, bending, and flexure problems for isotropic semiinfinite strips have been the subjects of numerous investigations (see [1, 3, 9, 14, 26] for examples). In this paper, we obtain analogous numerical solutions for the more general case of orthotropic strips. We do so with a new reduction of the boundary value problem in linear plane elasticity for orthotropic strips to a single integral equation for one scalar unknown. (In contrast, the reduction in [3] for

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example is to two simultaneous integral equations for two unknowns.) The numerical solution of this reduced problem is substantially more efficient than all known methods for isotropic strips.

In our formulation, the single integral equation is a Fredholm equation of the first kind. A significant step toward the existence theory of, and an efficient numerical procedure for, its solution is the recasting of this equation in the form of a singular integral equation with the singular part of the kernel being a linear combination of several Cauchy type kernels. A simple extension of the theory for Cauchy type singular integral equations developed in [4, 20] will be worked out in Appendix I to handle our generalized Cauchy integral equation.

Beyond their importance in linear elasticity theory, the three canonical problems for semiinfinite strips also have a fundamental role in the linear plate theory solution for thin plates with displacement edge data in plane strain deformation. It is known from [6–8, 22] that the solution of Kirchhoff's linear thin plate theory corresponds to the leading term of an interior (or outer asymptotic expansion of the exact) solution for the same problem in three dimensional linear elasticity theory. This leading term interior solution (or the actual interior solution itself) is generally insufficient for fitting the prescribed data at a cylindrical edge of the plate. For isotropic plates, it was shown by way of the reciprocal theorem of elastostatics that the correct interior solution fits the given edge data in a certain prescribed weighted average sense [11, 12]. When the prescribed displacement edge data give rise to a state of plane strain, the weighting functions in the averaging process are the stress components of the three canonical problems at the root of the cantilevered semiinfinite strip [10]. Given that the reciprocal theorem continues to hold, the same is true for orthotropic plates in the plane strain deformation. Certain moments of the stresses of the three canonical problems will again be useful, as they appear naturally in the formulation of appropriate displacement boundary conditions for plate theory solutions.

In this paper, we carry out the reduction of the plane strain elastostatic problem to a single singular integral equation in a form applicable to all three canonical orthotropic strip problems. Numerical solutions for the problem of axial extension will be obtained. The results will be applied to the formulation of appropriate displacement boundary conditions for plate theories and more generally for the interior solution. The relevant moments of stresses needed for this application will be calculated. Corresponding solutions for the bending and flexure problem and their applications will be reported in [18].

2. Three canonical problems for a cantilevered semiinfinite strip

We consider in this paper linear plane elasticity problems for orthotropic materials with the stress-strain relations

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \varepsilon_{xz} \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \end{pmatrix} \equiv \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_2} & 0 \\ -\frac{\nu_{21}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \end{pmatrix}, \tag{2.1}$$

where $a_{12} = a_{21}$ ($\nu_{12}/E_2 = \nu_{21}/E_1$). For an isotropic material, we have $E_1 = E_2 = E$, $\nu_{12} = \nu_{21} = \nu$, and $G = E/2(1+\nu)$.

The strain components are related to the displacement components u_x and u_z by

$$\varepsilon_{xx} = u_{x,x}, \qquad \varepsilon_{zz} = u_{z,z}, \qquad \varepsilon_{xz} = \varepsilon_{zx} = u_{z,x} + u_{x,z}. \tag{2.2}$$

Throughout this paper, $(\ )_{,t}$ indicates partial differentiation of $(\ )$ with respect to t.

The stress components must be in equilibrium with interior and surface loads. In the absence of interior loading, the differential equations of equilibrium are satisfied identically by expressing the stress components in terms of a stress function ϕ as follows:

$$\sigma_{xx} = \phi_{,zz}, \qquad \sigma_{zz} = \phi_{,xx}, \qquad \sigma_{xz} = \sigma_{zx} = -\phi_{,xz}. \tag{2.3}$$

To maintain strain compatibility, ϕ must satisfy the fourth order linear partial differential equation

$$\phi_{,yyyy} + (2+\delta_0)\phi_{,yyzz} + \phi_{,zzzz} = 0, \tag{2.4}$$

where

$$y = \frac{x}{\beta_0}, \qquad \beta_0 = \left(\frac{E_1}{E_2}\right)^{1/4},$$

$$\delta_0 = \frac{E}{G} - 2(1+\nu), \qquad E = \sqrt{E_1 E_2}, \qquad \nu = \sqrt{\nu_{21}\nu_{12}}. \tag{2.5}$$

The parameter δ_0 vanishes for an isotropic material (with E and ν reducing correctly to Young's modulus and the effective Poisson ratio, respectively) and is taken to be positive throughout this paper.

We are interested in problems involving the semiinfinite strip $\{|z| \le h, 0 \le x < \infty\}$ with two traction-free edges, so that

$$z = \pm h: \qquad \sigma_{zz} = 0, \ \sigma_{xz} = 0 \quad (y > 0). \tag{2.6}$$

The end $x = 0$ of the strip is fixed, so that

$$y = 0: \qquad u_x = 0, \ u_z = 0 \quad (|z| < h). \tag{2.7}$$

The strip is loaded at infinity in one of the following ways:

BVP-1. *The extension problem*:

$$\sigma_{xx} \to \frac{1}{2h}, \quad \sigma_{xz} \to 0 \qquad \text{as} \quad y \to \infty. \tag{2.8}$$

BVP-2. *The bending problem*:

$$\sigma_{xx} \to \frac{3z}{2h^3}, \quad \sigma_{xz} \to 0 \quad \text{as} \quad y \to \infty. \tag{2.9}$$

BVP-3. *The flexure problem*:

$$\sigma_{xx} \to \frac{3xz}{2h^3}, \quad \sigma_{xz} \to \frac{3}{4h}\left(1 - \frac{z^2}{h^2}\right) \quad \text{as} \quad y \to \infty. \tag{2.10}$$

As we shall see later, these are the three canonical problems whose solution will be needed in the plate theory solution of plane (strain) elasticity problems with displacement edge data. This same observation has already been noted in [10] for isotropic materials.

For the numerical solutions of these canonical problems, it is more convenient to subtract off a particular solution (different) for each case and work with the *residual* problems denoted by $\{\tilde{\phi}, \tilde{\sigma}, \tilde{u}\}$. For the extension problem, denoted by a superscript E, we let

$$\sigma_{xx}^E = \tilde{\sigma}_{xx}^E + \frac{1}{2h}, \qquad \sigma_{xz}^E = \tilde{\sigma}_{xz}^E, \qquad \sigma_{zz}^E = \tilde{\sigma}_{zz}^E,$$

$$u_x^E = \tilde{u}_x^E + \frac{x}{2hE_1} + u_0, \qquad u_z^E = \tilde{u}_z^E - \frac{\nu_{21}z}{2hE_1}. \tag{2.11}$$

Note that the inhomogeneous terms correspond to the *interior* (outer) solution of (2.4) with the proper behavior at infinity for the stress components. Hence, the residual stress and displacement components, $\{\tilde{\sigma}^E, \tilde{u}^E\}$, satisfy the same equilibrium equations, stress-strain relations, and strain-displacement relations as in the original problem. Along $z = \pm h$ we have also $\tilde{\sigma}_{xz}^E = \tilde{\sigma}_{zz}^E = 0$. As $y \to \infty$, both $\tilde{\sigma}_{xx}^E$ and $\tilde{\sigma}_{xz}^E$ tend to zero. Finally, we have now at the root of the strip

$$y = 0: \qquad \tilde{u}_x^E = -u_0, \quad \tilde{u}_z^E = \frac{\nu_{21}z}{2hE_1} = \frac{\nu z}{2hE\nu}. \tag{2.12}$$

Similarly, we set for the bending problem, denoted by a superscript B,

$$\sigma_{xx}^B = \tilde{\sigma}_{xx}^B + \frac{3z}{2h^3}, \qquad \sigma_{xz}^B = \tilde{\sigma}_{xz}^B, \qquad \sigma_{zz}^B = \tilde{\sigma}_{zz}^B,$$

$$u_x^B = \tilde{u}_x^B + \frac{3xz}{2h^3E_1} - \omega_0 z, \qquad u_z^B = \tilde{u}_z^B - \frac{3}{4h^3E_1}\left(x^2 + \nu_{21}z^2\right) + \omega_0 x + w_0,$$

and for the flexure problem, denoted by a superscript F,

$$\sigma_{xx}^F = \tilde{\sigma}_{xx}^F + \frac{3xz}{2h^3}, \qquad \sigma_{xz}^F = \tilde{\sigma}_{xz}^F + \frac{3}{4h}\left(1 - \frac{z^2}{h^2}\right), \qquad \sigma_{zz}^F = \tilde{\sigma}_{zz}^F,$$

$$u_x^F = \tilde{u}_x^F + \frac{1}{4h^3}\left(\frac{3x^2z}{E_1} + \frac{\nu_{21}z^3}{E_1} - \frac{z^3}{G}\right) - \omega_0 z,$$

$$u_z^F = \tilde{u}_z^F + \frac{1}{4h^3}\left(\frac{3h^2x}{G} - \frac{x^3}{E_1} - \frac{3\nu_{21}xz^2}{E_1}\right) + \omega_0 x + w_0. \tag{2.14}$$

In both cases, the residual problem is governed by (2.4) and the traction-free conditions along $z = \pm h$ with the stress components tending to zero as $y \to \infty$. At the root of the strip, we have for the bending problem

$$y = 0: \qquad \tilde{u}_x^B = \omega_0 z, \quad \tilde{u}_z^B = \frac{3\nu z^2}{4h^3E} - w_0, \tag{2.15}$$

and for the flexure problem

$$y = 0: \qquad \tilde{u}_x^F = \frac{1}{4h^3}\left(\frac{1}{G} - \frac{\nu}{E}\right)z^3 + \omega_0 z, \quad \tilde{u}_z^F = -w_0. \tag{2.16}$$

In subsequent developments, we will work with the residual problems to analyze their solution. However, numerical results will be given for the actual solution of the physical boundary value problems unless explicitly stated otherwise.

3. The extension problem

We work with the *residual* problem defined by (i) the partial differential equation (2.4), (ii) the edge conditions (2.6), written in terms of the stress functions $\tilde{\phi}$ as

$$z = \pm h: \qquad \tilde{\phi}_{,yy} = -\tilde{\phi}_{,yz} = 0, \tag{2.6$'$}$$

(iii) the end conditions (2.12) written as

$$y = 0: \qquad \tilde{u}_x = \tilde{u}(z), \quad \tilde{u}_z = \tilde{w}(z), \tag{2.12$'$}$$

where $y = x/\beta_0$, $\tilde{u}(z) = -u_0$, and $\tilde{w}(z) = \nu z/2hE$, and (iv) the limiting conditions at infinity

$$y \to \infty: \qquad \tilde{\sigma}_{xx} \to 0, \quad \tilde{\sigma}_{xz} \to 0. \tag{3.1}$$

We have omitted the superscript E which denotes quantities associated with the extension problem in this section; there can be no possible confusion given the above specification.

For isotropic materials, we know from the discussion of [10] that the solution of this residual problem must be an exponentially decaying state. Therefore, we have as $y \to \infty$ $(x \to \infty)$

$$\{\tilde{u}, \tilde{\sigma}\} = O(e^{-\alpha x/h}) \tag{3.2}$$

for some positive $O(1)$ constant α. The interior (outer) solution and boundary layer (residual) solution for orthotropic plane elasticity problems obtained in [17] suggest that the same is true for our residual orthotropic strip problem. The method of Fourier cosine and sine transforms is therefore applicable to the problem. Let

$$\phi(s, z) = \int_0^\infty \tilde{\phi}(y, z) \cos sy \, dy, \tag{3.3}$$

and transform (2.4) into

$$\Phi_{,zzzz} - s^2(2+\delta_0)\Phi_{,zz} + s^4\Phi = g(s, z), \tag{3.4}$$

where

$$g(s, z) = \tilde{\phi}_{,yyy}(0, z) - s^2\tilde{\phi}_{,y}(0, z) + (2+\delta_0)\tilde{\phi}_{,yzz}(0, z). \tag{3.5}$$

For boundary conditions, we note that the Fourier sine transform of the boundary condition (2.6b) is

$$\Phi_{,z}(s, \pm h) = 0, \tag{3.6a}$$

whereas the Fourier cosine transform of (2.6a) is $-s^2\Phi(s, \pm h) - \tilde{\phi}_{,y}(0, \pm h) = 0$, or

$$\Phi(s, \pm h) = 0 \tag{3.6b}$$

with $\tilde{\sigma}_{xz}(y, \pm h) = 0$ implying $\tilde{\phi}_{,y}(y, \pm h) = 0$ [17].

We now rewrite the expression for $g(s, z)$ in terms of the even function $\tilde{\sigma}_{xz}(0, z) \equiv \tilde{\tau}(z)$ and known quantities. For exponentially decaying states, the following relations hold:

$$\begin{aligned}
\tilde{\phi}_{,y}(0, z) &= -\beta_0 \int_{-h}^{z} \tilde{\tau}(t)\, dt = \beta_0 \int_z^h \tilde{\tau}(t)\, dt, \\
\tilde{\phi}_{,yzz}(0, z) &= -\beta_0 \tilde{\tau}'(z), \\
\tilde{\phi}_{,yyy}(0, z) &= -\beta_0 E\tilde{u}''(z) + \beta_0\left(\frac{E}{G} - \nu\right)\tilde{\tau}'(z),
\end{aligned} \tag{3.7}$$

where a prime indicates differentiation with respect to the argument of the function. To get the third relation in (3.7), we use the relation for the transverse normal strain ε_{zz} in the form

$$E_2\tilde{u}_{z,zx} = (\tilde{\sigma}_{zz} - \nu_{12}\tilde{\sigma}_{xx})_{,x} = \frac{1}{\beta_0^3}\tilde{\phi}_{,yyy} - \nu_{12}\tilde{\sigma}_{xx,x},$$

or

$$\tilde{\phi}_{,yyy}(y,z) = \beta_0^3[E_2\tilde{u}_{z,xz} + \nu_{12}\tilde{\sigma}_{xx,x}]$$
$$= -\beta_0 E\tilde{u}_{x,zz} + \beta_0\left(\frac{E}{G} - \nu\right)\tilde{\sigma}_{xz,z},$$

where we have used the equilibrium equation $\tilde{\sigma}_{xx,x} + \tilde{\sigma}_{xz,z} = 0$ to eliminate $\tilde{\sigma}_{xx,x}$, and the relations (2.1c) and (2.2c) for shear strain to eliminate $\tilde{u}_{z,x}$. Now, (3.7c) follows upon setting $y = 0$ $(x = 0)$ and keeping in mind $\tilde{u}_x(0,z) = \tilde{u}(z)$ and $\tilde{\sigma}_{xz}(0,z) = \tilde{\tau}(z)$. With (3.7), we can write (3.5) as

$$\frac{1}{\beta_0}g(s,z) = \nu\tilde{\tau}'(z) - E\tilde{u}''(z) - s^2\int_z^h \tilde{\tau}(t)\,dt. \tag{3.8}$$

The solution of the boundary value problem (3.4)–(3.6) may be obtained by the method of variation of parameters. For the extension problem, this solution is even in z and may be taken in the form

$$\Phi(s,z) = \int_0^h \overline{G}(s,z;t)g(s,t)\,dt, \tag{3.9}$$

where the Green's function $\overline{G}(s,z;t)$ is given by

$$\overline{G}(s,z;t) = \frac{\delta}{(\delta^4-1)s^3\Delta(\delta)}$$
$$\times\begin{cases} S_1(\delta) + \delta^4 S_1(1/\delta) + \delta^2[T_1(\delta) + T_1(1/\delta)] & (0 \le t \le z), \\ S_2(\delta) + \delta^4 S_2(1/\delta) + \delta^2[T_2(\delta) + T_2(1/\delta)] & (z \le t \le h) \end{cases} \tag{3.10}$$

with

$$\delta = \left[1 + \tfrac{1}{2}\delta_0 + \sqrt{\left(1 + \tfrac{1}{2}\delta_0\right)^2 - 1}\right]^{1/2} > 1,$$
$$\Delta(\delta) = \cosh(sh\delta)\sinh(sh/\delta) - \delta^2\sinh(sh\delta)\cosh(sh/\delta),$$
$$S_1(\delta) = \sinh(sh/\delta)\cosh(st\delta)\sinh(s\delta[z-h]),$$
$$S_2(\delta) = \sinh(sh/\delta)\cosh(sz\delta)\sinh(s\delta[t-h]), \tag{3.11}$$
$$T_1(\delta) = \cosh(s\delta t)\{\cosh(sh/\delta)\cosh(s\delta[z-h]) - \cosh(sz/\delta)\},$$
$$T_2(\delta) = \cosh(s\delta z)\{\cosh(sh/\delta)\cosh(s\delta[t-h]) - \cosh(st/\delta)\}.$$

Upon substituting (3.8) into (3.9) and integrating the resulting expression by parts, we get

$$\Phi(s,z) = -\beta_0 E\int_0^h \overline{G}\tilde{u}''(t)\,dt - \beta_0\int_0^h\left\{\nu\overline{G}_{,t} + s^2\int_0^t \overline{G}(s,z;\xi)\,d\xi\right\}\tilde{\tau}(t)\,dt. \tag{3.12}$$

By the inversion formula, we get from (3.12)

$$\begin{aligned}\tilde{\phi}(y,z) &= \frac{2}{\pi}\int_0^\infty(\cos sy)\Phi(s,z)\,ds\\ &= -\frac{2}{\pi}\beta_0 E\int_0^h G_0(y,z;t)\tilde{u}''(t)\,dt\\ &\quad - \frac{2\beta_0}{\pi}\int_0^h\{\nu G_1(y,z;t) + G_2(y,z;t)\}\tilde{\tau}(t)\,dt,\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}G_0(y,z;t) &= \int_0^\infty(\cos sy)\overline{G}(s,z;t)\,ds,\\ G_1(y,z;t) &= \int_0^\infty(\cos sy)\overline{G}_{,t}(s,z;t)\,ds,\\ G_2(y,z;t) &= \int_0^\infty(\cos sy)s^2\left[\int_0^t\overline{G}(s,z;\xi)\,d\xi\right]ds.\end{aligned} \tag{3.14}$$

The expression (3.13) is not yet a solution of our problem, as we do not know $\tilde{\tau}(z) \equiv \tilde{\sigma}_{xz}(0,z)$. To determine $\tilde{\tau}(z)$, we use the known end data for $\tilde{u}_z$ as follows. From the expression for the transverse normal strain component ε_{zz} in (2.2), we get

$$\tilde{u}_{z,z} = \frac{1}{E}\left\{\frac{\partial^2}{\partial y^2} - \nu\frac{\partial^2}{\partial z^2}\right\}\tilde{\phi} \equiv \frac{1}{E}L[\tilde{\phi}]. \tag{3.15}$$

Upon substituting (3.13) into the right hand side of the above relation, we get

$$\begin{aligned}&\frac{1}{\pi}\int_0^h\{\nu L[G_1] + L[G_2]\}\tilde{\tau}(t)\,dt\\ &\quad = -\frac{E}{2\beta_0}\tilde{u}_{z,z} - \frac{E}{\pi}\int_0^h L[G_0]\tilde{u}''(t)\,dt.\end{aligned} \tag{3.16}$$

With $\tilde{u}_z$ and $\tilde{\phi}$ being both even functions of y, we may set $y=0$ to get

$$\frac{1}{\pi}\int_0^h K(z;t)\tilde{\tau}(t)\,dt = -\frac{E}{2\beta_0}\tilde{w}'(z) - \frac{E}{\pi}\int_0^h L[G_0(y,z;t)]|_{y=0}\tilde{u}''(t)\,dt, \tag{3.17a}$$

where

$$K(z;t) = \{\nu L[G_1(y,z;t)] + L[G_2(y,z;t)]\}_{y=0}. \tag{3.17b}$$

Equation (3.17) is a Fredholm integral equation of the first kind for $\tilde{\tau}(z) \equiv \tilde{\sigma}_{xz}(0,z)$, as the right hand side is a known function. For the present extension problem, we have $\tilde{u}''(z) \equiv 0$ and $\tilde{w}'(z) = \nu/2Eh$, so that (3.17) simplifies to read

$$\frac{1}{\pi}\int_0^h K(z;t)\tilde{\tau}(t)\,dt = -\frac{\nu}{4h\beta_0} \tag{3.17$'$}$$

for the determination of $\tilde{\tau}(z)$. Once we have $\tilde{\tau}(z)$, Equation (3.13) gives $\tilde{\phi}(x,z)$ and hence the solution of the problem [with the displacement components to be computed from the strain-displacement relations (2.2)].

For the purpose of formulating the proper boundary conditions for plate theories with displacement edge data, we need only $\tilde{\sigma}(z) \equiv \tilde{\sigma}_{xx}(0,z)$ in addition to $\tilde{\tau}(z)$. To get $\tilde{\sigma}_{xx}(0,z)$, we note the relation

$$\begin{aligned}\tilde{\sigma}_{xx}(0,z) &= \lim_{y\to 0}\tilde{\phi}_{,zz}(y,z)\\ &= -\frac{2}{\pi}\beta_0 E\int_0^h G_{0,zz}(y,z;t)|_{y=0}\tilde{u}''(t)\,dt\\ &\quad -\frac{2}{\pi}\beta_0\int_0^h\{\nu G_{1,zz}(y,z;t)+G_{2,zz}(y,z;t)\}_{y=0}\tilde{\tau}(t)\,dt.\end{aligned} \tag{3.18}$$

For the extension problem, the above relation simplifies to read

$$\tilde{\sigma}(z) = -\frac{2}{\pi}\beta_0\int_0^h\{\nu G_{1,zz}+G_{2,zz}\}_{y=0}\tilde{\tau}(t)\,dt, \tag{3.19}$$

where the right hand side is a known quantity once we have $\tilde{\tau}(z)$ from the solution of (3.17$'$).

4. A generalized Cauchy integral equation

The expressions for G_1 and G_2 defined in (3.14) will now be cast in a form convenient for a further analysis of the kernel $K(z;t)$ in the integral equation

(3.17) [or (3.17′)]. Let

$$\begin{aligned}
\overline{P}_1(s,z;t;\delta) &= \sinh(sh/\delta)\sinh(s\delta t)\sinh(s\delta[z-h]),\\
\overline{P}_2(s,z;t;\delta) &= \sinh(sh/\delta)\cosh(s\delta z)\cosh(s\delta[t-h]),\\
\overline{Q}_1(s,z;t;\delta) &= \sinh(st/\delta)\{\cosh(s\delta h)\cosh(s[z-h]/\delta)-\cosh(s\delta z)\},\\
\overline{Q}_2(s,z;t;\delta) &= \cosh(sz/\delta)\cosh(s\delta h)\sinh(s[t-h]/\delta)\\
&\quad -\cosh(s\delta z)\sinh(st/\delta),
\end{aligned} \tag{4.1}$$

and

$$\{P_k(y,z;t;\delta),Q_j(y,z;t;\delta)\} = \int_0^\infty \frac{\cos sy}{s^2\Delta(\delta)}\{\overline{P}_k(s,z;t;\delta),\overline{Q}_j(s,z;t;\delta)\}\,ds \tag{4.2}$$

for k, $j=1,2$. We may then write G_1 and G_2 as

$$G_1(y,z;t;\delta) = \frac{\delta^2}{\delta^4-1}\begin{cases} [P_1(y,z;t;\delta)-Q_1(y,z;t;1/\delta)] \\ \quad -[P_1(y,z;t;1/\delta)-Q_1(y,z;t;\delta)] & (0\le t\le z),\\ [P_2(y,z;t;\delta)-Q_2(y,z;t;1/\delta)] \\ \quad -[P_2(y,z;t;1/\delta)-Q_2(y,z;t;\delta)] & (z\le t\le h),\end{cases} \tag{4.3}$$

$$G_2(y,z;t;\delta) = \frac{1}{\delta^4-1}\begin{cases} [P_1(y,z;t;\delta)-Q_1(y,z;t;1/\delta)] \\ \quad -\delta^4[P_1(y,z;t;1/\delta)-Q_1(y,z;t;\delta)] & (0\le t\le z),\\ [P_2(y,z;t;\delta)-Q_2(y,z;t;1/\delta)] \\ \quad -\delta^4[P_2(y,z;t;1/\delta)-Q_2(y,z;t;\delta)] & (z\le t\le h).\end{cases} \tag{4.4}$$

In the expressions for G_1 and G_2, we have dropped a term $-\sinh(sh/\delta)\cosh(sh\delta)$ in $\overline{P}_2$ and a term $\sinh(sh/\delta)\cosh(sh\delta)$ in $\overline{Q}_2$, since they make no contributions to the stresses. It is important to note that in studying the convergence of the improper integrals for G_1 and G_2, the entire integrand, not individual terms such as those given by (4.2), should be analyzed. A particular improper integral may converge even if some of its parts do not individually. The corresponding expression for G_0 obtained from (3.10) is straightforward.

The improper integrals P_k and Q_j which appear in the expressions for G_1 and G_2 may be split into a regular part, a singular part, and a part that is exponentially small. For $s \geq \eta \gg 1$, we have

$$\Delta(\delta) = \tfrac{1}{4}(1-\delta^2)e^{(\delta^2+1)hs/\delta}\{1+O(e^{-2hs/\delta})\} \equiv \Delta(s,\delta). \qquad (4.5)$$

It follows that for large s, we have

$$\bar{P}_1(s,z;t;\delta) = \tfrac{1}{8}e^{(\delta^2+1)hs/\delta} \times\{e^{-\delta s(z-t)} - e^{-\delta s(z+t)} - e^{-\delta s(2h-t-z)} + O(e^{-sh/\delta})\} \qquad (4.6)$$

and similar asymptotic expressions for $\bar{P}_2$, $\bar{Q}_1$, and $\bar{Q}_2$. They enable us to split P_k and Q_j as follows:

$$P_k(y,z;t;\delta) = \int_0^\eta \frac{\cos sy}{s^2\Delta(s,\delta)}\bar{P}_k(s,z;t;\delta)\,ds - \frac{1}{2(\delta^2-1)}\int_\eta^\infty \frac{\cos sy}{s^2}\{(-1)^k\gamma_1+\gamma_2+\gamma_3\}\,ds + O(e^{-\eta h/\delta}), \qquad (4.7a)$$

$$Q_j(y,z;t;\delta) = \int_0^\eta \frac{\cos sy}{s^2\Delta(s,\delta)}\bar{Q}_j(s,z;t;\delta)\,ds + \frac{1}{2(\delta^2-1)}\int_\eta^\infty \frac{\cos sy}{s^2}\{(-1)^j\gamma_5+\gamma_6-\gamma_7+2\gamma_8\}\,ds + O(e^{-\eta h/\delta}), \qquad (4.7b)$$

where the functions $\gamma_i(s,z;t;\delta)$ are given by the following elementary expressions:

$$\gamma_1 = e^{-\delta s|z-t|}, \qquad \gamma_2 = e^{-\delta s(z+t)}, \qquad \gamma_3 = e^{-\delta s(2h-t-z)},$$

$$\gamma_4 = e^{-s\delta(h-t)-s(h-t)/\delta}, \qquad \gamma_5 = e^{-s|t-z|/\delta},$$

$$\gamma_6 = e^{-s(z+t)/\delta}, \qquad \gamma_7 = e^{-s(2h-t-z)/\delta}, \qquad \gamma_8 = e^{-s\delta(h-z)-s(h-t)/\delta}. \qquad (4.7c)$$

With the expressions for P_k and Q_j in (4.7), we may rewrite (3.17′) as

$$\frac{1}{\pi}\int_0^h\left\{\frac{\lambda_1}{t-z}+\frac{\lambda_2}{t+z}+\frac{\lambda_3}{2h-t-z}+\frac{\lambda_4}{h-z+\delta^2(h-t)}\right.$$

$$\left.+\frac{\lambda_5}{\delta^2(h-z)+(h-t)}\right\}\tilde{\tau}(t)\,dt+\frac{1}{\pi}\int_0^h K_\eta(z;t)\tilde{\tau}(t)\,dt$$

$$=-\frac{\nu}{4h\beta_0}+O(e^{-\eta h/\delta}), \tag{4.8}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_5$ are independent of η and defined by

$$\lambda_1=\lambda_2=\frac{\delta^2(\delta^2+\nu)^2-(1+\nu\delta^2)^2}{2\delta(\delta^4-1)},\qquad \lambda_3=\frac{(1+\nu\delta^2)^2+\delta^2(\nu+\delta^2)^2}{2\delta(\delta^2-1)^2},$$

$$\lambda_4=-\frac{\delta(1+\nu\delta^2)(\nu+\delta^2)}{(\delta^4-1)(\delta^2-1)},\qquad \lambda_5=-\frac{\delta^3(\nu+\delta^2)(1+\nu\delta^2)}{(\delta^4-1)(\delta^2-1)}. \tag{4.9}$$

Let

$$\alpha_1=\alpha_2=-\frac{1}{2\delta}(1+\nu\delta^2),\qquad \alpha_3=\frac{(1+\delta^2)}{2\delta(\delta^2-1)}(1+\nu\delta^2), \tag{4.10a}$$

$$\alpha_4=-\frac{\delta}{(\delta^2-1)}(\delta^2+\nu),\qquad \alpha_5=\alpha_6=\frac{1}{2\delta}(\nu+\delta^2), \tag{4.10b}$$

$$\alpha_7=\frac{(1+\delta^2)}{2\delta(\delta^2-1)}(\delta^2+\nu),\qquad \alpha_8=-\frac{\delta}{(\delta^2-1)}(1+\nu\delta^2). \tag{4.10c}$$

Then the nonsingular portion of the kernel, $K_\eta(z;t)$, is given by

$$K_\eta(z;t)=\nu K_{1\eta}+K_{2\eta}+\frac{\nu\delta^2}{\delta^4-1}\sum_{i=1}^{8}\alpha_i f_{iR}+\frac{1}{\delta^4-1}\left(\sum_{i=1}^{4}\alpha_i f_{iR}+\delta^4\sum_{i=5}^{8}\alpha_i f_{iR}\right), \tag{4.11}$$

where

$$f_{1R} = \frac{e^{-\delta\eta|t-z|}-1}{t-z}, \qquad f_{2R} = \frac{e^{-\delta\eta(t+z)}-1}{z+t}, \tag{4.12a}$$

$$f_{3R} = \frac{e^{-\delta\eta(2h-t-z)}-1}{2h-t-z}, \tag{4.12b}$$

$$f_{4R} = \frac{e^{-\eta\{(h-z)+\delta^2(h-t)\}/\delta}-1}{(h-z)+\delta^2(h-t)}, \tag{4.12c}$$

$$f_{5R} = \frac{e^{-\eta|t-z|/\delta}-1}{(t-z)}, \qquad f_{6R} = \frac{e^{-\eta(z+t)/\delta}-1}{t+z}, \tag{4.12d}$$

$$f_{7R} = \frac{e^{-\eta(2h-t-z)/\delta}-1}{2h-t-z}, \tag{4.12e}$$

$$f_{8R} = \frac{e^{-\eta\{(h-t)+\delta^2(h-z)\}/\delta}-1}{\delta^2(h-z)+(h-t)}, \tag{4.12f}$$

and

$$K_{j\eta}(z;t) = \begin{cases} K_{j\eta 1}(z;t) & (0 \le t \le z), \\ K_{j\eta 2}(z;t) & (z \le t \le h) \end{cases} \tag{4.13a}$$

with

$$K_{1\eta k}(z;t) = -\frac{\delta^2}{\delta^4-1} \times\left\{\int_0^\eta\left[\overline{P}_k(\delta)+\overline{Q}_k(\delta)+\delta^2\overline{P}_k\left(\frac{1}{\delta}\right)+\delta^2\overline{Q}_k\left(\frac{1}{\delta}\right)\right]\frac{ds}{\Delta(s,\delta)}\right.$$
$$\left.+\nu\int_0^\eta\left[\delta^4\tilde{Q}_k\left(\frac{1}{\delta}\right)+\delta^2\overline{P}_k(\delta)+\overline{P}_k\left(\frac{1}{\delta}\right)+\frac{1}{\delta^2}\tilde{Q}_k(\delta)\right]\frac{ds}{\Delta(s,\delta)}\right\}, \tag{4.13b}$$

$$K_{2\eta k}(z;t) = -\frac{1}{\delta^4-1} \times\left\{\int_0^\eta\left[\overline{P}_k(\delta)+\delta^2\overline{Q}_k\left(\frac{1}{\delta}\right)+\delta^4\overline{Q}_k(\delta)+\delta^6\overline{P}_k\left(\frac{1}{\delta}\right)\right]\frac{ds}{\Delta(s,\delta)}\right.$$
$$\left.+\nu\int_0^\eta\left[\delta^2\overline{P}_k(\delta)+\delta^2\tilde{Q}_k(\delta)+\delta^4\overline{P}_k\left(\frac{1}{\delta}\right)+\delta^4\tilde{Q}_k\left(\frac{1}{\delta}\right)\right]\frac{ds}{\Delta(s,\delta)}\right\}, \tag{4.13c}$$

in which $\tilde{Q}_1$ and $\tilde{Q}_2$ are defined by

$$\begin{aligned}
\tilde{Q}_1(\xi) &= \sinh(st/\xi)\left\{\cosh(s\xi h)\cosh(s[z-h]/\xi)-\xi^4\cosh(s\xi z)\right\}\\
&\equiv \tilde{Q}_1(s,z;t;\xi)\\
\tilde{Q}_2(\xi) &= \cosh(sz/\xi)\cosh(s\xi h)\sinh(s[t-h]/\xi)-\xi^4\cosh(s\xi z)\sinh(st/\xi)\\
&\equiv \tilde{Q}_2(s,z;t;\xi).
\end{aligned} \tag{4.14}$$

In (4.13b) and (4.13c), we have suppressed the dependence of $\overline{P}_k$ and $\overline{Q}_k$ on s, z, and t so that the equations do not become unwieldy; in fact we have $\overline{P}_k(\xi)=\overline{P}_k(s,z;t;\xi)$ and $\overline{Q}_k(\xi)=\overline{Q}_k(s,z;t;\xi)$. Note that the regular kernel $K_\eta(z;t)$ contains contributions from both the proper integrals over $(0,\eta)$ and the improper integrals over (η,∞), $\nu K_{1\eta}+K_{2\eta}$ in (4.11) and the rest of the right hand side of (4.11).

The Fredholm integral equation of the first kind (3.17) [or (3.17′)] is now seen to be singular of the form (4.8). While the singularity of $K(z;t)$ is of the Cauchy type, the integral equation (4.8) is more general than those treated in [20] and [4]. With a slight modification, the method of [4] may also be used for extracting the corner singularity of our orthotropic strip. The details are effectively contained in Appendix I. An alternative method for treating singularities can be found in [16].

5. Numerical solution for the extension problem

The singular integral equation (4.8) is a special case of the more general equation (I.1) analyzed in Appendix I with $a=0$, $b=h$, $x=z$, $J=1$, $M=3$, and $c_0=\lambda_1$. The remaining constants are identified as follows:

$$\begin{aligned}
c_1 &= \lambda_2, & k_1 &= 1, & \theta_1 &= \pi,\\
d_1 &= -\lambda_3, & h_1 &= 1, & \omega_1 &= 0,\\
d_2 &= -\lambda_4/\delta^2, & h_2 &= 1/\delta^2, & \omega_2 &= 0,\\
d_3 &= -\lambda_5, & h_3 &= \delta^2, & \omega_3 &= 0.
\end{aligned} \tag{5.1}$$

Therefore, the singularity exponent β at $z=h$ is a root of the equation [cf. (I.14)]

$$F(\beta) = \lambda_1\cos\pi\beta-\lambda_3-\lambda_4\delta^{-2+2\beta}-\lambda_5\delta^{-2\beta}=0. \tag{5.2}$$

While we do not expect a singularity at $z=0$, it is of some interest to note that

Table 1
Elastic Moduli for Six Plate Materials

Material	E_1 (kg/cm^2)	E_2 (kg/cm^2)	G (kg/cm^2)	ν_{12}
(1) Pine wood 1	100,000	4,200	7,500	0.01
(2) Plywood 1	120,000	64.400	7,200	0.044
(3) Pine wood 2	4,200	100,000	7,500	0.238
(4) Plywood 2	64,400	120,000	7,200	0.082
(5) Isotropic 1	15,732	15,732	7,500	0.0488
(6) Isotropic 2	15,265	15,265	7,200	0.06

(I.13) gives for the present problem

$$\lambda_1 \cos \pi\alpha + \lambda_2 = 0. \tag{5.3}$$

With $\lambda_1 = \lambda_2$ for our case, (5.3) has no root inside the interval $(0,1)$. For the root $\alpha = 1$, the singularity would be too severe (as the strain energy would be unbounded). These observations are consistent with our expectation.

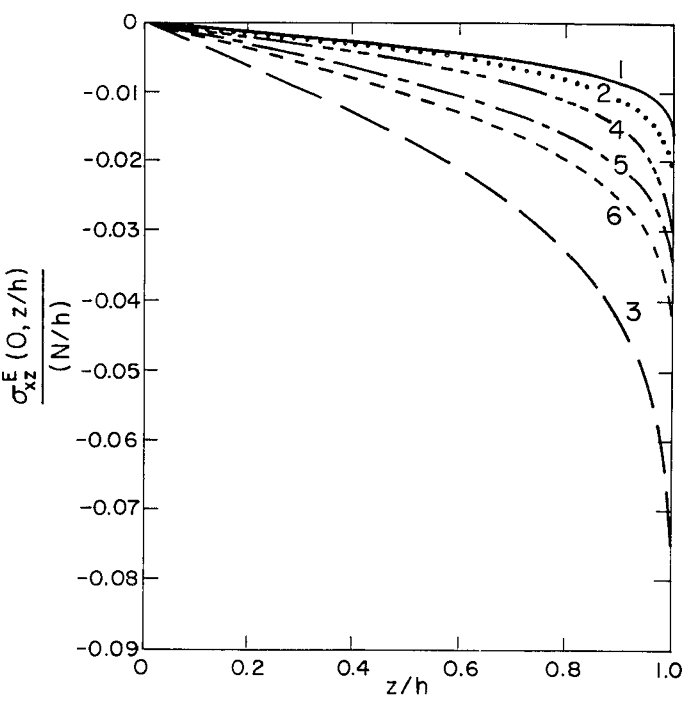

Figure 1. Distributions of transverse shear at the root of semiinfinite cantilevered strips under uniform tension at infinity.

The coefficients λ_i of (5.2) are not sensitive to the directional preference of the material; hence neither is the solution β. In the limit as $\delta \to 1$, Equation (5.2) reduces to the well-known equation for isotropic material. With $F'(\beta) < 0$ for $0 < \beta < 1$, $F(0) > 0$, and $F(1) < 0$, there is a unique real solution of (5.2) in the interval $0 < \beta < 1$ for an isotropic strip. By applying the argument principle of complex function theory over rectangles bounded by $\mathrm{Re}(\beta) = 0$, $\mathrm{Re}(\beta) = 1$ and $\mathrm{Im}(\beta) = \pm H$ for different H, it has been proved also that (5.2) has no complex solution in the strip $0 < \mathrm{Re}\beta < 1$ [15,16]. As indicated in the last section, the singularity of orthotropic strips can also be analyzed by a method similar to that of [16]. In fact, there is an excellent agreement between the numerical solutions by that method and by (5.2) obtained in [15].

For a numerical solution of (4.8), the singular integral equation is discretized by the method of orthogonal collocation [2]. With the singularity exponent β determined by (5.2), the unknown function $\tilde{\tau}(z)$ can be written as $(h - z)^{-\beta}\bar{\tau}(z)$, where $\bar{\tau}(z)$ is a bounded function and represented by a linear combination of Jacobi polynomials $J_n^{(0,-\beta)}(z)$. The solution can be made to satisfy the global equilibrium conditions and be evaluated at the zeros of the Jacobi polynomials [5].

The parameter η which splits the improper integral (4.2) [see also (4.7)] is taken to be 50; a larger η does not noticeably change the numerical results. Various integrals are evaluated numerically by Gaussian quadratures in Jacobi polynomials whose weights and roots are given by known procedures [23]. By

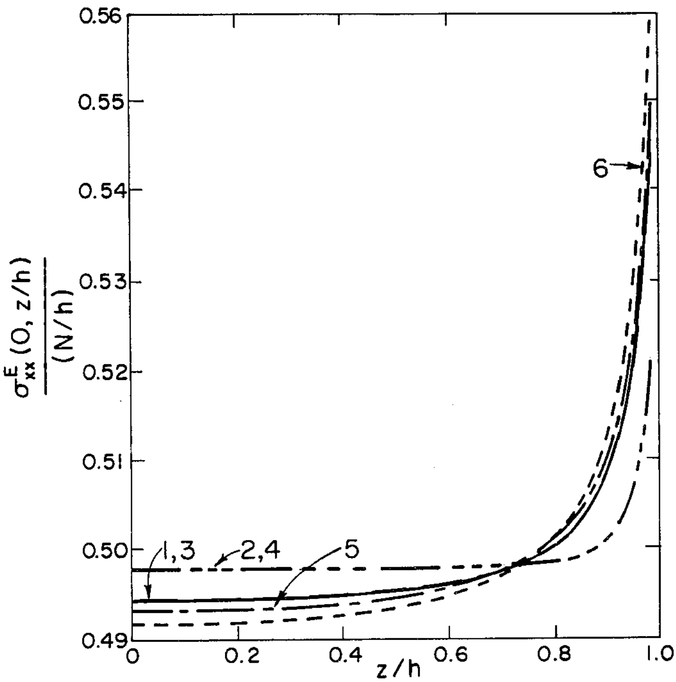

Figure 2. Distributions of axial stress at the root of semiinfinite cantilevered strips under uniform axial tension at infinity.

taking advantage of the symmetry property of $\tilde{\tau}$, a system of eight equations is sufficiently accurate for all cases treated in this paper.

The above numerical solution process has been implemented on the Amdahl 5850 in the Computing Centre of the University of British Columbia. A typical run for the complete solution with ten collocation points required less than 23.3 seconds of machine time (at a cost of \$2.50). The FORTRAN code written for this purpose was tested by comparing the results for an effectively isotropic strip (with $E_2 = 20{,}000$ kg/cm^2, $E_1 = 20{,}000.00001$ kg/cm^2, $G = 7500$ kg/cm^2, and $\nu = \frac{1}{3}$ corresponding to a Poisson's ratio of $\frac{1}{4}$) with the corresponding results in [9]. The agreement is excellent. In fact, with eight collocation constants, the moments of stress defined in (6.8) are close to (within 2% of) the results of [9] except for the quantity n_3^F; the latter is close to (within 7% of) the result of [9] with ten collocation constants.

New results are reported here for six different materials whose elastic parameter values are as given in Table I. Evidently, (3) and (4) are just materials (1) and (2), respectively, rotated in the plane by 90°. The value of ν for materials (5) and (6) is equal to $\sqrt{\nu_{12}\nu_{21}}$ for pinewoods and for plywoods, respectively.

With

$$N = \int_{-h}^{h} \sigma_{xx}^{E}\, dx,$$

the distributions of the dimensionless transverse shear stress $\sigma_{xz}^{E}/(N/h)$ at the

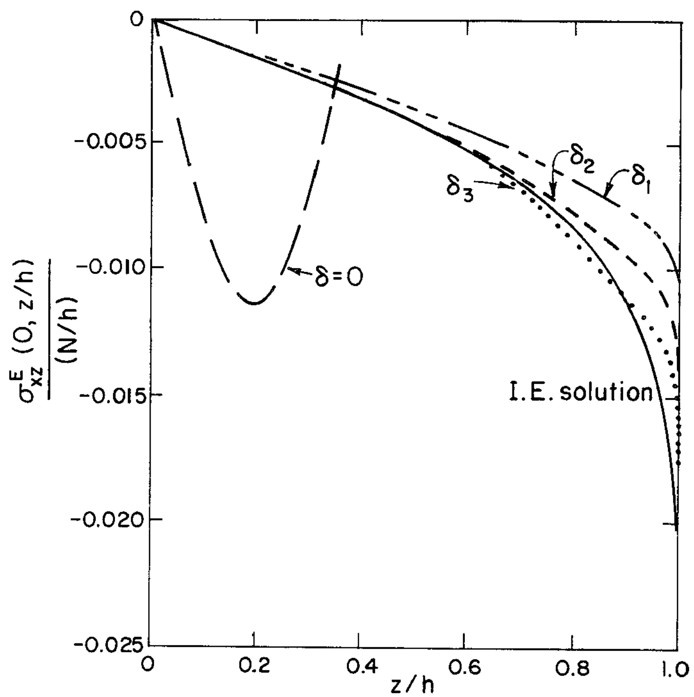

Figure 3. The truncated eigenfunction expansion solutions for $\sigma_{xz}^{E}(0, z)$ for different regularization parameter values.

root of the strip ($x=0$) for all six materials versus z/h are shown in Figure 1. The corresponding distributions for $\sigma_{xx}^{E}/(N/h)$ at the root are shown in Figure (2). We know from (3.17′) and (3.19) that $\beta_0\tilde{\tau}(z)$ and $\tilde{\sigma}(z)$ are not sensitive to the direction of orthrotropy.

The accuracy of these results for orthotropic strips (as well as other results presented herein) is verified by a second (less efficient) method of solution described in Appendix II of this paper. We show in Figure 3 a comparison of the distributions of $\sigma_{xz}^{E}(0,z/h)/(N/h)$ for material (1) by the singular integral equation approach and this second (eigenfunction expansion) method. We see from these results that the truncated eigenfunction expansion solution approaches the solution by the singular integral equation formulation as the regularization parameter δ approaches the optimal values (cf. Appendix II).

6. Displacement boundary conditions for the interior solution in plane strain deformations

We know from [17] that

$$\phi_I = \frac{N_0}{4h}z^2 + \frac{M_0}{4h^3}z^3 - \frac{3Q_0 x}{4h}\left(z - \frac{z^3}{3h^2}\right) \tag{6.1}$$

is an exact solution of (2.4) and (2.6). This is identical to the classical Lévy solution [10] for isotropic materials. For sufficiently thin plates, the residual solution in that case is a boundary layer effect [10]; hence (6.1) is more commonly known as the *interior* solution. The results of [17] suggest (and we assume) that the same is true for the present orthotropic problem. The stress and displacement components corresponding to (6.1) have been obtained in [17]; we list here the two displacement components needed later in this section:

$$\begin{aligned} E_1 u_x^I &= \frac{N_0}{2h}x + \frac{3M_0}{2h^3}xz \\ &\quad + \frac{Q_0}{4h^3}\left(3x^2 z + \nu_{21}z^3 - \frac{E_1}{G}z^3\right) + E_1(d - \omega z), \\ E_1 u_z^I &= -\frac{\nu_{21}N_0}{2h}z - \frac{3M_0}{4h^3}\left(\nu_{21}z^2 + x^2\right) \\ &\quad + \frac{Q_0}{4h^3}\left(\frac{3h^2E_1}{G}x - x^3 - 3\nu_{21}xz^2\right) + E_1(c + \omega x). \end{aligned} \tag{6.2}$$

Let $\tilde{u}_x \equiv u_x - u_x^I$, $\tilde{u}_z \equiv u_z - u_z^I$, etc. For a strip which spans the interval $(-l, l)$ in the x-direction, we apply the reciprocal theorem to the half strip $-l \le x \le 0$ with state (1) taken to be the residual state $\{\tilde{\sigma}_{ij}, \tilde{u}_k\}$ and state (2) taken to be the solution of any one of three canonical problems for a semiinfinite strip with its clamped end at $x = -l$. By an argument similar to that for isotropic materials [10], we get from the reciprocal theorem for the case $h \ll l$ (or more generally $\beta_0 h \ll l$)

$$\int_{-h}^{h} [\sigma_C(z)\tilde{u}_x(-l,z) + \tau_C(z)\tilde{u}_z(-l,z)]\, dz = O(e^{-\lambda_0 l/\beta_0 h}), \tag{6.3}$$

where λ_0 is an $O(1)$ constant; C may be E, B, or F; and σ_C and τ_C are the normal and shear stresses at $x = -l$ of the relevant strip problem. Upon omitting the exponentially small term, we rewrite (6.3) as

$$\int_{-h}^{h} [\sigma_C(z)\bar{u}(z) + \tau_C(z)\bar{w}(z)]\, dz$$

$$= \int_{-h}^{h} [\sigma_C(z)u_x^I(-l,z) + \tau_C(z)u_z^I(-l,z)]\, dz. \tag{6.4}$$

where $\bar{u}(z)$ and $\bar{w}(z)$ are the prescribed axial and transverse displacement data at $x = -l$. We may now use the expressions for u_x^I and u_z^I in (6.4) and the properties of the solution of the three canonical problems to get

$$\frac{N_0}{2E_1}\left[\frac{x}{h} - \nu_{21}t_1^E\right] + d = \int_{-h}^{h} \{\sigma_E(z)\bar{u}_x(z) + \tau_E(z)\bar{u}_z(z)\}\, dz \tag{6.5}$$

$$\frac{Q_0}{4E_1}\left[\frac{3x^2}{h^2} + \left(\nu_{21} - \frac{E_1}{G}\right)n_3^B - 3\nu_{21}\frac{x}{h}t_2^B\right] + \frac{3M_0}{4E_1 h}\left[2\frac{x}{h} - \nu_{21}t_2^B\right] - \omega h$$

$$= h\int_{-h}^{h} \{\sigma_B(z)\bar{u}_x(z) + \tau_B(z)\bar{u}_z(z)\}\, dz, \tag{6.6}$$

$$\frac{Q_0}{4E_1}\left[\left(\nu_{21} - \frac{E_1}{G}\right)n_3^F + \left(\frac{3xE_1}{Gh} - \frac{x^3}{h^3}\right) - 3\nu_{21}\frac{x}{h}t_2^F\right] - \frac{3M_0}{4E_1 h}\left[\nu_{21}t_2^F + \frac{x^2}{h^2}\right] + \omega x + c$$

$$= \int_{-h}^{h} \{\sigma_F(z)\bar{u}_x(z) + \tau_F(z)\bar{u}_z(z)\}\, dz, \tag{6.7}$$

where $x = -l$ and the dimensionless moments of stress t_1^E, n_3^B, t_2^B, etc., are defined by

$$n_3^F = \frac{1}{h^3 Q_0}\int_{-h}^{h} z^3 \sigma_F \, dz, \qquad t_1^E = \frac{1}{hN_0}\int_{-h}^{h} z\tau_E \, dz, \qquad t_2^F = \frac{1}{h^2 Q_0}\int_{-h}^{h} z^2 \tau_F \, dz$$

$$n_3^B = \frac{1}{h^2 M_0}\int_{-h}^{h} z^3 \sigma_B \, dz, \qquad t_2^B = \frac{1}{hM_0}\int_{-h}^{h} z^2 \tau_B \, dz \tag{6.8}$$

with

$$Q_0 = \int_{-h}^{h} \tau_F \, dz, \qquad M_0 = \int_{-h}^{h} \sigma_B z \, dz, \qquad N_0 = \int_{-h}^{h} \sigma_E \, dz. \tag{6.9}$$

Three other conditions are needed to completely determine the six unknown constants N_0, Q_0, M_0, ω, c, and d. They come from three necessary conditions on the difference between the interior solution and the exact solution at the second edge $x = l$. If the edge data at $x = l$ are also prescribed in terms of the displacement components, then (6.5)–(6.7) for $x = l$ again apply, with $\sigma_C(z)$ and $\tau_C(z)$ being for a semiinfinite strip whose end is at $x = l$ and with $\bar{u}_x$ and $\bar{u}_z$ being the data at that edge. The relevant three conditions for other sets of admissible edge data have already been discussed in [17]. Thus the interior solution of an orthotropic plate in plane strain deformation is determined (up to exponentially small terms) without any knowledge of the boundary layer solution components.

The constants $\{n_m^B, t_j^E, t_n^F\}$ defined in (6.8) depend only on the material properties of the plate and can be calculated once and for all for a given plate material. For completeness, we record in Table 2 their values for the six materials in Table 1 with t_1^E calculated from $\sigma_{xz}^E(0, z)$ obtained herein and the others taken from [18].

The given table of moments is for the edge $x = -l$ (or any edge with an outward normal in the negative x-direction). For an edge with an outward normal in the positive x-direction, such as $x = l$, the corresponding moments $\hat{t}_m^C$ and $\hat{n}_m^C$ are equal to t_m^C and n_m^C in magnitude but possibly with a change of sign.

Table 2

Moments of Stress for the Canonical Problems

Material	t_1^E	t_2^B	n_3^B	t_2^F	n_3^F
(1) Pine wood 1	−0.005975	−0.005276	0.6024	0.2685	−0.1884
(2) Plywood 1	−0.007584	−0.006534	0.6003	0.3041	−0.2258
(3) Pine wood 2	−0.02914	−0.02573	0.6024	0.2685	−0.03861
(4) Plywood 2	−0.01035	−0.008921	0.6003	0.3041	−0.1654
(5) Isotropic 1	−0.01436	−0.01276	0.6030	0.2603	−0.07205
(6) Isotropic 2	−0.01757	−0.01562	0.6037	0.2603	−0.07205

It is straightforward to show that

$$\hat{t}_1^E = -t_1^E, \qquad \hat{t}_2^B = -t_2^B, \qquad \hat{t}_2^F = t_2^F,$$

$$\hat{n}_3^B = n_3^B, \qquad \hat{n}_3^F = -n_3^F. \tag{6.10}$$

Appendix I. The theory of generalized Cauchy integral equations

The method for determining singularities in the solution of Cauchy type singular integral equations is worked out in [20]. The method has been applied in [4] to more general singular integral equations which arise in fracture mechanics problems involving several singular terms of the Cauchy type in the kernel. We briefly outline in this section the extension of the technique for the still more general equation (designated as a *generalized Cauchy integral equation*)

$$\frac{1}{\pi}\int_a^b \left[\frac{c_0}{t-x} + \sum_{j=1}^{J} \frac{c_j}{t-z_{1j}} + \sum_{m=1}^{m} \frac{d_m}{t-z_{2m}}\right] f(t)\,dt = F(x) \qquad (a<x<b), \tag{I.1}$$

where $\{c_j\}$ and $\{d_m\}$ are known constants, $F(x)$ satisfies a Hölder condition in $a \le x \le b$, and

$$z_{1j} = a + (x-a)k_j e^{i\theta_j}, \qquad z_{2m} = b + (b-x)h_m e^{i\omega_m}, \tag{I.2}$$

with $0<\theta_j<2\pi$ and $-\pi<\omega_m<\pi$. The quantities $\{k_j\}$ and $\{h_m\}$ are known positive constants; the special case $k_j = h_m = 1$ has been treated in [4]. Evidently the points $\{z_{1j}\}$ and $\{z_{2m}\}$ are not on the straight line segment $[a, b]$, but

$$\lim_{x\to a} z_{1j} = a, \qquad \lim_{x\to b} z_{2m} = b. \tag{I.3}$$

The more general cases of $k_j \neq 1$ and $h_m \neq 1$, not treated previously, are needed here in view of the form of (4.8).

The solution $f(x)$ of (I.1) is known to have singularities at the end points a and b for special cases already analyzed. This suggests that for the present more general case, we write

$$f(t) = \frac{q(t)}{(t-a)^{\alpha}(b-t)^{\beta}}, \tag{I.4}$$

where $q(t)$ satisfies a Hölder condition in $a \le t \le b$ and $0 < \operatorname{Re}\{\alpha, \beta\} < 1$. A sectionally holomorphic function may then be constructed by setting

$$\Phi(z) = \frac{1}{\pi}\int_a^b \frac{f(t)}{t-z}\,dt = \frac{1}{\pi}\int_a^b \frac{q(t)}{(t-z)(t-a)^{\alpha}(b-t)^{\beta}}\,dt, \tag{I.5}$$

which is analytic everywhere except at x on the line segment (a, b), where the two one-sided limits $\Phi^+(x)$ and $\Phi^-(x)$ as $z \to x$ are generally not equal. It has been proved in [20] that $\Phi(z)$ has an asymptotic expansion as z tends to an end point with the leading term given by

$$z \to a: \qquad \Phi(z) \sim \frac{q(a)e^{\pi i\alpha}}{(b-a)^{\beta}\sin\pi\alpha(z-a)^{\alpha}}, \tag{I.6}$$

$$z \to b: \qquad \Phi(z) \sim -\frac{q(b)}{(b-a)^{\alpha}\sin\pi\beta(z-b)^{\beta}}. \tag{I.7}$$

By the Plemelj formulas, the Cauchy integral in Equation (I.1) can be written as

$$\frac{1}{\pi}\int_a^b \frac{c_0 f(t)}{t-x}\,dt = c_0\Phi(x) = \tfrac{1}{2}c_0[\Phi^+(x)+\Phi^-(x)]. \tag{I.8}$$

As $x \to a$, we have

$$\begin{aligned}\tfrac{1}{2}c_0[\Phi^+(x)+\Phi^-(x)] &\sim \frac{c_0 q(a)e^{i\pi\alpha}}{2(b-a)^{\beta}\sin\pi\alpha}\left[\frac{1}{(z-a)_+^{\alpha}}+\frac{1}{(z-a)_-^{\alpha}}\right]\\ &= \frac{c_0 q(a)e^{i\pi\alpha}}{2(b-a)^{\beta}\sin\pi\alpha}\,\frac{1+e^{-2\pi i\alpha}}{(x-a)^{\alpha}} = \frac{c_0 q(a)\cot\pi\alpha}{(b-a)^{\beta}(x-a)^{\alpha}}.\end{aligned} \tag{I.9}$$

Similarly, we have as $x \to b$

$$\tfrac{1}{2}c_0[\Phi^+(x)+\Phi^-(x)] \sim -\frac{c_0 q(b)\cot\pi\beta}{(b-a)^{\alpha}(b-x)^{\beta}}. \tag{I.10}$$

The second type of integrals in (I.1) associated with constants $\{c_j\}$ are of the form (I.5). Since z_{1j} is not on the line (a, b), the corresponding $\Phi(z)$ has a leading term asymptotic expansion given by (I.6) as x tends to the end point a

(and hence z_{1j} also tends to a)

$$\frac{c_j}{\pi}\int_a^b \frac{f(t)}{t-z_{1j}}\,dt \sim \frac{c_j q(a) e^{i\pi\alpha} k_j^{-\alpha} e^{-i\alpha\theta_j}}{(b-a)^{\beta}(\sin\pi\alpha)(x-a)^{\alpha}}. \tag{I.11}$$

Note that the end point b is not a singular point for the kernel of this group of integrals.

Similarly, the third type of integrals in (I.1) associated with the constants $\{d_m\}$ are also of the form (I.5). As $x \to b$, we have

$$\frac{d_m}{\pi}\int_a^b \frac{f(t)}{t-z_{2m}}\,dt \sim -\frac{d_m q(b) h_m^{-\beta} e^{-i\beta\omega_m}}{(b-a)^{\alpha}(\sin\pi\beta)(b-x)^{\beta}}. \tag{I.12}$$

Now, with (I.9) and (I.11), the integral equation (I.1) requires

$$c_0\cos\pi\alpha + \sum_{j=1}^{J} c_j k_j^{-\alpha} e^{i(\pi-\theta_j)\alpha} = 0 \tag{I.13}$$

in order for the left hand side of (I.1) to remain nonsingular as $x \to a$. Similarly, with (I.10) and (I.12), the integral equation (I.1) requires

$$c_0\cos\pi\beta + \sum_{m=1}^{M} d_m h_m^{-\beta} e^{-i\beta\omega_m} = 0 \tag{I.14}$$

in order for the left hand side of (I.1) to remain nonsingular as $x \to b$. Equations (I.13) and (I.14) determine α and β, respectively. It suffices to limit the real part of α and β to be in (0,1).

Appendix II. Eigenfunction expansions and regularization

The solution for any one of the residual problems defined in Section 2 can also be solved by taking the stress function $\tilde{\phi}$ as a linear combination of a relevant set of eigenfunctions [9,17],

$$\tilde{\phi}(x,z) = \sum_{k=1}^{\infty}\left[a_k^S e^{\lambda_k^S x/\beta_0 h}\Phi_{Sk}(z) + a_k^B e^{\lambda_k^B x/\beta_0 h}\Phi_{Bk}(z)\right.$$

$$\equiv \sum_{n=1}^{\infty} A_n e^{\lambda_n x/\beta_0 h}\Phi_n(z). \tag{II.1}$$

It is known from [17] that the quantities $\{\lambda_k^S\}$ and $\{\Phi_{Sk}(z)\}$ in (II.1) are the eigenvalues and eigenfunctions of the eigenvalue problem

$$h^4\Phi'''' + (2+\delta_0)\lambda^2 h^2\Phi'' + \lambda^4\Phi = 0, \tag{II.2}$$

$$\Phi(\pm h) = \Phi'(\pm h) = 0 \tag{II.3}$$

associated with the boundary value problem for $\tilde{\phi}$ with the eigenvalues and eigenfunctions determined by

$$\cos\delta\lambda \sin\frac{\lambda}{\delta} - \delta^2 \sin\delta\lambda \cos\frac{\lambda}{\delta} = 0 \tag{II.4a}$$

and

$$\Phi(z) = \cos\frac{\lambda}{\delta}\cos\frac{\lambda\delta z}{h} - \cos\lambda\delta\cos\frac{\lambda z}{\delta h}, \tag{II.4b}$$

where

$$\delta^2 = 1 + \tfrac{1}{2}\delta_0 + \sqrt{(1+\tfrac{1}{2}\delta_0)^2 - 1}\,. \tag{II.5}$$

Correspondingly, $\{\lambda_k^B\}$ and $\{\Phi_k^B(z)\}$ are the eigenvalues and eigenfunctions of (II.2) and (II.3) determined by

$$\cos\delta\lambda \sin\frac{\lambda}{\delta} - \delta^{-2} \sin\delta\lambda \cos\frac{\lambda}{\delta} = 0 \tag{II.6a}$$

and

$$\Phi(z) = \sin\frac{\lambda}{\delta}\sin\frac{\lambda\delta z}{h} - \sin\lambda\delta\sin\frac{\lambda z}{\delta h}. \tag{II.6b}$$

The superscript S denotes quantities associated with the stretching action of the plates, while the superscript B is associated with the bending action of the plate.

The coefficients $\{a_k^S\}$ and $\{a_k^B\}$ are determined by the edge data. For this purpose we need the orthogonality relation

$$\int_{-h}^{h}\left\{h^4\Phi_m''\Phi_k'' - \lambda_k^2\lambda_m^2\Phi_m\Phi_k\right\}dz = 0 \qquad (k \neq m). \tag{II.7}$$

This relation is trivially satisfied if Φ_k and Φ_m are not from the same family $\{\Phi_k^S\}$ or $\{\Phi_k^B\}$, as the integrand will be an odd function of z in that case. When both Φ_k and Φ_m are from the same family, we consider the identity

$$\lambda_m^2\int_{-h}^{h}\left\{h^4\Phi_k'''' + (2+\delta_0)h^2\lambda_k^2\Phi_k'' + \lambda_k^4\Phi_k\right\}\Phi_m\,dz = 0. \tag{II.8}$$

By integration by parts and the boundary conditions (II.3), we may transform (II.8) into

$$\lambda_m^2 \int_{-h}^{h} \{ h^4 \Phi_m'' \Phi_k'' - (2+\delta_0) \lambda_k^2 h^2 \Phi_k' \Phi_m' + \lambda_k^4 \Phi_k \Phi_m \} \, dz = 0. \qquad \text{(II.9)}$$

A similar result can be obtained by reversing the role of m and k, giving

$$\lambda_k^2 \int_{-h}^{h} \{ h^4 \Phi_k'' \Phi_m'' - (2+\delta_0) \lambda_m^2 h^2 \Phi_m' \Phi_k' + \lambda_m^4 \Phi_m \Phi_k \} \, dz = 0. \qquad \text{(II.10)}$$

Upon subtracting (II.10) from (II.9), we get for $m \neq k$ the orthogonality relation (II.7).

To obtain the Fourier coefficients $\{a_k^S\}$ and $\{a_k^B\}$ (or $\{A_k\}$ after relabeling) in terms of the edge data, we limit ourselves to a discussion of the case of a semiinfinite strip, which is all that is needed for the present paper. In that case, the sum (II.1) is only over all eigenvalues of (II.4) and (II.6) with a negative real part.

The following identity for the Fourier coefficients follows from the orthogonality relation (II.7):

$$\int_{-h}^{h} \left\{ h^4 \left(\sum_{m=1}^{\infty} A_m \Phi_m'' \right) \Phi_k'' - \lambda_k^2 \left(\sum_{m=1}^{\infty} A_m \lambda_m^2 \Phi_m \right) \Phi_k \right\} dz$$

$$= \sum_{m=1}^{\infty} A_m \left\{ \int_{-h}^{h} \left[h^4 \Phi_m'' \Phi_k'' - \lambda_m^2 \lambda_k^2 \Phi_m \Phi_k \right] dz \right\} = c_k A_k, \qquad \text{(II.11)}$$

where

$$c_k = \int_{-h}^{h} \left[h^4 (\Phi_k'')^2 - \lambda_k^4 \Phi_k^2 \right] dz. \qquad \text{(II.12)}$$

But for a decaying (residual) solution, we have

$$\sum_{m=1}^{\infty} A_m \Phi_m'' = \tilde{\sigma}_{xx}|_{x=0} \equiv \tilde{\sigma}(z), \qquad \text{(II.13)}$$

and from the stress strain relation for ε_{zz}

$$\sum_{m=1}^{\infty} A_m \lambda_m^2 \Phi_m = \beta_0^2 h^2 \tilde{\sigma}_{zz}|_{x=0} = \beta_0^2 h^2 [E_2 \tilde{\varepsilon}_{zz} + \nu_{12} \tilde{\sigma}_{xx}]_{x=0}$$

$$= h^2 [E \tilde{w}'(z) + \nu \tilde{\sigma}(z)], \qquad \text{(II.14)}$$

where $\tilde{w} \equiv \tilde{u}_z(0, z)$ and $\tilde{\sigma} \equiv \tilde{\sigma}_{xx}(0, z)$. Upon substituting (II.13) and (II.14) into (II.11), we get

$$A_k = \frac{1}{c_k} \int_{-h}^{h} \{ h^4 \Phi_k'' \tilde{\sigma} - h^2 \lambda_k^2 \Phi_k [E \tilde{w} + \nu \tilde{\sigma}] \} \, dz$$

$$= \frac{h^2}{c_k} \int_{-h}^{h} \{ (h^2 \Phi_k'' - \nu \lambda_k^2 \Phi_k) \tilde{\sigma} - \lambda_k^2 \Phi_k E \tilde{w}' \} \, dz. \tag{II.15}$$

From the stress-strain relation for ε_{xx}, we have for a residual solution

$$\tilde{u}_x = \sum_{k=1}^{\infty} A_k e^{\lambda_k x/\beta_0 h} U_k(z), \qquad U_k(z) = \frac{h^2 \Phi_k''(z) - \nu \lambda_k^2 \Phi_k(z)}{E \lambda_k h \beta_0}. \tag{II.16}$$

Upon substituting (II.15) into (II.16) and setting $x = 0$, we get

$$\tilde{u}(z) = \int_{-h}^{h} K_2(z, y) \tilde{\sigma}(y) \, dy - \int_{-h}^{h} K_1(z, y) \tilde{w}'(y) \, dy \tag{II.17}$$

with

$$K_2(z, y) = \sum_{k=1}^{\infty} \frac{h^2}{c_k} [h^2 \Phi_k''(y) - \nu \lambda_k^2 \Phi_k(y)] U_k(z), \tag{II.18}$$

$$K_1(z, y) = \sum_{k=1}^{\infty} \frac{h^2}{c_k} E \lambda_k^2 \Phi_k(z) U_k(z). \tag{II.19}$$

For problems with prescribed edge displacements, $\tilde{w}(z) = \tilde{u}_z(0, z)$ and $\tilde{u}(z) = \tilde{u}_x(0, z)$ are known,[1] so that (II.17) is a Fredholm integral equation of the first kind for $\tilde{\sigma}(z)$. We have not been able to transform (II.17) into a singular integral equation as in the Fourier transform method of Section 4. An approximate solution was sought by truncating the kernel series (II.18) and (II.19) after a finite number of terms. The resulting Fredholm integral equation of the first kind is ill-posed [25]. In principle, it is possible to use the technique of regularization developed in [21, 24] to reduce the ill-posed problem to solving a stable minimization problem. In this approach, there is a parameter $\gamma(\delta)$ in the problem which is a function of the error level δ (not to be confused with the δ previously introduced in (3.11)) between the exact and the computed right hand member of the integral equation. The regularization technique chooses a proper γ by the so-called "discrepancy principle" [13, 19].

In our problem, the error level δ is associated with the remainder of the kernel series and is not known; the discrepancy principle therefore cannot be actually

[1]As seen in (2.12), the residual displacement edge data may involve constants of rigid body motion. Necessary conditions for decaying states developed in [15] should be used.

applied. Our solution process chooses δ to smooth the solution from oscillations. Evidently, the error level cannot be zero for any given approximating kernel; nor can it be too large if a large number of terms in the series of the kernel is kept. It is not surprising that there is an optimal value for δ which gives the best approximate solution for a given truncated kernel. This optimal δ is found by numerical experiments. While we obtained good agreements between the solution by the methods of this section and the one determined by the method in Sections 3 and 4, the present eigenfunction expansion method is inefficient and more costly than the Fourier transform–singular integral equation formulation. The former does serve as another validation of the latter.

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