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On shells of revolution with the Love–Kirchhoff hypotheses*

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Dedication

With fondness and appreciation, the authors dedicate this article to their teacher, collaborator and friend, Professor Eric Reissner, in the year of his seventy-fifth anniversary.

Abstract. On the occasion of the 100th anniversary of A.E.H. Love's fundamental paper on thin elastic shell theory, the present article summarizes a line of developments on shells of revolution related to the Love–Kirchhoff hypotheses which form the basis of Love's theory. The summary begins with the Günther–Reissner formulation of the linear theory which is shown to contain the classical first approximation shell theory as a special case. The static-geometric duality is deduced as a natural and immediate consequence of the more general theory. The repeated applications of this duality greatly simplify the solution process for boundary-value problems in shell theory, including the classical reduction of the axisymmetric bending problem and related recent reductions of shell equations for more general loadings to two simultaneous equations for a stress function and a displacement variable. In the nonlinear range, the article confines itself to Reissner's geometrically nonlinear theory of axisymmetric deformation of shells of revolution and Marguerre's shallow shell theory with special emphasis on recent results for elastic membranes, buckling of shells of revolution and applications of asymptotic methods.

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1. Introduction

A.E.H. Love's article "On the small free vibrations and deformations of thin elastic shells", published in the Philosophical Transactions of the Royal Society (London) one hundred years ago, is generally considered to be the first paper containing a complete and general linear theory of thin elastic shells [75]. However, two important earlier publications should be noted. In 1874, H. Aron [1] presented a set of equations for the bending of thin shells, generalizing a method of Kirchhoff and Clebsch for deriving equations for small strain but finite displacements of thin rods and plates from the general equations of elasticity. But Aron's development contained some errors that were corrected in the paper of Love. In 1881, Lord Rayleigh [86] had proposed a theory for the vibration of shells assuming the midsurface to remain unstretched (see also [68]).

Love derived a set of linear equations of motion and boundary conditions for shells undergoing *both* infinitesimal extensional and bending strains from three-dimensional elasticity theory by adopting Kirchhoff's assumptions for thin plate theory [56] and adding to them the *thin shell approximation* (see also [9]).This additional approximation judiciously neglects terms of the order of the thickness-to-radius of curvature ratio compared to unity. The resulting linear shell theory takes the form of a set of linear partial differential equations (PDE) in two middle surface coordinates; it has proved to be very useful in engineering applications [76].

Love's pioneering work stimulated considerable research effort on shell theory in several directions during the ensuing century. One natural direction has been concerned with the solution techniques for the relevant initial-boundary value problems in specific applications of the theory. Others include the derivation of thin and thick shell theories from three-dimensional elasticity theory without the Kirchhoff–Love hypotheses by asymptotic or iterative methods, and the formulation of nonlinear shell theories and their applications to finite deformation problems. In addition, there is the fundamental and difficult problem of justifying the adequacy and accuracy of shell theory solutions as approximations of the exact elasticity solutions, at least away from the edge of the shell. Most progress in these directions has been made for shells of revolution which are widely used in engineering designs. In this article, the writers summarize some of the developments for this important class of shells since Love's 1888 paper which fall within their own research interests.

With the benefit of hindsight, we now know that the developments of solution techniques for linear problems are greatly simplified by a certain symmetry inherent in the structure of the first approximation linear shell theory known as the *static-geometric duality*. The duality is an extension of the static-geometric analogy first observed by A.L. Goldenveizer [29, 31]. The analogy was simplified and made more complete by W. Günther [44] who followed the work of H. Schaefer [107] for cylindrical shells and formulated a general two-dimensional shell theory without any reference to three-dimensional elasticity theory. For the static-geometric duality, the critical element of this formulation is the introduction of a normal component in all vector stress and strain measures for the shell. The normal components of the stress couple vectors, not present in the classical shell theory, were eventually related to certain moment stresses of three-dimensional Cosserat media [17] by E. Reissner [94, 96, 97]. Given the importance of the static-geometric duality in linear shell theory, the Günther–Reissner shell theory will be outlined in Section 2. Its relation to three-dimensional elasticity theory will be discussed in Section 3, where the device of a transversely isotropic shell medium, first introduced in [47], will be used to circumvent the necessity of the Kirchhoff type assumptions. The static-geometric duality will be described in Section 4. Its applications in the solution techniques for the linear theory of shells of revolution will be found in Section 5. The principal feature of the solution process for this important class of shell problems is the reduction of the relevant system of shell equations to two simultaneous differential equations for two unknowns with the two equations being the static-geometric duals of each other. A further reduction to a single complex equation (of the same order as one of the original equations) is possible for specific shells and an exact solution in terms of elementary or special functions is immediate for some cases. In other cases, standard asymptotic methods can be applied if the shell is sufficiently thin.

Except for Marguerre's shallow shell theory [79], successful formulations of general nonlinear shell theories emerged only in the last forty years. In this article, we limit ourselves to a review in Section 6 of the main results of the vast literature on the axisymmetric finite deformation (but infinitesimal strain) theory of shells of revolution. While questions of existence and uniqueness of the solution for the linear theory can be answered by standard methods, the situation is quite different for the nonlinear theory even with the simplifications by axisymmetry. Thus far, only some specific classes of problems have yielded to a rigorous mathematical analysis. In Section 7, we summarize some known rigorous results in this area for circular membranes and plates. One of the most important applications of nonlinear shell theories is to the interesting nonlinear phenomenon of shell buckling. Some major (and by now) classical results in this area for isotropic and homogeneous shells will be described in Section 8. Finally, asymptotic methods for nonlinear shell problems will be briefly discussed in Section 9.

Only references relevant to the specific content of our sections are included in the bibliography at the end of the article. The choice of the topics discussed in these sections reflects the interests of the writers and not necessarily the importance of the topics chosen.

2. Two-dimensional formulation of the linear theory

2.1. Stress resultants and couples

For the purpose of analysis, a shell is modeled by a surface in space surrounded by an elastic substance which extends a distance $h/2$ in both directions normal to the surface. The thickness of the shell h may vary from point to point on the middle surface. The surface itself will be defined parametrically by a position vector $\mathbf{r}(\xi_1, \xi_2)$ which is a vector function of two curvilinear surface coordinates ξ_1 and ξ_2. With $(\cdot)_{,k} = \partial(\cdot)/\partial\xi_k$ and $\alpha_k = |\mathbf{r}_{,k}|$, the unit vectors $\mathbf{t}_k = \mathbf{r}_{,k}/\alpha_k$, $k = 1$ and 2, are *base vectors* tangent to the coordinate curves of the middle surface. For this article, the surface coordinates will be taken to be *orthogonal* so that $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$. In that case, the positive unit *normal* of the middle surface is given by $\mathbf{n} = \mathbf{t}_1 \times \mathbf{t}_2$.

A shell is physically a three-dimensional body. However, certain aspects of an approximate two-dimensional theory governing the mechanical behavior of such a body can be developed directly without any reference to the three-dimensional theory of elasticity. The relationship between the two-dimensional formulation and elasticity theory will be analyzed later. To develop a two-dimensional theory for shell structures, we will think of a shell as a surface (usually taken to be the middle surface of the shell structure) endowed with certain mechanical properties to be described below.

Let C be a directed curve on the shell surface. At a point P on C, we have a triad of unit vectors $\mathbf{v}$, $\mathbf{t}$, and $\mathbf{n}$ where $\mathbf{t}$ is tangent to C (positive in the direction of C), $\mathbf{n}$ is the unit normal to the shell surface and $\mathbf{v} = \mathbf{t} \times \mathbf{n}$. Let Δs be the arclength of an elemental arc of C with P as its centroid. Now, the elemental arc (and C itself) fictitiously cuts the surface into two parts. For the part with $\mathbf{v}$ pointing away from the surface, we associate with P a *stress resultant* vector $\mathbf{N}_\nu$ and a *stress couple* vector $\mathbf{M}_\nu$ with $\mathbf{N}_\nu \Delta s$ and $\mathbf{M}_\nu \Delta s$ being the (edge) force and (edge) moment, respectively, acting on the arc. On the other portion of the (cut) surface, we have correspondingly, $\mathbf{N}_{-\nu}$ and $\mathbf{M}_{-\nu}$. The principle of action and reaction requires that $\mathbf{N}_{-\nu} = -\mathbf{N}_\nu$ and $\mathbf{M}_{-\nu} = -\mathbf{M}_\nu$.

In terms of $\mathbf{N}_\nu$ and $\mathbf{M}_\nu$, we have the following fundamental postulate for our shell theory: *On any imaginary closed curve C in the interior of the shell surface, two vector fields $\mathbf{N}_\nu$ and $\mathbf{M}_\nu$ can be defined in such a way that their actions on the material occupying the surface area interior to C are equivalent to the action of the exterior material on the material inside C.*

We denote by C_i the coordinate curves ξ_i = constant and by $\mathbf{N}_i$ and $\mathbf{M}_i$ the corresponding stress resultant and couple, respectively. Equilibrium considerations give the Cauchy type formulas for $\mathbf{N}_\nu$ and $\mathbf{M}_\nu$: $\mathbf{N}_\nu = \nu_1\mathbf{N}_1 + \nu_2\mathbf{N}_2$ and $\mathbf{M}_\nu = \nu_1\mathbf{M}_1 + \nu_2\mathbf{M}_2$ where ν_1 and ν_2 are the directional cosines, $\nu_i = \boldsymbol{\nu}\cdot\mathbf{t}_i$. Let ΔS be an element of the shell surface with the point P as its centroid. A *surface force intensity* vector $\mathbf{p}$ and a *surface moment intensity* vector $\mathbf{q}$ are defined at P with $\mathbf{p}\Delta S$ and $\mathbf{q}\Delta S$ being the (surface) force and (surface) moment acting on the elemental surface. The two vector fields correspond to (external) *surface loads* and are usually known functions of ξ_1 and ξ_2.

2.2. *Equilibrium*

For our shell surface to be in static equilibrium, the resultant force and resultant moment for the entire surface as well as any of its (fictitiously) isolated parts must vanish so that

$$\int_C \mathbf{N}_\nu\,\mathrm{d}s + \iint_S \mathbf{p}\,\mathrm{d}S = \mathbf{0}, \quad \int_C (\mathbf{M}_\nu + \mathbf{r}\times\mathbf{N}_\nu)\,\mathrm{d}s + \iint_S (\mathbf{q} + \mathbf{r}\times\mathbf{p})\,\mathrm{d}S = \mathbf{0}. \quad (2.1,2)$$

In (2.1) and (2.2) S is an arbitrary portion or the whole of the shell surface with $\mathrm{d}S = \alpha_1\alpha_2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2$ for orthogonal surface coordinates, C is the boundary of S, and, in terms of an edge variable ξ_s, $\mathrm{d}s = \alpha_s\,\mathrm{d}\xi_s$ is an elemental arclength.

By our Cauchy type formulas for $\mathbf{N}_\nu$ and $\mathbf{M}_\nu$ and Green's theorem, we get from (2.1) and (2.2) the following two differential equations of equilibrium in orthogonal surface coordinates:

$$(\alpha_2\mathbf{N}_1)_{,1} + (\alpha_1\mathbf{N}_2)_{,2} + \alpha_1\alpha_2\mathbf{p} = \mathbf{0}, \tag{2.3}$$

$$(\alpha_2\mathbf{M}_1)_{,1} + (\alpha_1\mathbf{M}_2)_{,2} + \mathbf{r}_{,1}\times(\alpha_2\mathbf{N}_1) + \mathbf{r}_{,2}\times(\alpha_1\mathbf{N}_2) + \alpha_1\alpha_2\mathbf{q} = \mathbf{0}. \tag{2.4}$$

Six scalar equilibrium equations can be obtained from (2.3) and (2.4) with the help of the Gauss–Weingarten differentiation formulas in the theory of surfaces.

2.3. *Strain displacement relations*

The deformation of the shell is characterized by a translational displacement vector $\mathbf{u}$ and a rotational displacement vector $\boldsymbol{\phi}$. *Strain resultant* vectors $\boldsymbol{\varepsilon}_j$ and *strain couple* vectors $\boldsymbol{\kappa}_j$, $j = 1$ and 2, are introduced by way of the virtual work expression

$$\iint_S (\mathbf{N}_1\cdot\delta\boldsymbol{\varepsilon}_1 + \mathbf{N}_2\cdot\delta\boldsymbol{\varepsilon}_2 + \mathbf{M}_1\cdot\delta\boldsymbol{\kappa}_1 + \mathbf{M}_2\cdot\delta\boldsymbol{\kappa}_2)\,\mathrm{d}S$$

$$= \iint_S (\mathbf{p}\cdot\delta\mathbf{u} + \mathbf{q}\cdot\delta\boldsymbol{\phi})\,\mathrm{d}S + \int_C (\mathbf{N}_\nu^*\cdot\delta\mathbf{u} + \mathbf{M}_\nu^*\cdot\delta\boldsymbol{\phi})\,\mathrm{d}s$$

where an asterisk indicates a prescribed quantity, which may vary along the edge C. With equilibrium equations (2.3) and (2.4) and Green's theorem, this virtual work expression is an identity if the strain measures are related to the displacement vectors by

$$\alpha_j\boldsymbol{\varepsilon}_j = \mathbf{u}_{,j} + \mathbf{r}_{,j}\times\boldsymbol{\phi}, \quad \alpha_j\boldsymbol{\kappa}_j = \boldsymbol{\phi}_{,j}, \quad j = 1, 2. \tag{2.5}$$

We take (2.5) as the defining expressions for the strain resultant vectors $\boldsymbol{\varepsilon}_j$ and strain couple vectors $\boldsymbol{\kappa}_j$ in our theory.

2.4. Component representation

For the relations between the stress and strain measures, we need component representations of all stress and strain vectors. To be consistent with existing literature, we take these to be

$$\begin{aligned} \mathbf{N}_k &= N_{k1}\mathbf{t}_1 + \mathbf{N}_{k2}\mathbf{t}_2 + Q_k\mathbf{n}, \quad \mathbf{M}_k = \mathbf{n} \times (M_{k1}\mathbf{t}_1 + M_{k2}\mathbf{t}_2) + P_k\mathbf{n}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (k = 1, 2) \quad (2.6) \\ \boldsymbol{\varepsilon}_k &= \varepsilon_{k1}\mathbf{t}_1 + \varepsilon_{k2}\mathbf{t}_2 + \gamma_k\mathbf{n}, \quad \boldsymbol{\kappa}_k = \mathbf{n} \times (\kappa_{k1}\mathbf{t}_1 + \kappa_{k2}\mathbf{t}_2) + \lambda_k\mathbf{n}. \end{aligned}$$

The physical interpretation of the scalar stress resultants and couples is directly obtained from the definition of these quantities. It should be noted that the *drilling couples* P_1 and P_2 do not appear in any known shell theory derived from classical three-dimensional elasticity theory. We follow the approach of Günther [44] and E. Reissner [94, 96, 97] and regard the vanishing of the drilling couples to be a property of the shell material and therefore a consequence of specific stress strain relations.

The geometrical interpretation of the strain resultants and couples will be made when we establish a connection between the two-dimensional developments with three-dimensional elasticity theory in a later section. But by their association with the resultants Q_1 and Q_2 in the virtual work expression, γ_1 and γ_2 are evidently the *transverse shear strain* components. These two strain components are assumed to be negligibly small and are made to vanish in any shell theory deduced from elasticity theory by the Love–Kirchhoff hypotheses. The Günther–Reissner theory allows the stress strain relations to specify the proclivity of the shell for transverse shear deformation including the special case of its complete absence. The transverse strain couples λ_1 and λ_2 also do not appear explicitly in conventional shell theories; it will be seen later that they play the role of reactive quantities in these theories and are of no particular interest in practical application. However, their explicit appearance in the Günther–Reissner theory significantly simplifies the establishment of an important symmetry inherent in the structure of the shell equations in Section 4.

The following component representations of the displacement and load vectors will also be needed for the solution of specific problems:

$$\begin{aligned} \mathbf{u} &= u_1\mathbf{t}_1 + u_2\mathbf{t}_2 + w\mathbf{n}, \quad \boldsymbol{\phi} = \mathbf{n} \times (\phi_1\mathbf{t}_1 + \phi_2\mathbf{t}_2) + \omega\mathbf{n}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.7) \\ \mathbf{p} &= p_1\mathbf{t}_1 + p_2\mathbf{t}_2 + p_n\mathbf{n}, \quad \mathbf{q} = \mathbf{n} \times (q_1\mathbf{t}_1 + q_2\mathbf{t}_2) + q_n\mathbf{n}. \end{aligned}$$

2.5. Linear elastic stress strain relations

For a linear shell theory, the elasticity of the shell is given by twelve linear invertible stress strain relations. We will be interested in isotropic shells whose material properties have no directional preference in the tangent plane of the shell surface. In this article, we will be concerned mainly with developments related to Love's first approximation theory and will therefore work with a class of linearly elastic isotropic shells with stress strain relations of the form

$$\varepsilon_{11} = A(N_{11} - \nu_s N_{22}), \quad \varepsilon_{22} = \cdots, \quad \gamma_j = BQ_j \quad (j = 1, 2), \tag{2.8}$$

$$\varepsilon_{12} = A(1 + \nu_s)[(1 - \varrho_s)N_{12} + \varrho_s N_{21}], \quad \varepsilon_{21} = A(1 + \nu_s)[\varrho_s N_{12} + (1 - \varrho_s)N_{21}],$$

$$M_{22} = D(\kappa_{22} + \nu_b \kappa_{11}), \quad M_{11} = \cdots, \quad P_j = C\lambda_j \quad (j = 1, 2), \tag{2.9}$$

$$M_{21} = D(1 - \nu_b)[(1 + \varrho_b)\kappa_{12} - \varrho_b \kappa_{21}], \quad M_{12} = D(1 - \nu_b)[-\varrho_b \kappa_{21} + (1 + \varrho_b)\kappa_{12}].$$

The independent elastic parameters A, D, B, C, ν_s, ν_b, ϱ_s and ϱ_b are to be determined by experiments or related to the elastic moduli of three-dimensional elasticity. The shell is said to be *nonuniform* or *inhomogeneous* if these parameters vary with position on the shell surface. For ϱ_s not equal to 1/2, the relations (2.8) can be inverted to give the stress resultants in terms of the strain resultants, consistent with the form of relations (2.9). Similarly, the relations (2.9) can be inverted to give the strain couples in terms of the stress couples if $-\varrho_b$ is not equal to 1/2.

2.6. *Variational principles for the displacement method*

With the stress measures expressed in terms of the components of $\mathbf{u}$ and $\boldsymbol{\phi}$ by way of the stress strain relations, the six scalar equilibrium equations can be written as six second-order PDE for the six unknown displacement components. Evidently, six independent boundary conditions should be prescribed at each edge of the shell to completely specify the solution. With the notation $\Delta\mathbf{f} = \mathbf{f}^* - \mathbf{f}$, these six conditions may be given in terms of the stress measures in the form $\Delta\mathbf{N}_\nu = \mathbf{0}$ and $\Delta\mathbf{M}_\nu = \mathbf{0}$ which merely re-state the requirement of force and moment equilibrium at the edge. They may also be given in terms of displacement measures in the form $\Delta\mathbf{u} = \mathbf{0}$ and $\Delta\boldsymbol{\phi} = \mathbf{0}$. Other mixed conditions are also possible but will not be discussed here.

For further theoretical development and approximate solutions, it is of interest to obtain an equivalent variational formulation of the boundary value problems of our linear shell theory. For the purpose of establishing variational theorems, we note that the stress strain relations (2.9) and the inverted form of (2.8) can be expressed in terms of a scalar *stress potential* (or *strain energy density function*) $S(\varepsilon_{11}, \varepsilon_{12}, \ldots, \lambda_2)$ by the relations $N_{ij} = \partial S/\partial \varepsilon_{ij}$, $Q_j = \partial S/\partial \gamma_j$, $M_{ij} = \partial S/\partial \kappa_{ij}$ and $P_j = \partial S/\partial \lambda_j$ where

$$S = \frac{D}{2}[(\kappa_{11} + \kappa_{22})^2 - 2(1 - \nu_b)(\kappa_{11}\kappa_{22} - \kappa_{21}\kappa_{12})$$

$$+ (1 + \varrho_b)(1 - \nu_b)(\kappa_{12} - \kappa_{21})^2] + \frac{C}{2}(\lambda_1^2 + \lambda_2^2) + \frac{1}{2B}(\gamma_1^2 + \gamma_2^2) + \frac{1}{2A_0}$$

$$\times \left[(\varepsilon_{11} + \varepsilon_{22})^2 - 2(1 - \nu_s)(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}\varepsilon_{21}) + \frac{1 - \varrho_s}{1 - 2\varrho_s}(1 - \nu_s)(\varepsilon_{12} - \varepsilon_{21})^2\right] \tag{2.10}$$

with $A_0 = A(1 - \nu_s^2)$. The scalar relations in terms of S can be written more compactly as four vector equations:

$$\mathbf{N}_j = \frac{\partial S}{\partial \boldsymbol{\varepsilon}_j} \quad \text{and} \quad \mathbf{M}_j = \frac{\partial S}{\partial \boldsymbol{\kappa}_j} \quad (j = 1, 2) \tag{2.11}$$

where the vector derivatives are gradients in strain space:

$$\frac{\partial S}{\partial \boldsymbol{\varepsilon}_j} := \frac{\partial S}{\partial \varepsilon_{j1}} \mathbf{t}_1 + \frac{\partial S}{\partial \varepsilon_{j2}} \mathbf{t}_2 + \frac{\partial S}{\partial \gamma_j} \mathbf{n}, \quad \frac{\partial S}{\partial \boldsymbol{\kappa}_j} := \mathbf{n} \times \left(\frac{\partial S}{\partial \kappa_{j1}} \mathbf{t}_1 + \frac{\partial S}{\partial \kappa_{j2}} \mathbf{t}_2 \right) + \frac{\partial S}{\partial \lambda_j} \mathbf{n}. \tag{2.12}$$

For some variational principles, we need to have the strain measures in terms of the stress measures by way of a *strain potential* (or *complementary energy density function*) $\bar{S}(N_{11}, N_{12}, \ldots, P_2)$ in the form

$$\boldsymbol{\varepsilon}_j = \frac{\partial \bar{S}}{\partial \mathbf{N}_j}, \quad \boldsymbol{\kappa}_j = \frac{\partial \bar{S}}{\partial \mathbf{M}_j} \quad (j = 1, 2) \tag{2.13}$$

with the vector derivatives interpreted as gradient operators in the appropriate stress space. Given the stress potential for a set of (invertible) linear stress strain relations for a particular shell medium, it is not difficult to construct $\bar{S}$.

Variational principles for displacements, for stresses and displacements, and for stresses, strains and displacements have been established by E. Reissner [92] with the stipulation $P_1 \equiv P_2 \equiv 0$ ab initio. The extension of these results to allow for the presence of the drilling couples is straightforward. One important application of such variational principles is the derivation of appropriate reduced or contracted boundary conditions for lower order shell theories associated with special types of shell materials exhibiting limiting behaviors such as momentless behavior and inextensional bending behavior.

3. Shell theory and three-dimensional elasticity

3.1. Elasticity theory in shell coordinates

We now relate the strictly two-dimensional considerations of the last section to the three-dimensional elasticity theory to obtain expressions for the elastic parameters in (2.8) and (2.9) in terms of measurable quantities. To simplify the process, we take ξ_1 and ξ_2 to be *lines of curvature* (l.o.c.) coordinates of the shell (middle) surface. A point in the shell body may then be described by the position vector $\mathbf{x}(\xi_1, \xi_2, \zeta) = \mathbf{r}(\xi_1, \xi_2) + \zeta\mathbf{n}(\xi_1, \xi_2)$ where $\mathbf{n}$ is the unit normal at the point $\mathbf{r}(\xi_1, \xi_2)$ of the middle surface ($\zeta = 0$). The unit vectors $\mathbf{T}_i = \mathbf{x}_{,i}/\bar{\alpha}_i$ with $\bar{\alpha}_i = \alpha_i(1 + \zeta/R_i)$, for $i = 1, 2$, are tangent to the surface coordinate lines $\xi_k =$ constant, $\zeta =$ constant, $k \neq i$ respectively. Note that $R_1 \equiv R_{11}$ and $R_2 \equiv R_{22}$ are the two principal radii of curvature of the surface. The positive direction of the unit normal at $\mathbf{x}$ is defined by the convention $\mathbf{N} = \mathbf{T}_1 \times \mathbf{T}_2$.

For the orthogonal curvilinear space coordinates ξ_1, ξ_2 and $\xi_3 = \zeta$, the linear strain components e_{mn}, $m, n = 1, 2, 3$ of the three-dimensional linear elasticity are given in terms of the displacement components v_m, $m = 1, 2, 3$, by six linear strain displacement relations:

$$\left(1 + \frac{\zeta}{R_1}\right) e_{11} = \frac{v_{1,1}}{\alpha_1} + \frac{\alpha_{1,2} v_2}{\alpha_1 \alpha_2} + \frac{v_3}{R_1}, \quad \left(1 + \frac{\zeta}{R_2}\right) e_{22} = \cdots, \quad e_{33} = v_{3,3},$$

$$e_{12} = e_{21} = \frac{1}{1 + \zeta/R_1}\left[\frac{v_{2,1}}{\alpha_1} - \frac{\alpha_{1,2} v_1}{\alpha_1 \alpha_2}\right] + \frac{1}{1 + \zeta/R_2}\left[\frac{v_{1,2}}{\alpha_2} - \frac{\alpha_{2,1} v_2}{\alpha_1 \alpha_2}\right], \tag{3.1}$$

$$e_{13} = e_{31} = \frac{1}{1 + \zeta/R_1}\left[\frac{v_{3,1}}{\alpha_1} - \frac{v_1}{R_1}\right] + v_{1,3}, \quad e_{23} = e_{32} = \cdots$$

where $(\cdot)_{,m} = \partial(\cdot)/\partial\xi_m$. The three linear differential equations of equilibrium for the stress components σ_{mn}, $m, n = 1, 2, 3$, take the form

$$(\bar{\alpha}_2 \sigma_{11})_{,1} + (\bar{\alpha}_1 \sigma_{21})_{,2} + \left(1 + \frac{\zeta}{R_2}\right) \alpha_{1,2} \sigma_{12} - \left(1 + \frac{\zeta}{R_1}\right) \alpha_{2,1} \sigma_{22}$$

$$+ \frac{\alpha_1 \bar{\alpha}_2}{R_1} \sigma_{13} + (\bar{\alpha}_1 \bar{\alpha}_2 \sigma_{31})_{,3} + \bar{\alpha}_1 \bar{\alpha}_2 f_1 = 0, \tag{3.2}$$

$$\cdots\cdots$$

$$(\bar{\alpha}_2 \sigma_{13})_{,1} + (\bar{\alpha}_1 \sigma_{23})_{,2} - \frac{\alpha_1 \bar{\alpha}_2}{R_1} \sigma_{11} - \frac{\bar{\alpha}_1 \alpha_2}{R_2} \sigma_{22} + (\bar{\alpha}_1 \bar{\alpha}_2 \sigma_{33})_{,3} + \bar{\alpha}_1 \bar{\alpha}_2 f_3 = 0,$$

where f_m, $m = 1, 2, 3$, are the components of the distributed body force intensity vector. Moment equilibrium in the three mutually orthogonal directions requires the symmetry conditions $\sigma_{mn} - \sigma_{nm} = 0$, $m \neq n$.

3.2. *Stress results and couples*

When the drilling couples P_1 and P_2 are absent, five of the six scalar equilibrium equations obtained from the vector equilibrium equations (2.3) and (2.4), specialized to l.o.c. coordinates, may be identified with appropriate (weighted) integrals of (3.2) across the shell thickness if we set for $i \neq k$

$$\{N_{ij}, M_{ij}\} = \int_{-h/2}^{h/2} \sigma_{ij}\left(1 + \frac{\zeta}{R_k}\right)\{1, \zeta\}\, d\zeta, \quad Q_i = \int_{-h/2}^{h/2}\left(1 + \frac{\zeta}{R_k}\right)\sigma_{i3}\, d\zeta \tag{3.3}$$

where subscripts i, j and k take only the values 1 and 2 in this paper. N_{ij} and M_{ij} are generally not symmetric in subscripts even though σ_{ij} is symmetric.

For the remaining (sixth) equilibrium equation from (2.4), we integrate the identity $(\sigma_{12} - \sigma_{21})(1 + \zeta/R_1)(1 + \zeta/R_2) = 0$ across the thickness to get

$$N_{12} - N_{21} + \frac{M_{21}}{R_2} - \frac{M_{21}}{R_1} = 0. \tag{3.4}$$

The algebraic equation (3.4) is identical to the *sixth* equilibrium equation (in l.o.c. coordinates), again without the drilling couple terms and with $q_n = 0$.

It has been shown [94, 96, 97] that the drilling couples in (2.6) are induced by moment stresses in a Cosserat type of three-dimensional elasticity theory [17]. Moreover, the six scalar two-dimensional equations associated with (2.3) and (2.4) are in fact exact consequences of the equilibrium equations in this theory. The drilling couple terms are therefore expected to be absent in any shell theory derived from classical elasticity theory.

3.3. Transversely rigid elastic shells

Approximations of the three-dimensional elasticity theory for shell structures occur mainly in the stress strain relations. In 1888, A.E.H. Love proposed an approximate theory for shells based effectively on the following simplifying assumptions: (1) *Normals to the undeformed middle surface remain normal to the deformed middle surface without change in length*, (2) *Transverse normal stress may be neglected in the in-plane stress strain relations*, (3) *The shell is thin so that ζ/R_j terms may be omitted compared to unity*. The first two assumptions are identical to those used by Kirchhoff for his thin plate theory [56]. The third is called *the thin shell approximation.*

Love's first assumption requires the deformed shell to be in a state of plane strain so that $e_{3m} = 0$. It follows immediately from (3.1) that $v_3 = W(\xi_1, \xi_2)$ and $v_k = V_k(\xi_1, \xi_2) + \zeta\Psi_k(\xi_1, \xi_2)$, $k = 1, 2$, with $\Gamma_k \equiv \Psi_k - (V_k/R_k) + (W_{,k}/\alpha_k) = 0$. The remaining three non-vanishing strain components become

$$\left(1 + \frac{\zeta}{R_k}\right) e_{kk} =: \bar{\varepsilon}_{kk} + \zeta\bar{\kappa}_{kk}, \quad e_{12} = e_{21} =: \frac{\bar{\varepsilon}_{12} + \zeta\bar{\kappa}_{12}}{1 + \zeta/R_1} + \frac{\bar{\varepsilon}_{21} + \zeta\bar{\kappa}_{21}}{1 + \zeta/R_2} \tag{3.5}$$

where the expressions for $\bar{\varepsilon}_{ij}$ and $\bar{\kappa}_{ij}$ in terms of V_k and W are identical to the expressions for the corresponding scalar strain resultants and couples obtained from (2.5) (in l.o.c. coordinates) with terms involving ω omitted if we identify W, V_k, Ψ_k with w, u_k, ϕ_k, respectively, and take $\gamma_j = 0$.

For an isotropic shell, the conditions $e_{13} = e_{23} = 0$ require the transverse shear stress components (and hence Q_j) to vanish. But for problems involving transverse bending action, we need nonvanishing transverse shear resultants Q_j to maintain equilibrium. This inconsistency can be circumvented by considering a transversely isotropic shell with the following orthotropic stress strain relations [47]:

$$\begin{aligned} e_{11} &= \frac{1}{E}(\sigma_{11} - \nu\sigma_{22} - \nu_3\sigma_{33}), \quad e_{22} = \cdots, \quad e_{12} = \frac{1}{G}\sigma_{12}, \\ e_{33} &= \frac{1}{E_3}\sigma_{33} - \frac{\nu_3}{E}(\sigma_{11} + \sigma_{22}), \quad e_{j3} = \frac{1}{G_3}\sigma_{j3}, \quad (j = 1, 2) \end{aligned} \tag{3.6}$$

where $G = E/2(1 + \nu)$. Love's first assumption is formally equivalent to taking the shell material to be *transversely rigid* so that $1/G_3 = 1/E_3 = \nu_3 = 0$. In that case, e_{13} and e_{23} vanish without any implication on the magnitude of σ_{13} and σ_{23}. With $\nu_3 = 0$ in the expressions for e_{11} and e_{22} of (3.6) the plane stress requirement of the second assumption is satisfied. Though usually made in practice, this assumption is not critical to the formulation of a shell theory.

3.4. The Flügge–Lurje–Byrne stress strain relations

To get a set of stress strain relations for the basic field variables of a shell theory, N_{ij}, Q_j, and M_{ij}, we need the stress components σ_{ij} in terms of e_{ij}. The dependence of e_{ij} on ζ is known explicitly so that the integration with respect to ζ in the definitions of stress resultants and couples in (3.3) can be executed. We get σ_{ij} in terms of the in-plane strain components by inverting the relations (3.6). We use them to form expressions for N_{ij} and M_{ij} as defined in (3.3) and get for transversely rigid shells with E and ν independent of the thickness coordinate ζ:

$$N_{11} = C_{11}\bar{\varepsilon}_{11} + C_{12}\bar{\varepsilon}_{22} + B_{11}\bar{\kappa}_{11}, \quad M_{11} = B_{11}\bar{\varepsilon}_{11} + D_{11}\bar{\kappa}_{11} + D_{12}\bar{\kappa}_{22},$$

$$N_{12} = C_{s1}\bar{\varepsilon}_{12} + C_{ss}\bar{\varepsilon}_{21} + B_{s1}\bar{\kappa}_{12}, \quad M_{12} = B_{s1}\bar{\varepsilon}_{12} + D_{s1}\bar{\kappa}_{12} + D_{ss}\bar{\kappa}_{21},$$

and similar expressions for N_{22}, M_{22}, N_{21} and M_{21}. The coefficients C_{ij}, B_{ij}, D_{ij}, etc., are given in terms of E and ν by

$$(C_{kk}, B_{kk}, D_{kk}) = \int_{-h/2}^{h/2} \frac{1 + \zeta/R_j}{1 + \zeta/R_k} \cdot \frac{E}{1 - \nu^2}(1, \zeta, \zeta^2)\, d\zeta \quad (j \neq k),$$

$$(C_{jk}, D_{jk}) = \int_{-h/2}^{h/2} \frac{\nu E}{1 - \nu^2}(1, \zeta^2)\, d\zeta \quad (j \neq k), \quad (C_{ss}, D_{ss}) = \int_{-h/2}^{h/2} G(1, \zeta^2)\, d\zeta,$$

$$(C_{sk}, B_{sk}, D_{sk}) = \int_{-h/2}^{h/2} \frac{1 + \zeta/R_j}{1 + \zeta/R_k} G(1, \zeta, \zeta^2)\, d\zeta \quad (j \neq k).$$

With $[(1 + \zeta/R_1)/(1 + \zeta/R_2)]^{\pm 1} = 1 \pm \varrho\zeta + \cdots$ where $\varrho = (1/R_2) - (1/R_1)$, we carry out term-by-term integration to get "asymptotic" series for the nonvanishing coefficients C_{ij}, D_{ij}, B_{ij}, etc. Upon omitting cubic or higher order terms in h/R (with $R = \min[R_1, R_2]$), we get the Flügge–Lurje–Byrne (FLB) approximate system of stress strain relations in l.o.c. coordinates [8, 24, 77, 82]:

$$N_{11} = C(\bar{\varepsilon}_{11} + \nu\bar{\varepsilon}_{22}) + \varrho D\left(\bar{\kappa}_{11} - \frac{\bar{\varepsilon}_{11}}{R_1}\right), \quad M_{11} = D(\bar{\kappa}_{11} + \nu\bar{\kappa}_{22} + \varrho\bar{\varepsilon}_{11}),$$

$$N_{12} = Gh(\bar{\varepsilon}_{12} + \bar{\varepsilon}_{21}) + \tfrac{1}{2}\varrho D(1 - \nu)\left(\bar{\kappa}_{12} - \frac{\bar{\varepsilon}_{12}}{R_1}\right), \tag{3.7}$$

$$M_{12} = \tfrac{1}{2}D(1 - \nu)(\bar{\kappa}_{12} + \bar{\kappa}_{21} + \varrho\bar{\varepsilon}_{12})$$

with a similar set for N_{22}, M_{22}, N_{21} and M_{21}.

The relations (3.7) should be transformed into relations for the strain resultants and couples as defined in Section 2.4. For this purpose, we note that the relations between $(\bar{\varepsilon}_{ij}, \bar{\kappa}_{ij})$ and

(V_k, W, Ψ_k) are identical to the relations between $(\varepsilon_{ij}, \kappa_{ij})$ and (u_k, w, ϕ_k) except for terms involving ω in the in-plane shear strain resultants and twisting strain couples (keeping in mind $\gamma_j = 0$). Upon identifying (V_k, W, Ψ_k) and (u_k, w, ϕ_k) we can in fact replace $\bar{\varepsilon}_{ij}$ and $\bar{\kappa}_{ij}$ in (3.7) by the corresponding strain resultants and couples as terms involving ω cancel out in these relations. It is then possible to construct the corresponding S from these relations for the formulation of variational principles [102]. The more general stress potential for general orthogonal coordinates can also be obtained by known transformation laws.

With $\bar{\varepsilon}_{ij}$ and $\bar{\kappa}_{ij}$ given in terms of V_1, V_2 and W, the stress resultants and couples are also given by the same three displacement quantities by way of the FLB relations. The first two scalar moment equilibrium equations (with no drilling couples) can be used to express the transverse shear resultants Q_j in terms of V_1 and V_2 and W. In that case the three scalar force equilibrium equations become three PDE for the three middle surface displacement components V_1, V_2 and W. It would seem then that the remaining (sixth) equilibrium equation (3.4) would impose an additional restriction on these three displacement components, possibly overdetermining them. Fortunately, this is not the case! Instead, the so-called "sixth equilibrium equation" is identically satisfied by the expression given by the FLB relations for N_{ij} and M_{ij} in terms of V_1, V_2 and W.

3.5. First approximation stress strain relations

We may write an in-plane stress component of the shell as the sum of an even part and an odd part. The even part is associated with the stretching and (in-plane) shearing actions of the shell and is called the *direct stress*. The odd part is associated with the bending and twisting actions of the shell and is called the *bending stress*. From the definition of the stress resultants and stress couples, we see that direct stresses are of the order of N_{ij}/h and bending stresses are of the order of M_{ij}/h^2. By comparing the magnitude of the direct and bending stresses, it can be argued that FLB relations contain many terms which are negligible for thin shells. Allowing for an error of the order of h/R, we can simplify the normal resultants and couples immediately to

$$N_{11} = C(\bar{\varepsilon}_{11} + \nu\bar{\varepsilon}_{22}), \quad N_{22} = \cdots. \quad M_{11} = \cdots, \quad M_{22} = D(\bar{\kappa}_{22} + \nu\bar{\kappa}_{11}). \tag{3.8}$$

With a little more care, the in-plane shear resultants and twisting couples can be simplified (with an $O(h/R)$ error) to

$$\begin{aligned} N_{12} &= Gh(\bar{\varepsilon}_{12} + \bar{\varepsilon}_{21}) + \tfrac{1}{2}\varrho M_{12}, \quad N_{21} = Gh(\bar{\varepsilon}_{21} + \bar{\varepsilon}_{12}) - \tfrac{1}{2}\varrho M_{21}, \\ M_{12} &= \tfrac{1}{2}D(1-\nu)[\bar{\kappa}_{12} + \bar{\kappa}_{21} + \tfrac{1}{2}\varrho(\bar{\varepsilon}_{12} - \bar{\varepsilon}_{21})] = M_{21}. \end{aligned} \tag{3.9}$$

Given $\bar{\varepsilon}_{ii} = \varepsilon_{ii}$ and (in l.o.c. coordinates) $\bar{\kappa}_{ii} = \kappa_{ii}$, the stress strain relations (3.8) can be written immediately in terms of the normal strain resultants and couples leaving the form of the relations unchanged. On the other hand, we have $(\varepsilon_{12}, \varepsilon_{21}) = (\bar{\varepsilon}_{12}, \bar{\varepsilon}_{21}) + (-1, 1)\omega$ and $(\kappa_{12}, \kappa_{21}) = (\bar{\kappa}_{12}, \bar{\kappa}_{21}) + \omega(-1/R_1, 1/R_2)$ so that we can also replace $\bar{\varepsilon}_{ij}$ and $\bar{\kappa}_{ij}$ in the remaining stress strain relations (3.9) by ε_{ij} and κ_{ij} as terms involving ω in the latter cancel out. Finally, if we want ε_{ij} to be identical to the midsurface strain components, then we must require $\varepsilon_{12} = \varepsilon_{21}$. In that case, the $\varepsilon_{12} - \varepsilon_{21}$ term in the expressions for M_{12} and M_{21} drops out so that the relations for the stress and strain couples become

$$M_{12} = M_{21} = \tfrac{1}{2}D(1 - \nu)(\kappa_{12} + \kappa_{21}),$$
$$M_{22} = D(\kappa_{22} + \nu\kappa_{11}), \quad M_{11} = D(\kappa_{11} + \nu\kappa_{22}). \tag{3.10}$$

We may then form the sum $N_{12} + N_{21}$ and invert the normal strain relations in (3.8) to get

$$\varepsilon_{12} = \varepsilon_{21} = \tfrac{1}{2}A(1 + \nu_s)(N_{12} + N_{21}), \tag{3.11}$$

$$\varepsilon_{11} = A(N_{11} - \nu N_{22}), \quad \varepsilon_{22} = A(N_{22} - \nu N_{11}). \tag{3.12}$$

The relations (3.10)–(3.12) constitute the stress strain relations for a *first approximation shell theory* as a consequence of Love's assumptions. Note that the thin shell approximation was used to delete terms of order h/R compared to unity only in the stress strain relations. These stress strain relations are seen to be a special case of the stress strain relations (2.8) and (2.9) with $B = 0$, $C = 0$ and $-\varrho_b = \varrho_s = 1/2$. In other words, the expressions (2.8) and (2.9) are appropriate for some stress strain relations derivable from three-dimensional elasticity theory.

The first approximation stress strain relations for shells (3.10)–(3.12) are considerably simpler than the FLB relations. In fact, it has been argued that for isotropic shells, more refined approximate stress strain relations such as the FLB relations are inconsistent with the Kirchhoff–Love hypotheses [59]. More specifically, higher order relations do not improve upon the accuracy of the first approximation theory given by (3.10)–(3.12) unless we also remove Love's restrictive assumptions on transverse shear strains and on the transverse normal stress. Having the approximate three dimensional stress strain relations for Love's theory in terms of the strain resultants and strain couples, a stress potential can now be obtained for different variational principles in l.o.c. shell coordinates [101]. The construction of the scalar functional S requires some finesse as the expressions for the in-plane shear strain resultants are not invertible. Once we have S for l.o.c. coordinates, the corresponding result for general orthogonal surface coordinates can be obtained by known transformation laws.

3.6. Correct interior solution for isotropic shells

The transversely rigid shell model has enabled us to circumvent in an elegant way the internal inconsistencies in Love's approximating set of assumptions for constructing a theory of thin elastic shells from three-dimensional theory of elasticity. However, the model is, strictly speaking, not applicable to isotropic materials encountered in most applications. Therefore, we still have the task of justifying the appropriateness and adequacy of Love's theory for isotropic shells as it was originally intended. The justification process is further complicated by the fact that the approximate (shell theory) solution cannot fit most prescribed edge data for the three-dimensional problem and the method of matched asymptotic expansion is usually ineffective for an explicit solution of shell problems. A decision must be made on what portion of the prescribed data should be assigned to the approximate solution for its determination.

The situation is similar to, but more complicated than the corresponding flat plate problem. We know from an asymptotic analysis that the exact solution of the plate problem consists of an interior component and a boundary layer component (e.g., [27]). The leading term approximate interior solution corresponds to a solution of the transversely rigid plate theory and is significant throughout the plate. The boundary layer solution is only significant in a narrow region adjacent to the edge(s) of the plate and may be neglected in the plate interior. Both solution components are needed to fit the prescribed edge data for the three-dimensional problem. The boundary layer solution usually cannot be obtained in terms of simple functions. There has been considerable effort over the years to assign an appropriate portion of the edge data for the determination of the interior solution without a simultaneous consideration of the boundary layer solution. For stress edge data, Saint Venant's principle has been invoked to obtain a set of boundary conditions for the two-dimensional stress resultants and couples. However, recent results show that this approach does not always lead to the correct interior solution [40–42]. For displacement or mixed edge data, it is now known that certain weighted averages of the prescribed data should be satisfied by the approximate (plate theory) solution [39, 73, 74].

With the flat plate being a limiting case of shell structures, we expect that many of the results for isotropic thin plates will also apply to isotropic thin shells, possibly with some added complications. The technique used for flat plates can in fact be extended to obtain the proper edge conditions for a first approximation shell theory [43]. On the other hand, we have for shells a second small parameter h/R in addition to the parameter h/L in plate theory (where L is a characteristic length of the plate such as the wavelength of the external loads). The asymptotic expansions of the interior solution are therefore double asymptotic series which are known to be considerably more difficult to obtain [38].

A correct leading term asymptotic *interior solution* depends on the correct set of two-dimensional shell equations (as well as the correct specification of the edge conditions for the shell solution). Discussions on interior solution expansions such as [31, 50, 51, 98] and others indicate that a *consistent* first approximation shell theory would provide this correct set of shell equations. Not all shell theories derived from the Kirchhoff–Love hypotheses are adequate for the correct leading term asymptotic interior solution. For example, a first approximation theory which omits the ζ/R_j term in (3.3) to symmetrize N_{ij} and M_{ij} (see [76]) is not only inconsistent with the sixth equilibrium equation but also leads to incorrect (interior) solutions for specific problems (such as shown in [91] for the axial torsion of noncircular cylindrical shells). For these problems, the errors incurred by Love's symmetrization of N_{ij} are not $O(h/R)$ or $O(h^2/L^2)$ compared to unity. The work of Koiter [59] has made it possible for us to identify what constitutes an adequate set of equations for a first approximation shell theory. It remains to determine the correct edge conditions for the shell equations from the given edge data for the three-dimensional problem. This will be done in [43].

4. The structure of the linear theory

4.1. Stress functions and compatibility

The two vector differential equations of equilibrium (2.3) and (2.4) for four unknown vector fields may be solved by two (vector) stress functions $\mathbf{U}$ and $\mathbf{\Phi}$. In the absence of surface loads, the two equations are satisfied identically by setting

$$\begin{aligned} \alpha_2\mathbf{N}_1 &= \mathbf{\Phi}_{,2}, & -\alpha_1\mathbf{N}_2 &= \mathbf{\Phi}_{,1}, \\ \alpha_2\mathbf{M}_1 &= \mathbf{U}_{,2} + \mathbf{r}_{,2} \times \mathbf{\Phi}, & -\alpha_2\mathbf{M}_2 &= \mathbf{U}_{,1} + \mathbf{r}_{,1} \times \mathbf{\Phi} \end{aligned} \tag{4.1}$$

with $\mathbf{U} = U_1\mathbf{t}_1 + U_2\mathbf{t}_2 + F\mathbf{n}$, and $\mathbf{\Phi} = \mathbf{n} \times (\Phi_1\mathbf{t}_1 + \Phi_2\mathbf{t}_2) + \Omega\mathbf{n}$. From the definition of $\mathbf{N}_j$ and $\mathbf{M}_j$, we see that $\mathbf{\Phi}$ and $\mathbf{U}$ are in fact measures of the resultant force and moment, respectively, acting on a portion of the shell.

The four (vector) strain measures defined in terms of two displacement vectors $\mathbf{u}$ and $\boldsymbol{\phi}$ in (2.5) are related by two vector differential compatibility equations:

$$(\alpha_2\boldsymbol{\kappa}_2)_{,1} - (\alpha_1\boldsymbol{\kappa}_1)_{,2} = \mathbf{0}, \tag{4.2}$$

$$(\alpha_2\boldsymbol{\varepsilon}_2)_{,1} - (\alpha_1\boldsymbol{\varepsilon}_1)_{,2} + \mathbf{r}_{,1} \times (\alpha_2\boldsymbol{\kappa}_2) - \mathbf{r}_{,2} \times (\alpha_1\boldsymbol{\kappa}_1) = \mathbf{0}. \tag{4.3}$$

It should be noted that the present Günther–Reissner formulation of the strain displacement relations has enabled us to deduce the compatibility equations essentially by inspection. Furthermore, the corresponding six scalar compatibility equations are all first order partial differential equations. This should be compared with conventional formulations for a theory with $C = B = 0$ in which λ_1 and λ_2 do not appear explicitly. There are only four compatibility equations in these conventional formulations; one is a second order PDE, two are first order PDE's and one is an algebraic equation. Their derivation requires considerable ingenuity [29, 31].

For $-\varrho_b \neq 1/2$, the stress strain relations (2.8) and (2.9) can be taken in the inverted form, giving individual strain measures in terms of linear combinations of the stress measures. Moreover, these inverted relations can be expressed in terms of a scalar strain potential $\bar{S}(N_{11}, N_{12}, \ldots, P_2)$ by (2.13). The expression for $\bar{S}$ can be obtained in a straightforward manner and will not be given here. For our first approximation theory (for which $-\varrho_b = \varrho_s = 1/2$ and $B = C = 0$), the stress and strain potentials can still be constructed with some finesse [101].

Upon expressing the stress measures in the inverted stress strain relations in terms of stress functions, the two vector compatibility equations can be written as two second order vector partial differential equations for $\mathbf{U}$ and $\mathbf{\Phi}$ or six second order linear scalar PDE for the six scalar stress functions. These six equations, together with an appropriate set of six boundary conditions at each edge of the shell, define a boundary value problem in linear PDE. One-, two- and three-field variational principles have been formulated for this method of solution for shell problems [123]. Among their applications are derivations of appropriate displacement boundary conditions in terms of stress functions for the first approximation shell theory, membrane theory, as well as a simple form of the stress boundary conditions for an inextensional bending theory previously considered in [92].

4.2. Static-geometric duality

The rather symmetric development of the displacement method and the stress function method of solution for specific shell problems suggests the possibility of a symmetry inherent in our formulation of linear shell theory. A closer examination of the equations of the theory confirms that there is in fact such a symmetry known as the *static-geometric duality* [7, 31, 44, 94, 103]. Evidently, if we replace all strain measures in the compatibility equations (4.2) and (4.3) by the dual stress measures according to Table I below, we will get the homogeneous equilibrium equations (2.3) and (2.4) and vice versa. The stress function representations (4.1) are also the duals of the strain displacement relations (2.5) if we observe the additional dual relations $(\mathbf{U}, \mathbf{\Phi}) \leftrightarrow (\mathbf{u}, \boldsymbol{\phi})$. The presence of drilling couples and transverse shearing strains in the Günther–Reissner formulation of shell theory has made it much easier to establish the s–g duality. The development leading up to the duality in this article should be compared with the original approach used by A.L. Goldenveizer [29, 31] (see also [7]).

Table I

$\mathbf{N}_1$	$\mathbf{N}_2$	$\mathbf{M}_1$	$\mathbf{M}_2$	$\mathbf{\Phi}$	$\mathbf{U}$
$\boldsymbol{\kappa}_2$	$-\boldsymbol{\kappa}_1$	$\boldsymbol{\varepsilon}_2$	$-\boldsymbol{\varepsilon}_1$	$\boldsymbol{\phi}$	$\mathbf{u}$

Table II

A	B	ν_s	ϱ_s	S
$-D$	$-C$	$-\nu_b$	$-\varrho_b$	$-\bar{S}$

The duality between kinematic and static aspects of the theory can be extended to include the stress strain relations. It is rather remarkable that the relations (2.8) for the strain resultants are in fact the s–g duals of the relations (2.9) for the stress couples if we observe the additional dual relations among the elastic parameters given in Table II. These dual material relations also give a correspondence between the stress potential $S(\varepsilon_{11}, \varepsilon_{12}, \ldots, \lambda_2)$ and the strain potential $\bar{S}(N_{11}, N_{12}, \ldots, P_2)$. As an example, we get the expression for $-\bar{S}$ from the expression (2.10) for S by replacing all quantities in the latter by their dual quantities according to Tables I and II. It follows that there is now also an s–g duality between the variational principles for the displacement method of Section 2.7 and the principles for the stress function method of Section 4.1.

4.3. Dual boundary value problems

While various types of commonly encountered boundary conditions at an edge of the shell can be formulated in terms of the primary unknowns in the displacement method of solution, the same does not seem to be the case in the stress function formulation. For example, the displacement edge conditions $\Delta\mathbf{u} = \mathbf{0}$ and $\Delta\boldsymbol{\phi} = \mathbf{0}$ are not expressible in terms of the stress functions and their derivatives. At the same time, the Euler boundary conditions of the variational principles for the stress function method are either $\Delta\boldsymbol{\varepsilon}_s = \mathbf{0}$ and $\Delta\boldsymbol{\kappa}_s = \mathbf{0}$ or $\Delta\mathbf{U} = \mathbf{0}$ and $\Delta\mathbf{\Phi} = \mathbf{0}$. The two types of Euler boundary conditions are, not surprisingly, the static-geometric duals of stress and displacement conditions, respectively, of the principles for the displacement method. While they are not explicitly conditions on the displacements themselves, we note that in terms of an edge variable ξ_s (with $ds = \alpha_s\, d\xi_s$), we have $\alpha_s\boldsymbol{\kappa}_s = \boldsymbol{\phi}_{,s}$ and $\alpha_s\boldsymbol{\varepsilon}_s = \mathbf{u}_{,s} + \mathbf{r}_{,s} \times \boldsymbol{\phi}$. Given the edge displacements $\mathbf{u}^*(\xi_s)$ and $\boldsymbol{\phi}^*(\xi_s)$ the edge values of the strain resultant $\boldsymbol{\varepsilon}_s^*$ and strain couple $\boldsymbol{\kappa}_s^*$ are completely determined by differentiations with respect to ξ_s. Hence, the displacement edge conditions can be reformulated as conditions on strain measures as suggested by the variational principles [123].

With this reformulation, we can now write the displacement edge conditions in terms of stress functions and their derivatives. This is accomplished by expressing $\boldsymbol{\varepsilon}_s$ and $\boldsymbol{\kappa}_s$ in terms of the stress resultants and couples by way of the stress strain relations and then using the stress function representations for the stress resultants and couples. For the conventional first approximation theory, the appropriate displacement conditions (in terms of strain measures) have been shown to be the static-geometric duals of the Kirchhoff–Bassett contracted stress boundary conditions [123].

With the added duality between the two sets of (stress and displacement) boundary conditions, we have enlarged the scope of the elegant s–g duality to include duality between two boundary value problems. In particular, we have established the duality between the two fundamental problems in shell theory. The extended duality is computationally more useful. The solution of a specific boundary value problem becomes the solution of the dual problem simply by interpreting all quantities as the dual quantities [124, 125, 130]. Moreover, a computer program written to solve one boundary value problem can be used to solve the dual problem without any modification whatsoever, simply by using as input parameter values the values of the dual parameters and interpreting any output of the program as the solution of the dual quantity [78, 127].

4.4. Solution methods with an intrinsic s–g duality

The governing equations in the stress function method for linear shell problems are now seen to be the s–g duals of the equations in the displacement method. It is of considerable interest to investigate other methods of solution with an intrinsic s–g duality. That is, the dual of the set of governing equations for these methods is to be the set itself. For such methods, half of their governing equations must be the dual of the other half. Marguerre's linear shallow shell theory is an example of such methods. We list here three solution methods with an intrinsic duality for our general linear theory for later applications.

If the equilibrium equations (2.3) and (2.4) are parts of the governing equations, then so must be the compatibility equations (4.2) and (4.3) for an intrinsic duality. The remaining four vector equations for the eight vector unknowns must come from the stress strain relations taken in the form (2.8) and (2.9), with the latter being the s–g dual of the former. These stress strain relations may be expressed in terms of a mixed potential $\tilde{S}(\mathbf{N}_1, \mathbf{N}_2, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)$ with

$$\boldsymbol{\varepsilon}_j = \frac{\partial \tilde{S}}{\partial \mathbf{N}_j}, \quad \mathbf{M}_j = \frac{\partial \tilde{S}}{\partial \boldsymbol{\kappa}_j}. \tag{4.4}$$

The construction of $\tilde{S}$ is straightforward. Note that ϱ_s and $-\varrho_b$ taking on the value of $1/2$ does not cause any complication for $\tilde{S}$. The above *equilibrium-compatibility method* of solution may be regarded as a system of twelve scalar first order PDE for six scalar stress resultants and six dual scalar strain couples.

Another possible method of solution for specific shell problems with an intrinsic s–g duality employs as the governing equations the stress function representation (4.1), the strain displacement relations (2.5) and the stress strain relations (4.4). In this *displacement-stress function method*, we may use (4.1) and (2.5) to transform (4.4) (or (2.8) and (2.9)) into twelve first order linear PDE for six scalar stress functions and six scalar displacement variables.

A third method with an intrinsic s–g duality uses a part each of stress function representations, equilibrium equations, compatibility equations and strain displacement relations as well as all of the stress strain relations. Our experience with Marguerre's linear shallow shell theory suggests that considerable benefits can be gained from this *mixed method*. For the case of no surface loads, the governing equations in this mixed method are:

$$\alpha_2 \mathbf{N}_1 = \mathbf{\Phi}_{,2}, \quad -\alpha_1 \mathbf{N}_2 = \mathbf{\Phi}_{,1}, \quad \mathbf{M}_i = \frac{\partial \tilde{S}}{\partial \boldsymbol{\kappa}_i} \quad (i = 1, 2),$$
$$(\alpha_2 \mathbf{M}_1)_{,1} + (\alpha_1 \mathbf{M}_2)_{,2} + \mathbf{r}_{,1} \times (\alpha_2 \mathbf{N}_1) + \mathbf{r}_{,2} \times (\alpha_1 \mathbf{N}_2) = \mathbf{0} \tag{4.5}$$

and their s–g dual equations. Note that the equilibrium equation (2.3) and the compatibility equation (4.2) are automatically satisfied by this approach. The first three equations of (4.5) and their duals can be used to transform the last equation and its dual into two second order vector equations for $\mathbf{\Phi}$ and $\boldsymbol{\phi}$.

5. Linear theory of shells of revolution

5.1. Fourier decomposition

For shells of revolution which are often used in actual designs, the linear shell theory of Section 2 admits certain simplifications with significant consequences. We summarize the major results for the linear theory of shells of revolution in this section while the remainder of this article will be concerned with results for the nonlinear theory of this same class of shell structures.

In cylindrical coordinates (r, θ, z), we take the parametric representation of the surface of revolution in the form $r = r(\xi)$ and $z = z(\xi)$ where ξ is a variable along a meridian (a constant θ curve) of the surface. With $\xi_1 = \xi$ and $\xi_2 = \theta$, we have $\mathbf{r} = r(\xi)\mathbf{i}_r + z(\xi)\mathbf{i}_z$, and

$$\alpha_1^2 =: \alpha^2 = (r')^2 + (z')^2, \quad \alpha_2 = r, \quad r' = \alpha \cos \varphi, \quad z' = \alpha \sin \varphi, \tag{5.1}$$

$$1/R_{11} = (r''z' - r'z'')/\alpha^3 = -\varphi'/\alpha, \quad 1/R_{22} = -z'/r\alpha, \quad 1/R_{12} = 0 \tag{5.2}$$

where $(\cdot)' = \partial(\cdot)/\partial\xi$ and φ is the angle made by the meridional tangent with the base plane of the surface of revolution. It follows from $1/R_{12} = 1/R_{21} = 0$ that ξ and θ form a system of l.o.c. surface coordinates.

As α_i and R_{ij} are functions of ξ only, it is readily seen that the coefficients of the scalar linear PDE of equilibrium and compatibility for a shell of revolution are independent of the polar angle θ. If shell properties are also independent of θ, the loads and response of a circumferentially complete shell of revolution (in the form of a dome or a shell frustum) may be represented by their Fourier series in θ. Linearity then allows for a separate analysis of each Fourier coefficient. Not so obvious however are the different additional simplifications and reductions associated with the different harmonics and with problems involving circumferentially incomplete shells of revolution including the special case of a shell slit along a meridian. We summarize in this section the main theoretical results essential to a true understanding of the linear theory of shells of revolution and effective solution techniques for specific problems governed by this theory.

5.2. Axisymmetric bending and stretching

For problems with load and response distributions independent of θ, the equations for the elastostatics of the shell decouple into two groups. Each group consists of three scalar equilibrium equations, their dual compatibility equations and six stress strain relations. One group involves the stress resultants and couples (N_{11}, N_{22}, Q_1, M_{11}, M_{22}, P_2), the dual strain measures, the displacement variables (u_1, w, ϕ_1) and their dual stress functions. It describes the axisymmetric bending and stretching action of the shell. The second group, consisting of the remaining field variables and equations, characterizes the axisymmetric torsion and twisting shell actions and will be discussed in Section 5.3.

The first significant breakthrough for the axisymmetric bending of shells of revolution was for spherical shells by H. Reissner in 1912 [104]. The requirement of overall axial force equilibrium gives a first integral of (2.3) which enables us to transform the only moment equilibrium equation for our problem into a second order ODE for a stress function and an angular displacement variable (a strain function). Not having the benefit of the static geometric duality at this time, it was a stroke of genius that H. Reissner succeeded in obtaining an appropriate compatibility equation for a second ODE for the same two unknowns. The two simultaneous second order equations were further reduced to a single second order equation for a complex function. This second order equation in turn shows that the general solution for the axisymmetric bending of spherical shells can be expressed in terms of associated Legendre functions.

Shortly after H. Reissner's pioneering work for spherical shells, E. Meissner [80] used a similar approach to reduce the axisymmetric bending problem for general shells of revolution also to two simultaneous second order ODE's for a stress function and an angular displacement variable. Meissner's results are esthetically unsatisfactory. They do not preserve the static-geometric duality inherent in the shell equations (though not known at the time). Also, it is not possible to take his results to the flat plate limit. Both of these shortcomings were removed later by a different choice of stress function [103].

The more natural choice of stress function first introduced by E. Reissner [88] is $\Phi = rH$, where H is the radial stress resultant. Let the axial stress resultant be V, then N_{11} and Q_1 are related to H and V by

$$N_{11} = H\cos\varphi + V\sin\varphi, \quad Q_1 = -H\sin\varphi + V\cos\varphi. \tag{5.3}$$

In terms of V and H, the three equilibrium equations for the problem become

$$(rV)' + r\alpha p_V = 0, \quad (rH)' - \alpha N_{22} + r\alpha p_H = 0, \qquad (5.4, 5.5)$$

$$(rM_{11})' - \alpha\cos\varphi M_{22} + r\alpha(H\sin\varphi - V\cos\varphi) - \alpha\sin\varphi P_2 + r\alpha q_1 = 0 \tag{5.6}$$

where p_V and p_H are the vertical and radial surface load component, respectively. Equation (5.4) yields the first integral mentioned above; thus, V is determined up to a constant of integration C_V:

$$rV = r(N_{11}\sin\varphi + Q_1\cos\varphi) = -\int r\alpha p_V\,d\xi + C_V. \tag{5.7}$$

Equation (5.5) can be used to express N_{22} in terms of the stress function Φ. By duality, we have immediately a first integral for the compatibility equations (4.2) $r(\kappa_{22} \sin \varphi - \lambda_2 \cos \varphi) = -C_B$, where C_B is another constant of integration. We also have $\alpha\kappa_{11} = \phi'$ as the dual of equation (5.5) with the strain function ϕ being the dual of the stress function Φ. The mixed stress strain relations are now used to express M_{kk} and P_2 in (5.6) in terms of κ_{jj} and λ_2, and ε_{ii} and γ_1 in the dual equation of (5.6) in terms of N_{jj} and Q_1. It follows that all unknown stress and strain measures in (5.6) and the dual compatibility equation can be expressed in terms of ϕ, ϕ', Φ and Φ'. In this way, (5.6) becomes a second order equation in ϕ and Φ [103]

$$\left(\frac{rD}{\alpha}\phi'\right)' - \left(\frac{rD}{\alpha}\right)\left[\left(\frac{r'}{r}\right)^2 - \frac{(\nu_b Dr'/\alpha)'}{rD/\alpha} + \frac{C}{D}\left(\frac{z'}{r}\right)^2\right]\phi - z'\Phi$$

$$= -\frac{rD}{\alpha}\left[\frac{r'z'}{r^2}\left(1 - \frac{C}{D}\right)C_B - \frac{(\nu_b z' DC_B/\alpha)'}{rD/\alpha}\right] - r'(rV) - \alpha r q_1. \tag{5.8}$$

Except for load terms, the second equation for ϕ and Φ is the s–g dual of (5.8). For the simpler classical first approximation theory, we have $B = C = 0$; however, the two equations remain a fourth order system (see also [81]).

For isotropic conical, circular cylindrical, spherical and toroidal shells of uniform properties and with $B = C = 0$, equations (5.8) and its dual can be combined into a second order complex differential equation for a complex potential $x = \phi + \lambda\Phi$ for some complex constant λ. Considerably more mathematical techniques are available for the analysis of a single second order differential equation, e.g., [15, 46, 111].

5.3. *Axisymmetric torsion and twisting*

Overall axial moment equilibrium allows us to obtain a first integral of the three differential equations of equilibrium for this class of problems [103]:

$$N_{12} - \sin\varphi M_{12} + \cos\varphi P_1 = \frac{1}{r}T(\xi) \tag{5.9}$$

where $T(\xi)$ is the resultant axial torque over the latitudinal circular edge located at ξ. By the s–g duality, we get a first integral of the relevant compatibility equations as well. The two first integrals reduce the order of the system from six to four. As in the case of axisymmetric bending and stretching, we can further reduce the fourth order system to two simultaneous second order equations for the stress function Ψ and its dual strain function ψ. The actual reduction to two simultaneous equations begins with the introduction of the stress function $\Psi = rN_{12}$ and the dual strain function $\psi = r\kappa_{21}$. We then use the relevant stress strain relations in (2.8) and (2.9) to transform the first integral (5.9), the only force equilibrium equation, and the two dual relations into a system of four simultaneous linear equations for N_{21}, Q_2, κ_{12} and λ_1. These equations can be solved to give each of the four quantities in terms of Ψ, ψ, their first derivatives and load terms. Upon inserting the results into the "sixth" equilibrium equation and the dual compatibility equation, we get two second order equations for Ψ and ψ [103].

For the classical first approximation theory, we have $B = C = 0$ and $-\varrho_b = \varrho_s = 1/2$ (as well as $q_2 = q_n = 0$). In that case, the expressions for N_{12}, Q_2 and the dual strain measures simplify to read

$$\begin{aligned} D_s \sin\varphi\kappa_{12} &= -\frac{T}{r} - D_s \sin\varphi\,\frac{\psi}{r} + \frac{\Psi}{r}, \quad & \alpha \sin\varphi\lambda_1 &= \alpha\cos\varphi\kappa_{12} - \psi', \\ A_s \sin\varphi N_{21} &= \frac{C_T}{r} - A_s \sin\varphi\,\frac{\Psi}{r} - \frac{\psi}{r}, \quad & \alpha \sin\varphi Q_2 &= -\alpha\cos\varphi N_{21} + \Psi' \end{aligned} \tag{5.10}$$

where $D_s = D(1 - \nu_b)/2$, $A_s = A(1 + \nu_s)/2$, and C_T is a constant of integration in the first integral of the compatibility equations (giving the relative displacement of the shell along a meridional slit). The sixth equilibrium equation and its dual compatibility equation are now algebraic equations

$$\begin{aligned} r\alpha(N_{12} - N_{21}) - r\varphi' M_{12} + \alpha\sin\varphi M_{21} &= 0, \\ r\alpha(\kappa_{21} - \kappa_{12}) + r\varphi'\varepsilon_{21} - \alpha\sin\varphi\varepsilon_{12} &= 0. \end{aligned} \tag{5.11}$$

It is important to note that the expressions in (5.10) for N_{21} and κ_{12} do not involve any derivative of the two primary unknowns. Hence the two equations for Ψ and ψ obtained by substituting these expressions into (5.11) are also algebraic equations. This is consistent with the fact that a consistent first approximation theory of shells is an eighth order theory. With four first integrals and a fourth order system of differential equations for Φ and ϕ in axisymmetric bending and stretching, the governing equations for torsion and twisting problems in this theory are necessarily algebraic (see p. 567 of [76]).

5.4. *Lateral deformations*

Problems associated with the first harmonics in the Fourier decomposition in the polar angle θ are also amenable to simplifications and reductions similar to the axisymmetric case. First integrals for lateral (or wind) load problems were derived by V.S. Chernina [13] (see also [108a]), and the system of shell equations were reduced to two simultaneous second order ODE's. Similar to Meissner's results for axisymmetric bending, Chernina's reduction also does not preserve static-geometric duality and her two governing equations take on an indeterminate form in the flat plate limit. These shortcomings were removed in [128, 130]. Within the inherent error of a first approximation theory, the two new second order simultaneous equations are remarkably similar to (5.8) and its dual. The new reduction also allows for nonperiodic displacement fields relevant to problems for shells slit along a meridian or circumferentially incomplete shell sectors.

We consider here the typical load distributions $\{p_1, p_n\} = \{\bar{p}_1(\xi), \bar{p}_n(\xi)\}\cos\theta$, $p_2 = \bar{p}_2(\xi)\sin\theta$ and for simplicity $q_1 = q_2 = q_n = 0$. It is then consistent to take the stress resultants and couples in the form

$$\begin{aligned} \{N_{11}, N_{22}, Q_1, M_{11}, M_{22}, P_2\} &= \{\bar{N}_{11}(\xi), \ldots, \bar{P}_2(\xi)\}\cos\theta, \\ \{N_{12}, N_{21}, Q_2, M_{12}, M_{21}, P_1\} &= \{\bar{N}_{12}(\xi), \ldots, \bar{P}_1(\xi)\}\sin\theta, \end{aligned} \tag{5.12}$$

with a corresponding set of formulas for the strain measures. Again, we use the equilibrium-compatibility method and limit our discussion to the classical first approximation theory of Section 3. To get two simultaneous equations for the stress function $\Psi = rN_{12}$ and the dual strain function $\psi = r\kappa_{21}$, we use the stress strain relations $M_{12} = M_{21} = D_s(\kappa_{12} + \kappa_{21})$ and $\varepsilon_{12} = \varepsilon_{21} = A_s(N_{12} + N_{21})$, the sixth equilibrium equation and the sixth compatibility equation as four simultaneous linear equations for $\bar{\varepsilon}_{12} = \bar{\varepsilon}_{21}$, $\bar{M}_{12} = \bar{M}_{21}$, $\bar{N}_{21}$ and $\bar{\kappa}_{12}$ and solve them to get

$$\begin{aligned} \bar{\varepsilon}_{12} &= \bar{\varepsilon}_{21} = \frac{A(1 + \nu_s)}{r(1 + \varepsilon_0^2)}[\Psi - \tfrac{1}{2}\varrho D(1 - \nu_b)\psi], \\ \bar{N}_{21} &= -\left[\frac{1 - \varepsilon_0^2}{1 + \varepsilon_0^2}\Psi - \frac{D(1 - \nu_b)}{1 + \varepsilon_0^2}\varrho\psi\right] \end{aligned} \tag{5.13}$$

and two static-geometric dual relations, where $\varrho = (1/R_2) - (1/R_1)$ and $\varepsilon_0^2 = DA(1 - \nu_b)(1 + \nu_s)\varrho^2/4$.

With the help of (5.13), we can use all the remaining equations in our formulation (including the four first integrals) except one moment equilibrium (corresponding to the governing equation for axisymmetric bending problems) and its dual compatibility equation to express all the unknown stress and strain measures in terms of Ψ, Ψ', ψ and ψ' and load terms alone. Upon substituting these expressions into the two yet unused first order differential equations, we get two second order ODE's for Ψ and ψ in the form

$$\psi'' - \frac{A(1 + \nu_s)}{R_2(1 + \varepsilon_1^2)}\Psi'' = f_1, \quad \Psi'' + \frac{D(1 - \nu_b)}{R_2(1 + \varepsilon_1^2)}\psi'' = f_2 \tag{5.14}$$

where $\varepsilon_1^2 = O(DA/R^2)$ and the functions f_i are linear combinations of Ψ', ψ', Ψ, ψ and load terms. We solve them to get ψ'' and Ψ'' each as a linear combination of Ψ', Ψ, ψ' and ψ; except for load terms, the two combinations are s–g duals of each other. Within the inherent error of the theory, these two equations may be simplified to read

$$\psi'' + \frac{(Dr/\alpha)'}{(Dr/\alpha)}\psi' - \left[4\left(\frac{r'}{r}\right)^2 - \frac{\{(1 - \nu_b)Dr'/\alpha\}'}{(Dr/\alpha)} + 2\left(\frac{z'}{r}\right)^2\right]\psi - \frac{z'}{(Dr/a)}\Psi = g_\psi(\xi) \tag{5.15}$$

and its static-geometric dual. Equation (5.15) is remarkably similar to the corresponding equation for axisymmetric bending (5.8).

When $z = 0$, the two equations for Ψ and ψ reduce to the exact equations for a flat plate. For a homogeneous circular cylindrical shell of constant thickness, they differ from the exact equations only by terms already neglected in the classical shell theory. When $z \neq 0$ and r not a constant, it is possible to further reduce the two equations to a single second order complex ODE for a complex potential in the case of spherical, conical and toroidal shells. Similar to axisymmetric problems, exact, asymptotic and numerical methods are available for the analysis of both the single complex equation and the two simultaneous equations for Ψ and ψ, e.g. [109, 130].

5.5. Self-equilibrating loads

Fifty years ago, the equations of the classical linear (and nonlinear) shallow shell theory for general unsymmetric loads were formulated and reduced to two simultaneous fourth order PDE for the transverse midsurface displacement component w and the (dual) stress function F [79]. Encouraged by the results for shallow shells, several successful independent investigations were undertaken to obtain two similar equations for nonshallow spherical shells [44a, 2, 23, 61]. In the first approximation theory of Section 3, the two equations for w and F are particularly attractive as they are free of $O(h/R)$ terms. Omitting the surface load terms, we have $(\nabla^2 + 2)\{[\nabla^2 + (1 + \nu_b)]w - (a/D)F\} = 0$, and its dual as the governing equations [134]. A more general reduction allowing for transverse shear deformability can be found in [23].

The expectation of two fourth order PDE for w and F for the first approximation shell theory generated by the success for spherical shells was further heightened by a similar reduction for circular cylindrical shells due to J.G. Simmonds [110]. Upon omitting surface load terms, Simmonds' two equations take the form $\nabla^2\nabla^2 w + w^{\cdot\cdot} = (a/D)F''$, and its dual. The reduction for spherical shells is exact; in contrast, Simmonds' reduction for cylindrical shells was accomplished by adding and deleting terms in the stress strain relations expected to be small of order of the error inherent in the first approximation shell theory. The accuracy of Simmonds' two equations for w and F has in fact been confirmed by an exact reduction [126].

The method of reduction of [126] does not rely on the particular shell geometry involved. It works with a specific harmonic in the Fourier decomposition. For the displacement-stress function method in the first approximation theory of Section 3 with $\{u_1, w, U_1, F\} = \{u_n(\xi), w_n(\xi), U_n(\xi), F_n(\xi)\}\cos(n\theta)$ and $\{u_2, U_2\} = \{v_n(\xi), V_n(\xi)\}\sin(n\theta)$, $n \geqslant 2$, four of the six nontrivial stress strain relations may be rearranged to give u_n', v_n', U_n' and V_n' separately in terms of u_n, v_n, w_n, w_n' and the dual stress functions. They can be used to eliminate all derivatives of u_n, v_n, U_n and V_n from the remaining two stress strain relations leaving

$$w_n'' = L_{w2}(u_n, v_n, U_n, V_n; w_n, w_n', F_n, F_n'), \quad F_n'' = L_{F2}(u_n, \ldots, F_n') \tag{5.16}$$

where L_{w2} and L_{F2} are linear combinations of their arguments and the appropriate Fourier components of the surface loads. Evidently, the two relations are the s–g dual of each other. For a circular cylindrical shell of radius a with no surface loads, these two relations are

$$w_n'' = -\nu_b n v_n - \frac{a}{D} n V_n + \nu_b n^2 w_n + \frac{a}{D} F_n, \quad F_n'' = \nu_s n V_n + \cdots. \tag{5.17}$$

To obtain two equations for w_n and F_n alone, we differentiate both sides of the two equations in (5.16) with respect to ξ, and use the expressions for u_n', v_n' and their duals to eliminate these derivatives from the resulting expressions to get

$$w_n''' = L_{w3}(u_n, v_n, U_n, V_n, w_n, w_n', w_n'', F_n, F_n', F_n''), \quad F_n''' = L_{F3}(u_n, \ldots). \tag{5.18}$$

Differentiate (5.18) once more and again eliminate u_n', v_n' and their duals to get

$$w_n'''' = L_{w4}(u_n, \ldots, v_n, w_n, \ldots, w_n''', F_n, \ldots, F_n'''), \quad F_n'''' = L_{F4}(u_n, \ldots). \tag{5.19}$$

We solve (5.16) and (5.18) for u_n, v_n and their duals in terms of w_n, F_n and their derivatives up to third order. We then use the results to eliminate u_n, v_n and their duals from (5.19) leaving us with two fourth order ODE's for w_n and F_n. For a circular cylindrical shell, these two equations are identical to Simmonds' equations except for terms of order of the inherent error in our shell theory (keeping in mind that $-n^2(\cdot)$ corresponds to $(\cdot)_{,\theta\theta}$) [126].

The above method of reduction may be applied to any shell of revolution, in particular to a conical shell with $r = r_0 + c\xi$, $z = z_0 + s\xi$. For this shell, r_0, z_0, $c = \cos\varphi$ and $s = \sin\varphi$ are fixed geometrical parameters (the meridional slope angle φ being a constant). Except for negligibly small terms of order of the inherent error in our shell theory, the final equations for w_n and F_n (in the absence of surface loads) may be taken in the form [129]:

$$w_n'''' + \frac{2c}{r} w_n''' - \frac{2n^2 + c^2}{r^2} w_n'' + \frac{c(2n^2 + c^2)(n^2 - 1)}{r^3(n^2 - c^2)} w_n' + \frac{n^2(n^2 - 4c^2)(n^2 - 1)}{r^4(n^2 - c^2)} w_n = \frac{s}{rD} F_n'' \tag{5.20}$$

and its static-geometric dual. These equations reduce correctly to the equations for flat plates, shallow shells, and circular cylindrical shells upon specializing the parameters s and c. Similar to the spherical and the cylindrical shell case, the two equations for conical shells ((5.20) and its dual) can be further reduced to a single fourth order complex differential equation for a complex function. This fourth order equation has been shown to have as its complementary solutions a combination of generalized hypergeometric functions with appropriate arguments.

With $-n^2(\cdot)$ corresponding $(\cdot)_{,\theta\theta}$, equation (5.20) is really a sixth order PDE in θ. Only for the special cases of flat plate ($c = 1, s = 0$), shallow shells ($c \simeq 1$) and cylindrical shells ($c = 0, s = 1$) can the equation be integrated into a fourth order equation. This observation is consistent with a similar conclusion in [30] on the reduction of shell equations in general. That is, it is generally not possible to reduce the classical linear shell equations to two simultaneous fourth order PDE for w and F. On the other hand, the method described in [126] allows for a reduction to two simultaneous fourth order ODE for the coefficients of the eigenfunction expansions of w and F in either the ξ or θ variable.

6. Nonlinear theory of shells

6.1. *Introduction*

In the linear theory of elastic shells, the deformation at a given point of the shell is proportional to the magnitude of the applied load. As in the classical three-dimensional elasticity theory, the load magnitude must be sufficiently small for the linear shell theory to be applicable; otherwise, a nonlinear shell theory is required. If the strain-displacement relations are nonlinear but linear stress–strain relations are adequate, we have a *geometrically* nonlinear theory. When the latter are nonlinear, one speaks of a *physically* or *materially* nonlinear theory. Many conventional materials for engineering shell designs are linearly elastic for small strain so that linear stress strain relations are appropriate. Thus for thin shells there is a good reason to formulate geometrically nonlinear theories with the assumption of finite deflection but small strains. More recently, materially nonlinear behavior has been studied for rubber-like shells and membranes; but this will not be included in this review.

A general finite deflection theory in three-dimensional elasticity was developed by Cauchy in 1820 [11]; but it was not until this century that nonlinear strain-displacement relations were introduced in shell theory. In 1859 [57], Kirchhoff proposed a nonlinear plate theory which was later reduced to two simultaneous equations for the normal displacement w and a stress function F by von Kármán in 1910 [119]. Before von Kármán's reduction, A. Föppl had derived a nonlinear theory for flat membranes [25]. All these are geometrically nonlinear theories with small finite deflections: the displacement components u and v tangential to the plane of the membrane or plate are assumed small compared to the normal displacement w (so that the strain-displacement relations are linear with respect to u and v), and only quadratic terms in the change of angles of rotation are retained whenever they appear explicitly in the theory.

The von Kármán equations for flat plates have been studied extensively by both mathematicians and engineers because their simple structure makes them amenable to theoretical and numerical analysis. In cartesian coordinates, these equations are:

$$D\nabla^4 w - [F, w] = p(x, y), \quad A\nabla^4 F + \tfrac{1}{2}[w, w] = 0 \tag{6.1}$$

where F corresponds to the Airy stress function in two-dimensional elasticity, p is the surface load normal to the (undeformed) plate, $\nabla^4 = \nabla^2\nabla^2$ is the biharmonic operator and $[f, g] = f_{xx}g_{yy} + f_{yy}g_{xx} - 2f_{xy}g_{xy}$. The Föppl membrane equations [25] are obtained by setting the flexural rigidity D equal to zero in (6.1).

The earliest papers considering nonlinear behavior of *shells* appear to be those of Donnell [20, 21] on circular cylindrical shells, followed about one year later by the work of Biezeno [3] on shallow spherical shells subject to a point load at the apex. These papers are concerned with one of the most important practical applications of nonlinear shell theory, the prediction of buckling loads and the analysis of the post-buckling behavior. Along this line, the work of von Kármán and Tsien [120] on the buckling of spherical shells under external pressure must also be mentioned (see also [117, 121]). All these papers are for specific shell geometries and small finite deflection behavior.

A nonlinear theory of shallow shells with a middle surface of arbitrary shape given by $z = z(x, y)$ is due to Marguerre [79]; again it is a small finite deflection theory. Formally, the basic equations of that theory can be obtained from (6.1) by replacing the bilinear terms $[F, w]$ and $[w, w]$ by $[F, z + w]$ and $[z + w, z + w] - [z, z]$, respectively. In this way, we obtain

$$\begin{aligned} D\nabla^4 w - [z, F] - [w, F] &= p(x, y), \\ A\nabla^4 F + [z, w] + \tfrac{1}{2}[w, w] &= 0. \end{aligned} \tag{6.2}$$

Due to the linear curvature terms $[z, F]$ and $[z, w]$, the structure of the solution for (6.2) changes drastically from that of (6.1). In particular, it is now possible to describe the phenomenon of buckling under normal pressure.

In a related development, the need for a stress analysis of thin inflatable sheets led Bromberg and Stoker to develop a geometrically *nonlinear membrane theory* [5]. The theory is for axisymmetrical deformations of elastic membranes of revolution and was applied to shallow spherical membranes (see also [25, 108]).

The first geometrically nonlinear theory, based on the Kirchhoff–Love hypotheses for a general class of thin shells and without making a small finite deflection assumption, was given in 1949/50 by E. Reissner [88, 89]. This work is concerned with axisymmetric bending (and stretching) of shells of revolution. The basic nonlinear equations of shell theory were derived with the help of the Kirchhoff–Love hypotheses. These equations were then reduced to a system of two coupled second order ODE for the meridional angle of rotation and a stress function similar to the linear theory. New approaches to the more difficult problem of constructing a geometrically nonlinear two-dimensional theory for unsymmetric deformation of shells of arbitrary shape were undertaken by Leonard [69], Sanders [106], Naghdi and Nordgren [83], and were further developed by Koiter [62], Simmonds and Danielson [113, 114], and E. Reissner [98], among others.

In this section, Reissner's equations for axisymmetric bending of shells of revolution will be reviewed, and some further simplifications due to the small strain assumption will be pointed out. These equations contain some important special cases such as shallow shells, plates and membranes. Some of the recent rigorous mathematical analyses for these special cases will be discussed in Section 7.

6.2. *Reissner's equations for shells of revolution*

The discussion of forces, moments and equilibrium in the (infinitesimal strain) nonlinear theory is formally the same as in Section 2.1–2.3, except that equilibrium of forces and moments must hold for any portion of the *deformed* shell, described by a vector $\mathbf{R}(\xi_1, \xi_2)$. If $\mathbf{N}_i$ and $\mathbf{M}_i$ are taken as stress resultants and couples that measure force and moment per unit length of the *undeformed* shell, then equations (2.3) and (2.4) remain valid if $\mathbf{r}_{,1}$ and $\mathbf{r}_{,2}$ in (2.4) are replaced by $\mathbf{R}_{,1}$ and $\mathbf{R}_{,2}$, respectively. Similarly, equilibrium considerations in Section 5 for shells of revolution carry over to the nonlinear theory. However, the angle φ in (5.3) must be replaced by the angle ϕ made by the *deformed* meridional tangent with the base plane of the undeformed surface of revolution. Since equations (5.4) and (5.5) represent force equilibrium in the (fixed) axial and radial direction, they remain the same for finite deformations. Hence, if p_v is given, a first integral is obtained from force equilibrium in the axial direction in the same way as in the linear theory. Similarly, upon introducing the stress function $\Phi = rH$, N_{22} can be expressed in terms of H by (5.5) (with φ replaced by ϕ). There is only one other equilibrium equation for our problem, namely, the nonlinear moment equilibrium equation

$$(rM_{11})' - \alpha M_{22} \cos \phi + r\alpha(H \sin \phi - V \cos \phi) - \alpha P_2 \sin \phi + r\alpha q_1 = 0. \tag{6.3}$$

In [88, 89, 93, 95, 99], Reissner has obtained expressions for the strains and curvature changes (strain couples) in terms of the radial and axial displacement components u and w, the meridional angle ϕ and the shearing angle γ by a direct geometrical argument. In the present derivation strains ε_{ik}, γ as well as terms of order $O(h/R)$, are neglected compared to unity, but otherwise the strain-displacement relations are geometrically exact. For a shell theory based on the Kirchhoff–Love hypotheses, these relations are [88]

$$\varepsilon_{11} = \frac{\cos \varphi}{\cos \phi}\left(1 + \frac{u'}{r'}\right) - 1, \quad \varepsilon_{22} = \frac{u}{r}, \tag{6.4}$$

$$\kappa_{11} = \frac{1}{\alpha}(\varphi' - \phi'), \qquad \kappa_{22} = \frac{1}{r}(\sin \varphi - \sin \phi). \tag{6.5}$$

The compatibility equation $(r\varepsilon_{22})' - \alpha\varepsilon_{11} \cos \phi = \alpha(\cos \phi - \cos \varphi)$ follows immediately from (6.4) upon elimination of u. For transverse-shear-deformable shells, the corresponding compatibility equation was found in [93, 95, 99] to be

$$(r\varepsilon_{22})' - \alpha\varepsilon_{11} \cos \phi + \alpha\gamma \sin \phi = \alpha(\cos \phi - \cos \varphi) \tag{6.6}$$

where $\alpha\gamma = w' \cos \phi - u' \sin \phi - \alpha \sin (\phi - \varphi)$ and now $\alpha(1 + \varepsilon_{11}) = u' \cos \phi + w' \sin \phi + \alpha \cos (\phi - \varphi)$ (which reduces to the expression in (6.4) if $\gamma = 0$).

The situation is similar to that of the linear theory. The moments M_{ii} and P_2 in (6.3) can be expressed in terms of the variable ϕ via the stress strain relation (2.9), the equations (6.5), and the relation $\lambda_2 = (\cos \varphi - \cos \phi)/r$ derived in [95, 99]. Hence (6.3) reduces to a second order equation in the two unknowns ϕ and $\Phi = rH$. A second equation in the variables ϕ and Φ is given by (6.6), after ε_{ii} and γ are expressed in terms of N_{jj} and Q, which in turn are given in terms of Φ and rV. Thus the equations for finite axisymmetric bending of shells of revolution are reduced to two simultaneous second order ODE for the variables Φ and ϕ. We list Reissner's equations below for the first approximation stress strain relations (3.10)–(3.12) with uniform shell properties [89]

$$\left[\frac{r}{\alpha}(\phi - \varphi)'\right]' + \nu\varphi'(\cos \phi - \cos \varphi) - \frac{\alpha}{r} \cos \phi(\sin \phi - \sin \varphi)$$

$$= \frac{\alpha}{D}(\Phi \sin \phi - rV \cos \phi), \tag{6.7}$$

$$\left[\frac{r}{\alpha}\Phi'\right]' - \left(\frac{\alpha}{r}\cos^2\phi - \nu\phi' \sin \phi\right)\Phi - \left(\frac{\alpha}{r}\cos \phi \sin \phi + \nu\phi' \cos \phi\right) rV$$

$$= \frac{\alpha}{A}(\cos \phi - \cos \varphi) + \nu \sin \phi(rV)' - (r^2 p_H)' - \nu r\alpha \cos \phi p_H. \tag{6.8}$$

A generalization of these equations which incorporates the deformational effect of transverse normal stress may be found in [99].

6.3. *Simplified Reissner's equations*

A simplification of (6.8) was proposed in [93] by observing that in most terms ϕ may be replaced by φ on account of the small strain assumption inherent in the derivation of (6.7) and (6.8). For example, combining the second term in (6.8) $\alpha\Phi \cos^2\phi/r$ with the first term on the right hand side, we have

$$\frac{\alpha}{A}\left\{(\cos\phi - \cos\varphi) + A\frac{\Phi}{r}\cos^2\varphi + \left[A\frac{\Phi}{r}(\cos\phi + \cos\varphi)\right](\cos\phi - \cos\varphi)\right\}.$$

Since the term within the brackets is of order $\varepsilon = \max(|\varepsilon_{11}|, |\varepsilon_{22}|)$, the third term may be neglected in comparison with the first term, which amounts to replacing the term $\alpha\Phi\cos^2\phi/r$ in (6.8) by $\alpha\Phi\cos^2\varphi/r$. A more elegant way of simplifying (6.8) in this fashion was effected in [93] through a variational principle. The result is that ϕ may be replaced by φ in all terms of equation (6.8) except in the term $(\alpha/A)(\cos\phi - \cos\varphi)$.

Another simplification of (6.7), (6.8) was proposed by Koiter [64]. He observed that when a set of intrinsic equations for arbitrary shells derived earlier [65] is reduced to the case of the linear theory of shells of revolution with uniform properties, these equations are simpler than the ones given by (5.8) and its dual. In particular, the terms on the left hand side of (5.8) multiplied by Poisson's ratio disappear. The resulting equations agree with the equations of Simmonds [112] who has given a rigorous justification of neglecting the terms in question. Although in the nonlinear theory a rigorous justification is not available, one might argue heuristically in a similar fashion to arrive at a simplified set of nonlinear equations which is virtually free of Poisson's ratio [64]. More recently, an alternative derivation of these simplified equations has been given in [71]. If ξ is taken as the arclength along a meridian ($\alpha = 1$), these equations take the form

$$r[(\phi - \varphi)' + r^{-1}(\sin\phi - \sin\varphi)]' = \frac{1}{D}(\Phi\sin\phi - rV\cos\phi),$$

$$r[\Phi' + r^{-1}\Phi\cos\phi]' + r[V\sin\phi + rp_H]' = \frac{1}{A}(\cos\phi - \cos\varphi) - (1 + \nu)p_\xi. \tag{6.9}$$

We note that the second equation of (6.9) contains $\cos\phi$ and $\sin\phi$ on the left hand side where the simplified version of (6.8) has only $\cos\varphi$ and $\sin\varphi$.

A somewhat different simplified version of Reissner's equations, obtained by neglecting $O(\varepsilon)$ terms compared to unity, was also derived in [71]. They are virtually free of Poisson's ratio and are significantly simpler than equations (6.9). They may be taken in the form

$$r(\phi - \varphi)'' + (\phi - \varphi)'\cos\varphi - r^{-1}\cos\phi = \frac{1}{D}(\Phi'\sin\phi - rV\cos\phi),$$

$$r\Phi'' + \Phi'\cos\varphi - r^{-1}\Phi = \frac{1}{A}(\cos\phi - \cos\varphi) + (r^2 p_H)' + r\nu p_\xi. \tag{6.10}$$

In contrast to (6.9), ϕ does not appear on the left hand side of the second equation of (6.10). Thus for vanishing tangential load p_ξ, the simplified equations are free of ν. However, it should be recalled that for finite rotations one has relations

$$p_H = p_\xi\cos\phi + p_n\sin\phi, \quad p_v = p_\xi\sin\phi - p_n\cos\phi, \tag{6.11}$$

which involve the independent variable ϕ. Depending on the type of loading, the effect of Poisson's ratio terms in (6.9) and (6.10) may be significant for moderate to large values of the load intensity [145]. The same effect can be significant also for meridionally nonuniform shell properties [99].

In order to solve the basic equations (6.7) and (6.8) or any of the simplified versions thereof, we must formulate boundary conditions consistent with the small strain assumption. In practically all cases of interest, the boundary conditions are nonlinear. For instance, suppose the tangential edge load N_{11}, the moment M_{11} or the horizontal displacement is prescribed at the edge. In terms of ϕ and Φ, these quantities are

$$N_{11} = r^{-1}\Phi\cos\phi + V\sin\phi, \quad M_{11} = D[\alpha^{-1}(\phi - \varphi)' + \nu r^{-1}(\sin\phi - \sin\varphi)],$$
$$u = rA[\alpha^{-1}(\Phi' + rp_H) - \nu(r^{-1}\Phi\cos\phi + V\sin\phi)]. \tag{6.12}$$

It appears that these relations must also be used in conjunction with the simplified versions (6.9) and (6.10) of Reissner's equations.

6.4. Membranes, plates and shallow shells

Very thin shells have negligible bending stiffness. For such shells, neglecting stress couples often yields a good approximation of the shell response under smoothly varying external loads except near the shell edge. Therefore, an important special case of shell theory is a nonlinear membrane theory. Formally, the equations for this theory are obtained by setting $D = 0$ in (6.7), which yields $rV\cos\phi = \Phi\sin\phi$ or $\tan\phi = rV/\Phi$. Thus (6.8) is transformed into a second order equation in the single variable Φ. This equation simplifies considerably if ϕ is replaced by φ in all terms except $(\alpha/A)(\cos\phi - \cos\varphi)$, as discussed above. Then the only nonlinear term in (6.8) is $(\alpha/A)[\Phi/(\Phi^2 + r^2V^2)^{1/2} - \cos\varphi]$. Koiter's version (6.9) is rather inconvenient for membrane theory. On the basis of Simmonds' version (6.10), nonlinear membrane theory is governed by the equation

$$(r\Phi')' - r^{-1}\Phi = \frac{1}{A}\left\{\frac{\Phi}{[\Phi^2 + (rV)^2]^{1/2}} - \cos\varphi\right\} + (r^2p_H)' + r\nu p_s. \tag{6.13}$$

Recalling that the prime in (6.10) means differentiation with respect to the arclength s, it is seen that (6.13) is identical to the equation derived earlier in [16]. Here $\Phi = rN_{11}\cos\phi$, as the assumptions of membrane theory imply $Q = 0$.

In many applications, including buckling problems, shell geometry and prescribed loading are such that the changes in the angle of rotation $\phi - \varphi$ are relatively small. Introducing $\beta = \varphi - \phi$, one may write $\cos\phi = \cos\varphi + \beta\sin\varphi - \frac{1}{2}\beta^2\cos\varphi + \cdots$ and a similar expansion for $\sin\phi$. Retaining only terms up to the second degree in β and in Φ in the equations (6.7), (6.8), we get an approximate theory referred to as a "*moderate rotation theory*" or a "*small finite deflection theory*", which is consistent with the earlier definition of small finite deflections. We only record here the small finite deflection equations resulting from (6.10), as given in [71]

$$D[(r\beta')' - r^{-1}\beta] - \Phi(\sin\varphi + \beta\cos\varphi) + rV(\cos\varphi - \beta\sin\varphi) = 0,$$
$$A[(r\Phi')' - r^{-1}\Phi] + \tfrac{1}{2}\beta^2\cos\varphi + \beta\sin\varphi + A[(r^2p_H)' + \nu rp_s] = 0. \tag{6.14}$$

The equations (6.12) for N_{11}, M_{11} and u also simplify, but they do remain quadratic in β, leading to nonlinear boundary conditions for (6.14).

A special case of moderate rotation theory is the *shallow shell theory* as defined earlier. It is based on the assumption that φ^2 and $\varphi\beta$ are small compared to unity, so that $\cos\varphi = r' \simeq 1$ and $\sin\varphi = z' \simeq dz/dr$. Thus all differentiations in the following equations are with respect to r:

$$D[r\beta'' + \beta' - r^{-1}\beta] - \Phi(z'(r) + \beta) + rV = 0,$$
$$A[r\Phi'' + \Phi' - r^{-1}\Phi] + \beta z'(r) + \tfrac{1}{2}\beta^2 + Ar[rp_H' + (2+\nu)p_H] = 0. \tag{6.15}$$

Now equations (6.12) become linear, that is,

$$N_r = N_{11} = \frac{1}{r}\Phi, \quad M_{11} = D\left(\beta' + \frac{\nu}{r}\beta\right), \quad u = Ar\left(\Phi' + rp_H - \frac{\nu}{r}\Phi\right). \tag{6.16}$$

In the case of a flat circular plate, equations (6.15) agree with the integrated form of the von Kármán equations (6.1). In particular, for vanishing p_H, we have

$$DK[\beta] - \Phi\beta + rV = 0, \quad AK[\Phi] + \tfrac{1}{2}\beta^2 = 0 \tag{6.17}$$

where $K[\cdot] = r[\cdot]'' + [\cdot]' - r^{-1}[\cdot]$. Finally, the small finite deflection equations for a circular membrane under a vertical load are obtained from (6.14) by setting $D = 0$ and eliminating β:

$$AK[\Phi] + \frac{1}{2}\left(\frac{rV}{\Phi}\right)^2 = 0. \tag{6.18}$$

7. Circular membranes and plates

7.1. Small finite deflections of circular membranes

The first approximate solution of Föppl's small finite deflection equations for a circular membrane under uniform pressure with a fixed edge was obtained by Hencky [45] and for annular membranes under various loading and edge conditions by Schwerin [108]. These authors calculated formal power series solutions of the relevant differential equation. Various treatments of these two problems by other numerical methods have appeared in the more recent engineering literature (e.g., see [52], where other references are given). The first rigorous result on Hencky's problem was given in [139]; an existence theorem was proved by establishing the convergence of the power series derived in [45]. A simpler convergence proof for both a fixed edge and a radially stressed edge was given in [140], where uniqueness was also established. Similar results for Schwerin's series solutions were obtained in [141].

In terms of dimensionless variables, equation (6.18) can be written in the form

$$L[y] := y'' + \frac{3}{x}y' = -\frac{2}{y^2}Q^2(x), \quad 0 < x < 1, \quad Q(x) = \frac{2}{x^2}\int_0^x t\bar{p}(t)\,dt, \tag{7.1}$$

where $x = r/a$, $N_r Eh = k^2 y/4$, $k = (2p_0 a/hE)^{1/3}$, $p(r) = p_0\bar{p}(x)$, and $p_0 = \max|p(r)|$ so that $|\bar{p}(x)| \leqslant 1$. $\bar{p}$ is assumed to be piecewise continuous for $0 \leqslant x \leqslant 1$. A solution of (7.1) is sought satisfying the boundary conditions

$$y'(0) = 0, \quad \text{and} \quad \text{(a) } y(1) = S > 0 \quad \text{or} \quad \text{(b) } y'(1) + (1 - \nu)y(1) = H, \tag{7.2}$$

depending on whether N_r or the radial displacement u is prescribed at the edge $r = a$. The BVPs (7.1), (7.2a) and (7.1), (7.2b) define Problem S and Problem H, respectively. Hencky considered Problem H for $H = 0$ and uniform load, that is, $\bar{p} = 1 = Q$. The following theorems hold [140].

(7.3) Problem H for $H = 0$, $Q = 1$ has a unique solution $y(x)$. The solution is positive and can be represented by a series

$$y(x) = \sum_{n=0}^{\infty} a_n x^{2n}, \quad \tfrac{20}{19} < a_0 < 2 \text{ for } \nu = \tfrac{1}{3}.$$

The series $y(x)$ and the series $y'(x)$ obtained by term-wise differentiation of $y(x)$ are both uniformly convergent for $|x| \leqslant 1$.

(7.4) Problem S for $Q = 1$ and $S > S_0 = 4/5$ has a unique solution, which is positive and admits a uniformly convergent power series solution as in (7.3) with $a_0 > 0$.

Numerical experiments indicate convergence for all $S > 0$, but we have not attempted to prove this.

A different and more elegant existence proof for Problems S and H, removing some of the restrictions in (7.3) and (7.4), can be obtained by an integral equation method. The Green's functions G_S and G_H of the operator $-L$ in (7.1) and the boundary conditions $y'(0) = y(1) = 0$ and $y'(0) = y'(1) + (1 - \nu)y(1) = 0$, respectively, are easily calculated. Problems S and H are then equivalent to the integral equations

$$y(x) = C + \int_0^1 G(x, t)\,\frac{2Q^2(t)}{y^2(t)}\,dt =: (Ty)(x) \tag{7.5}$$

where

$$C = \begin{cases} S \\ H(1 - \nu)^{-1} \end{cases}, \quad G(x, t) = \begin{cases} G_S(x, t) \\ G_H(x, t) \end{cases} \text{ in } \begin{cases} \text{Problem } S \\ \text{Problem } H \end{cases}.$$

The special case $Q = 1$, Problem S, was first considered by Dickey [18] who proved that the iteration $y_{n+1} = Ty_n$ converges for $S > 0.648$. The more general case $Q^2(x) > 0$ for both Problems S and H was treated in [143], based on properties of antitone operators. It is easily seen that T is antitone in the sense that if $0 < y(x) \leqslant z(x)$, then $(Ty)(x) \geqslant (Tz)(x)$. A general theorem then implies that the iteration $y_{n+1} = Ty_n$, $y_0 = C$, has the property

$$y_0 \leqslant y_2 \leqslant y_4 \cdots \leqslant y_{2n} \leqslant y_{2n+1} \leqslant y_{2n-1} \leqslant \cdots \leqslant y_3 \leqslant y_1 \tag{7.6}$$

provided that $0 < y_0 \leqslant y_2 \leqslant y_1$. For Problem S, one calculates from (7.5) that $y_1 = S + (1 - x^2)/4S^2$. If $y_1 - y_0$ is sufficiently small, T is contractive and the Banach fixed point theorem implies that both sequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ converge to the same limit $y(x)$, which is a solution of (7.5). In this way, the following result was established [143]:

(7.7) Assume $0 < Q^2(x) \leqslant c$ (by definition $c \leqslant 1$), then Problem S has a unique solution for all $S > 0.5609c^{1/3}$, Problem H has a unique solution for all $H > H_0(v)c^{1/3}$, where $H_0(v)$ is a positive constant depending on v, e.g. $H_0(1/3) \doteq 0.7918$.

The restrictions on S and H can be removed by employing a more powerful albeit non-constructive technique: the Schauder fixed point theorem. It can be shown that the condition (7.6), which is satisfied for all $S > 0$ in Problem S and for all $H > 0$ in Problem H implies that the Schauder theorem is applicable. As a consequence, one obtains the following result [143]:

(7.8) Let $Q^2(x) > 0$, then Problem S (Problem H) has a unique solution $y(x)$ for all $S > 0$ ($H > 0$). The solution satisfies $0 < y_{2n} \leqslant y(x) \leqslant y_{2n+1}$ for all positive integers n.

In order to settle the case $H \leqslant 0$ for Problem H, one must show that for any real H there is a value $S > 0$ such that the solution of Problem S satisfies the condition $y'(1) + (1 - v)y(1) = H$. This has also been done, yielding the result [143]:

(7.9) Let $Q^2(x) > 0$, then Problem H has a unique positive solution $y(x)$ for all real H.

It should be mentioned that the results (7.8) and (7.9) can also be obtained by the "shooting method" [9a]. However, it appears that the proofs are more complicated than the ones given in [143], as one must cope with the singularity of (7.1) at $x = 0$.

The solutions of (7.1), (7.2) discussed thus far are all positive. The question of negative solutions has been fully settled for uniform pressure ($\bar{p} = 1$) in [10], where partial results for arbitrary $\bar{p}(x)$ are also given. Negative solutions are unstable, as a membrane cannot support compressive radial stresses. In fact, even positive solutions will lose stability if $N_\theta < 0$. As shown in [145], the onset of instability is given by $N_\theta = 0$ at the edge $r = a$; the critical value is $S_c \doteq 0.729$. For all $S > S_c$ the circumferential stress is tensile throughout the membrane, but for $S < S_c$ it is compressive in a region adjacent to the edge; hence, wrinkling of the membrane will occur for $S < S_c$ [71].

7.2. *Small finite deflections of annular membranes*

We now report some mathematical results for annular membranes subject to vertical pressure $\bar{p}(x)$. The Föppl small deflection equations reduce to

$$L[y] = -\frac{2}{y^2} R^2(x, \varepsilon), \quad \varepsilon < x < 1, \quad R(x, \varepsilon) = \frac{2}{x^2} \int_\varepsilon^x t\bar{p}(t)\, dt \tag{7.10}$$

where $\varepsilon = r_i/a$, and r_i is the radius of the inner edge of the annulus. Prescribing either the radial traction or the radial displacement at the edges, the boundary conditions are

$$\begin{aligned} y(\varepsilon) &= s \quad \text{or} \quad \varepsilon y'(\varepsilon) + (1 - v)y(\varepsilon) = h \quad \text{at inner edge,} \\ y(1) &= S \quad \text{or} \quad y'(1) + (1 - v)y(1) = H \quad \text{at outer edge.} \end{aligned} \tag{7.11}$$

Schwerin [108] obtained power series solutions for Problem (s, H) with $s = 0$, $H = 0$ in (7.11). Furthermore, he found a closed form solution for a membrane subjected to axial edge load, fixed at the inner and outer edges ($h = H = 0$) in the absence of surface loads. The solutions for a free inner edge ($s = 0$) and uniform pressure ($\bar{p} = 1$) can be expressed in terms of solutions of the circular membrane. More precisely, we have [36]:

(7.12) Problem (s, S), $s = 0$: Let $z(\xi)$ be the solution of (7.1) with $Q = 1$ and $z(1) = S_\varepsilon := (1 - \varepsilon^2)^{-4/3} S > 0$. Then there exists a unique solution $y(x)$ of (7.10), (7.11) with $R = 1 - (\varepsilon/x)^2$. This solution is positive and has the form

$$y(x) = (1 - \varepsilon^2)^{1/3} \left(1 - \frac{\varepsilon^2}{x^2}\right) z(\xi), \quad \xi := \left(\frac{x^2 - \varepsilon^2}{1 - \varepsilon^2}\right)^{1/2}. \tag{7.13}$$

(7.14) Problem (s, H), $s = 0$: Let $z(\xi)$ be the solution of (7.1) with $Q = 1$ and $z'(1) + (1 - \nu_\varepsilon) z(1) = H_\varepsilon := (1 - \varepsilon^2)^{-1/3} H$ where $\nu_\varepsilon = \nu - \varepsilon^2(1 + \nu)$. Then there exists a unique solution $y(x)$ of (7.10), (7.11) with $R = 1 - (\varepsilon/x)^2$. This solution is positive and has the form (7.13).

Proposition (7.14) covers the Schwerin problem $s = H = 0$. Employing formal power series solutions, Schwerin argued that the stress concentration factor at a traction-free hole should be 2 in the limit $\varepsilon \to 0$, that is,

$$\lim_{\varepsilon \to 0} \frac{\sigma_\theta(x = \varepsilon)}{\sigma_{\theta,0}(x = 0)} = 2 \tag{7.15}$$

where σ_θ refers to the circumferential stress at the inner edge of the annular membrane and $\sigma_{\theta,0}$ refers to the circumferential stress at the center of a circular membrane, both fixed at the outer edge ($H = 0$). The relation (7.15) can be shown to be an exact consequence of (7.12)–(7.14) for both Problems (s, S) and (s, H), with $s = 0$ but arbitrary $S > 0$ and real H. There σ_θ and $\sigma_{\theta,0}$ refer to solutions of Problem $(0, S)$ and Problem S, respectively (or to Problem $(0, H)$ and Problem H, respectively), with the same radial tension S (or displacement H) being applied at the outer edge of the membranes [36].

The analysis of the annular membrane problems defined by (7.10) and (7.11) for $s \geqslant 0$, $S > 0$, h, H real and arbitrary load $\bar{p}(x)$ is more complicated than in the circular membrane problems. Some preliminary results using the integral equation method were obtained in [141], but for quite restricted ranges of the boundary parameters s, S, h, and H. Multiplying (7.1) by x^3, integrating from 0 to x and using $x^3 L[y] = (x^3 y')'$ and $y'(0) = 0$, it is readily seen that $y'(x) \leqslant 0$ for any solution of equation (7.1). The solutions of the differential equation (7.11) in general do not have this property. However, a transformation introduced by Schwerin [108] for the purpose of simplifying his numerical solutions turns out to be crucial for the theoretical analysis. The change of variable $y(x) = (1 - \varepsilon^2)^{4/3} g(z)/x^2$ and $z := (x^2 - \varepsilon^2)/(1 - \varepsilon^2)$ maps any positive regular solution $y(x)$ of (7.11), with $\varepsilon < x < 1$, into a positive regular solution $g(z)$ of the differential equation

$$\frac{d^2 g}{dz^2} + \frac{1}{g^2} \tilde{R}^2(z, \varepsilon) = 0, \quad 0 < z < 1, \quad (1 - \varepsilon^2)\tilde{R}(z, \varepsilon) = \int_\varepsilon^{x(z)} t\bar{p}(t)\, dt \tag{7.16}$$

where $x(z) := [z(1 - \varepsilon^2) + \varepsilon^2]^{1/2}$. Similarly the boundary conditions (7.11) can be expressed in terms of the new variables z and $g(z)$. It is clear from (7.16) that any solution $g(z)$ must be concave. Using this property and transforming (7.16) and the boundary conditions for g into an integral equation, the problem of the existence of solutions was fully resolved in [36] for Problem (s, S), $s > 0$ and for Problem (s, H), provided the condition $s + H(1 - \varepsilon^2)/2\varepsilon^2 > 0$ holds. This condition has been removed by Grabmüller and Novak [33], who also treated the more difficult cases of Problems (h, S) and (h, H).

7.3. *Finite rotation of circular and annular membranes*

A mathematical analysis of the boundary value Problem S posed by Reissner's finite rotation theory for circular and annular membranes under axisymmetrical surface loads was virtually nonexistent until 1980. Without surface load, the problem simplifies considerably. Two cases were treated by Clark and Narayanaswamy [16]: a membrane of revolution with a uniform radial edge load and a membrane with uniform edge load parallel to the axis of revolution. The first problem leads to a linear equation solvable in closed form. The second problem is genuinely nonlinear, it includes the special case of a flat annular membrane with transverse edge load, treated by Schwerin for the Föppl small deflection theory. The nonlinear differential equation is solved by numerical and asymptotic methods.

In the presence of surface load, the integral equation technique used for the Föppl membrane equations was extended in [141] to obtain existence results for a rather restricted range of the boundary data. In terms of the dimensionless variables x and y, equation (6.13) for an annular membrane with given vertical load can be written as

$$L[y] = \frac{4}{(kx)^2}\left(1 - \frac{y}{[y^2 + k^2x^2R^2(x, \varepsilon)]^{1/2}}\right) - \underline{\frac{2\nu k^2 R(x, \varepsilon)}{[y^2 + k^2x^2R^2(x, \varepsilon)]^{1/2}}}, \quad \varepsilon < x < 1, \tag{7.17}$$

where x, y and k are defined in Section 7.1. The circular membrane problem corresponds to (7.17) in the formal limit as $\varepsilon \to 0$. In that case, solutions of (7.17) are sought satisfying the boundary conditions

$$y'(0) = 0 \quad y(1) = S \quad \text{or} \quad y'(1) + y(1) - \nu[y(1) + k^2R^2(1, \varepsilon)]^{1/2} = H. \tag{7.18}$$

In the case of an annular membrane, $\varepsilon > 0$, the condition $y'(0) = 0$ is replaced by

$$y(\varepsilon) = s \quad \text{or} \quad \varepsilon y'(\varepsilon) + (1 - \nu)y(\varepsilon) = h.$$

Again the four boundary value Problems (s, S), (h, S), (s, H) and (h, H) result, two of them involve a nonlinear boundary condition at $x = 1$, if ν is different from zero.

In the case of a circular membrane, Problem S for $S > 0$ and Problem H for $H > 0$ were recently solved in the special case $\nu = 0$, using the Schauder fixed point theorem [143]. As pointed out in Section 6.2, it is generally believed that the term multiplying ν in (7.17) is small compared to the other terms and therefore has little influence on the solution. It is seen that, if the underlined term in (7.17) is dropped, the right hand side of (7.17) has the same monotonicity properties as in equation (7.1). Introducing appropriate Green's functions $G_i(x, t)$ with $i = 1$ for Problem S and $i = 2$ for Problem H, the boundary value problems

(7.17), (7.18), $\varepsilon = 0$, are seen to be equivalent to the integral equations

$$y(x) = C_i + \int_0^1 G_i(x, t)F(t, y(t))\,\mathrm{d}t + D_i[y^2(1) + k^2R^2(1, 0)]^{1/2}, \quad i = 1, 2, \tag{7.19}$$

where $C_1 = S$, $D_1 = 0$ in Problem S and $C_2 = H$, $D_2 = \nu$ in Problem H, and where

$$F(t, y) = \frac{1}{t}\left(1 - \frac{y}{[y^2 + k^2t^2R^2(t, 0)]^{1/2}}\right).$$

In order to find solutions of (7.19), for $\nu \neq 0$, and solutions to a corresponding integral equation for annular membranes for the full range of physically admissible parameters s, S, h, H, a novel theorem on the existence of positive solutions of integral equations of the type (7.19) due to Novak [33] is needed. Grabmüller and Pirner [34, 35] have succeeded in obtaining a complete existence and uniqueness theory of positive solutions of both circular and annular membranes based on equation (7.17), without the underlined term. The results differ markedly from those for the Föppl membrane model.

7.4. Small finite deflections of circular and annular plates

An approximate solution in terms of power series of the van Kármán equations for a circular plate under uniform normal pressure fixed at the edge was first given by Way [136]. In terms of the dimensionless variables $f = -m^2w'(x)/hx$, $g = m^4a^2N_r/Eh^3$, $x = r/a$, $m^4 = 12(1 - \nu^2)$ equations (6.17) reduce to

$$L[f] = 2\gamma Q(x) + fg, \quad L[g] = -\tfrac{1}{2}f^2, \quad 0 < x < 1, \tag{7.20}$$

where L and Q have been defined in (7.1) and γ is the dimensionless load parameter $\gamma = m^6a^4p/4Eh^4$. At the edge $r = a$ ($x = 1$) we may prescribe either the angular deflection β or the bending moment M_r and either the radial stress N_r or the radial displacement u. Together with regularity conditions at the center of the plate, we thus have the boundary conditions

$$f'(0) = g'(0) = 0, \tag{7.21}$$

$$\begin{aligned} f(1) &= C \quad \text{or} \quad f'(1) + (1 + \nu)f(1) = M, \\ g(1) &= S \quad \text{or} \quad g'(1) + (1 - \nu)g(1) = H. \end{aligned} \tag{7.22}$$

Hence we obtain four boundary value problems for (7.20), denoted by Problems *CS*, *CH*, *MS*, and *MH*, depending on the choice among conditions (7.22). For example, a clamped edge corresponds to Problem *CH* with $C = H = 0$, while a simply-supported edge corresponds to Problem *MH* with $M = H = 0$. Formal power series of (7.20) and (7.21) are easily computed, but the problem is to determine the leading coefficients of the series for f and g such that (7.22) is satisfied. An existence proof based on this or other shooting technique is not known. After the leading coefficients have been computed, the convergence of the series can be proved a posteriori, for sufficiently small γ, in order to ascertain the validity of Way's numerical solutions [137].

The first rigorous mathematical analysis of equation (7.20) has been given by Friedrichs and Stoker [28] for the buckling of a circular plate under radial thrust. This will be discussed in the next section dealing with buckling problems. An existence theory for the circular plate under normal pressure, for a restricted range of γ, was obtained by Keller and Reiss [55], applying the Banach fixed point theorem to an integral equation formulation of the problem. By introducing Green's function of L for homogeneous boundary conditions for $g(x)$, the second of equation (7.20) can be solved for g in terms of f. Substituting this expression into the first of equations (7.20), an integral equation for f is obtained in the form

$$V(x) = J\{V(x)\} + \Gamma(x), \quad V(x) := xf(x), \tag{7.23}$$

where J is a nonlinear (double) integral operator. In view of the positivity of $g(x)$, the operator J turns out to be antitone in the sense defined above for the operator T in (7.5). Consequently, the known results for such operators are applicable and yield the following theorem [55]:

(7.24) If $Q(x) \geqslant 0$ and $J\{\Gamma(x)\} + \Gamma(x) \geqslant 0$ then the sequence defined by $V_{n+1} = J\{V_n\} + \Gamma$, $V_0 = 0$ is alternating as in (7.6). Moreover, if $0 \leqslant \gamma < \gamma_0$, the sequence V_n converges to the unique solution of (7.23). The number γ_0 depends on the boundary data C, H, M, and S. For a clamped edge $\gamma_0 = 34.1$.

There are two alternative existence proofs available for the circular plate problems, one using a variational technique [122] and one using Hammerstein's method of proving existence of nonliner integral equations by expanding the solution in a series of eigenfunctions of associated linear problems [19]. The uniqueness of the solution for a restricted range of γ is implied in (7.24). For arbitrary γ, uniqueness was established in [87], using the results of [122]. Recently, an elementary proof was given [144], which at the same time covers uniqueness of solutions for a variety of annular plate boundary value problems.

A solution (f, g) of (7.20) will be called tensile, if $g(x) \geqslant 0$, $0 \leqslant x \leqslant 1$. It follows from a general theorem for shallow shells discussed in Section 8.1 that tensile solutions of (7.20)–(7.22) are unique. Multiplying the second of equations (7.20) by x^3 and integrating shows that $g'(x) \leqslant 0$. If $g(1) = S \geqslant 0$, then we have $g(x) \geqslant S$. If $g'(1) + (1 - \nu)g(1) = H \geqslant 0$, then $g'(1) \leqslant 0$ implies $g(1) \geqslant 0$ and therefore $g(x) \geqslant 0$. Thus the solutions are tensile and we conclude

(7.25) The solutions of the circular plate Problems CS, CH, MS, and MH are unique provided that $S \geqslant 0$ or $H \geqslant 0$. (Buckling may occur if $S < 0$ or $H < 0$).

For the annular plate problems, $Q(x)$ in (7.20) must be replaced by $R(x, \varepsilon)$, using the notation introduced in (7.10), and (7.21) must be replaced by the boundary conditions

$$\begin{aligned} f(\varepsilon) &= c \quad \text{or} \quad \varepsilon f'(\varepsilon) + (1 + \nu) f(\varepsilon) = m, \\ g(\varepsilon) &= s \quad \text{or} \quad \varepsilon g'(\varepsilon) + (1 - \nu) g(\varepsilon) = h. \end{aligned} \tag{7.26}$$

The boundary value problems for an annular plate are then defined by (7.20), (7.22) and (7.26). Under certain restrictions on the constants S, H, s and h, the solutions of the annular plate problems are unique [144].

7.5. Comparison between moderate and large rotation solutions

For a given membrane or plate problem, it is of interest to determine whether small finite deflection theory (for *moderate rotation*) will yield results of sufficient accuracy, or whether the finite deflection equations (for *large rotation*) should be used. As rigorous error estimates are usually not available, a number of comparisons based on numerical solutions have been made. For membranes subjected to edge loads and point loads, differences in the solutions have been analyzed [16, 71]. Some results for membranes under uniform surface loads have been obtained in [141]. The differences in both problems are measured by a parameter proportional to $(P/Eh)^{1/3}$ where P is either the edge load or the surface load intensity. While the differences are small for sufficiently small $(P/Eh)^{1/3}$, the range of existence of solutions for the Föppl membrane may be entirely different from the existence range for the Reissner membrane. As shown in [34, 35] the geometrically exact theory may permit only bounded solutions that are in some sense small, while the approximate theory has large solutions under the same loading conditions.

Comparisons for circular plates have also been made in a number of papers and are summarized in [70, 71], where references can be found. It appears that the differences between moderate and large rotation theory are less than for membranes. Once the error is in the 10%-range, the strains usually become so large that small strain theory is no longer applicable.

8. Buckling problems for plates and shells

8.1. Uniqueness of positive solutions for shallow shells

Assuming the solutions of the membrane problems described by the differential equations (7.1), (7.10), (7.19) to be positive in the sense that $N_r \geqslant 0$, it was shown that the solutions are necessarily unique. We now ask the corresponding question for the more complex problem of axisymmetric bending and stretching of shallow shells of revolution. When loads are applied to the surface and to the edge, we may expect, intuitively, and by computational experience with special cases, that uniqueness of solutions will hold so long as the stresses throughout the shell are predominantly tensile. Shells may buckle, but buckling is usually caused by compressive stresses and in that case uniqueness can be expected only for sufficiently small loads. These intuitive conclusions have been confirmed in [144]. It is assumed that the meridional stress resultant $N_{11} = N_\xi$ is positive throughout the shell except possibly at an edge where $N_\xi = 0$ is admitted. No assumptions are made concerning the circumferential stress resultant $N_{22} = N_\theta$ or the bending moments M_ξ and M_θ. It turns out that $N_\xi \geqslant 0$ implies uniqueness of the axisymmetric solutions for a class of shell problems including shallow shells, open or closed at the apex. We note that $N_\xi \geqslant 0$ does not imply global uniqueness, as shells of revolution may buckle nonsymmetrically if N_θ is negative.

The basic small finite deflection equations for shallow shells are given in Section 6.4. We take them in the dimensionless form

$$L[f] = -Z(x)g + fg + 2\gamma R(x, \varepsilon), \quad L[g] = Z(x)f - \tfrac{1}{2}f^2, \tag{8.1}$$

where $Z(x)$ relates to the geometry of the undeformed middle surface. All other symbols in (8.1) were explained in Section 7. At the outer edge, we have the conditions given in (7.22). For a dome-shell we have the conditions (7.21) at the apex while for annular shells we have (7.26) at the inner edge. Selecting two boundary conditions for f and g from (7.22) we obtain Problems *CS*, *CH*, *MS* and *MH* defined earlier. By selecting two conditions at the inner edge and two conditions at the outer edge for f and g, we obtain 16 different boundary value problems for open (annular) shells, denoted by (*cs*, *CS*), (*cs*, *CH*), . . . , (*mh*, *MH*). Positivity is again defined by $g(x) \geqslant 0$.

The uniqueness of positive solutions (f, g) is a simple consequence of the mathematical structure of equations (8.1) and of a detailed study of the boundary terms obtained by integration by parts. We briefly indicate the procedure. Let (f_1, g_1) and (f_2, g_2) be two solutions of one of the boundary value problems defined above. By setting $v = f_1 - f_2$ and $w = g_1 - g_2$, we have from (8.1)

$$\begin{aligned} L[v] &= -Zw + f_1 g_1 - f_2 g_2 = -Zw + g_1 v + f_2 w, \\ L[w] &= Zv - \tfrac{1}{2}(f_1^2 - f_2^2) = -\tfrac{1}{2}(f_1 + f_2 - 2Z)v, \end{aligned} \tag{8.2}$$

with homogeneous boundary conditions for v and w. Now we multiply $vLv + wLw$ by x^3 and integrate. After some algebra, we get

$$\begin{aligned} \int_\varepsilon^1 (vL[v] + wL[w])x^3 \,\mathrm{d}x &= -\int_\varepsilon^1 x^3(v'^2 + w'^2)\,\mathrm{d}x + B_\varepsilon + B_1 \\ &\quad + \int_\varepsilon^1 [-Zvw + g_1 v^2 - f_2 vw - \tfrac{1}{2}vw(f_1 + f_2 - 2Z)]x^3\,\mathrm{d}x \\ &= \frac{1}{2}\int_\varepsilon^1 x^3 v^2 (g_1 + g_2)\,\mathrm{d}x. \end{aligned} \tag{8.3}$$

For a dome shell, $\varepsilon = 0$ and the boundary term B_ε at the apex is zero while $B_1 = v(1)v'(1) + w(1)w'(1)$ which is nonpositive in all problems *CS*, *CH*, *MS* and *MH*. Since the last term in (8.3) is nonnegative by assumption, the identity, together with the homogeneity of the boundary conditions for v and w, implies that $v = w = 0$. For shells with an inner edge, the same result is obtained only if $B_\varepsilon = -\varepsilon^3[v(\varepsilon)v'(\varepsilon) + w(\varepsilon)w'(\varepsilon)]$ is zero, that is, if the boundary conditions are $f(\varepsilon) = c$ and $g(\varepsilon) = s$. In all other cases $B_\varepsilon \geqslant 0$. In order to get the proper sign for the boundary term at $x = \varepsilon$, we apply the Schwerin type transformation for f and g:

$$z = \frac{x^2 - \varepsilon^2}{1 - \varepsilon^2}, \quad f(x) = \frac{1}{x^2}\bar{f}(z), \quad g(x) = \frac{1}{x^2}\bar{g}(z). \tag{8.4}$$

Let (f_i, g_i), $i = 1, 2$, again be two positive solutions (that is, $g_i(x) \geqslant 0$) of the relevant boundary value problem, then taking the weighted differences $v(z) = (\bar{f}_1 - \bar{f}_2)/p(z)$, $w(z) = (\bar{g}_1 - \bar{g}_2)/q(z)$ with $p, q > 0$, it turns out that p and q can be so chosen that the uniqueness proof for all 16 cases defined above can be carried out on the basis of an identity similar to (8.3). The details are found in [144].

8.2. Circular plates under radial load

A two-dimensional generalization of the simplest buckling problem for an elastic rod under axial compression (Euler column) is the thin circular elastic plate under a uniform radial compressive thrust P applied along its edge in the plane of the plate. One equilibrium state is a uniform radial compression where the plate remains planar; it is called the *unbuckled state*. This state becomes unstable when P is sufficiently large, and new equilibrium states arise called *buckled states*, which occur in pairs with equal and opposite axial deflections. Introducing the new variables $F = f$, $G = g + \lambda$ where λ is the dimensionless radial thrust, equations (8.1) for $Z = 0$ and $\gamma = 0$ take the form

$$L[F] + \lambda F = FG, \quad L[G] = -\tfrac{1}{2}F^2. \qquad (8.5, 8.6)$$

If the edge of the plate is clamped, we have the boundary conditions:

$$F(1) = 0, \quad G(1) = 0. \qquad (8.7)$$

The trivial solution $(F, G) = (0, 0)$ represents the unbuckled state. If (F, G) is a nontrivial solution for some λ, then clearly $(-F, G)$ also is a solution for this value of λ. The boundary value problem (8.5)–(8.7) constitutes a bifurcation problem. From each of the eigenvalues of the linearized problem (all simple), there bifurcates a branch of nontrivial solutions representing buckled states. The linearized problem can be solved in terms of the Bessel function $J_1(\lambda x)$, where $\lambda = \lambda_n = j_{1,n}$, the n-th zero of J_1. Friedrichs and Stoker [28] proved that the trivial solution of (8.5)–(8.7) is the only solution for $\lambda \leqslant \lambda_1$, that there is a pair of buckled states with no internal node for all $\lambda > \lambda_1$ and that there are no other solutions when $\lambda \leqslant \lambda_2$. The problem was taken up again by Keller, Keller and Reiss [54], who conjectured that, in addition to the trivial solution, there should be exactly n pairs of buckled states when $\lambda_n < \lambda \leqslant \lambda_{n+1}$, where one pair has no internal node (bifurcating from λ_1), the next pair has one internal node (bifurcating from λ_2), and so on up to $n - 1$ internal nodes. They showed by the Poincaré perturbation method that for any positive integer n, a pair of buckled states with $n - 1$ internal nodes exists (locally), when λ is slightly larger than λ_n. Wolkowisky [148] was able to prove, by using the Schauder fixed point theorem, that these pairs of buckled states continue to exist for all $\lambda > \lambda_n$ (see also [12, 149]).

8.3. Shallow spherical shells under external pressure

This problem has an extensive literature. Along with a cylindrical shell under axial compression, it is one of the most frequently discussed buckling problems in shell theory. The basic equations for this problem are given by (8.1), with $Z(x) = \mu^2$ where Reissner's number $\mu^2 = \sqrt{12(1 - \nu^2)}H/h$ characterizes the shell geometry with H being the rise of the shell above the base plane. Boundary conditions for a simply supported edge or a clamped edge are given by appropriate equations in (7.22). Formal power series solutions of this boundary value problem were obtained in [116]; their convergence was verified later in a posteriori sense for a limited range of values of the parameters involved [137]. An existence proof for an unrestricted range of μ^2 and γ was obtained in [122], using a variational formulation of the BVP. Of special interest is the possibility of axisymmetric snap-through buckling.

Mathematically, it is characterized by a limit point and by multiple solutions of the boundary value problem formulated above. Though an exact analysis of solution multiplicity is not available, we describe here some known results, as we believe that the spherical shell problem constitutes an important chapter in the general theory of buckling of elastic shells.

Up to 1958 there had been a number of theoretical investigations using approximate analytical methods; but the results on the buckling loads were in severe disagreement with each other, and with experimental values. In 1959/60 two papers appeared, written independently, in which two different numerical methods, implemented on a high speed digital computer, were employed to calculate accurate values for the buckling load γ_c as a function of μ^2. The results were in excellent agreement with each other [6, 137]. They show that the solution of equations (8.1), for $\varepsilon = 0, R = 1, Z = \mu^2$ and clamped edge conditions, is unique for $\mu^2 \leqslant \mu_0^2$; but for $\mu^2 > \mu_0^2$ at least three equilibrium states exist in some range of γ, and the shell buckles at some critical value γ_c (limit point). In a load deflection diagram for a fixed μ^2, the curves are monotone increasing for $\mu^2 \leqslant \mu_0^2$, but they show the S-shape or more complicated behavior typical for limit points if $\mu^2 > \mu_0^2$. The investigation of this problem has shown that traditional approximate methods used in engineering may fail for a more complex problem. The important role of accurate and reliable methods of modern numerical analysis was recognized in connection with this genuinely nonlinear problem. In fact, this problem has often been used in recent years as a test example, to demonstrate the performance of a new numerical method [105, 147].

The results of [6, 137] for the axisymmetric buckling load are still significantly higher than the experimental values. The question then arose whether the axisymmetric states are stable with respect to nonsymmetric perturbations. This problem was investigated as follows: Assume

$$w(r, \theta) = w_0(r) + w_n(r) \cos n\theta, \quad F(r, \theta) = F_0(r) + F_n(r) \cos n\theta, \tag{8.8}$$

where n is an integer and w_0, F_0 refer to the above axisymmetric solution. Assuming $|w_n| \ll |w_0|, |F_n| \ll |F_0|$, we substitute (8.8) into the Marguerre shallow shell equations (6.2), transformed into polar coordinates. Upon linearizing the resulting equations with respect to w_n and F_n, we obtain a system of linear homogeneous differential equations for $w_n(r)$ and $F_n(r)$ with homogeneous boundary conditions. Given n, one must find the smallest value γ_n for which this system has a nontrivial solution. For a given value of μ^2, the nonsymmetric buckling load γ_A is then taken as the minimum of all γ_n, varying n. These calculations were first performed independently in [48] and [138], by two entirely different methods. The results agree and show that we have $\gamma_A > \gamma_c$ only in a narrow range $\mu_0 < \mu < \mu_1$, whereas for $\mu > \mu_1$, γ_A drops significantly below the symmetrical critical load γ_c. As is seen from the above outline, the determination of γ_A amounts to solving a nonlinear eigenvalue problem for each n. In modern terminology, one has symmetry-breaking bifurcation points at the value γ_n. According to the results in [48, 138], the eigenvalues γ_A are either simple or double (depending on μ), but this has not been proved rigorously. The values of γ_n have been re-calculated by a recently developed numerical algorithm, using concepts from bifurcation theory [142].

There still remains a gap between theory and experiment, although there is no doubt that nonsymmetric deformations do play a dominant role in the buckling process. Carefully conducted experiments on accurately manufactured almost perfect spherical shell caps have narrowed the gap considerably [53]. The remaining difference can be attributed to a phenomenon called *imperfection sensitivity*, a property exhibited by many shell problems for certain type of loadings. It leads to a certain unavoidable scatter in the experimental data. Imperfection sensitivity will be described in the next section. It turns out that a shallow spherical shell under external pressure is indeed imperfection sensitive [49].

8.4. Complete spherical shells under external load

A complete thin spherical shell under external pressure contracts uniformly until a critical load is reached, when deflections of a nonspherical shape appear. This classical buckling load, defined by a linear eigenvalue problem has been known for a long time [150]. Consideration of nonsymmetric deformations [118] yields the same lowest eigenvalue as in the case of axisymmetric buckling [150]. This buckling load is $p_c = 4Eh^2\sqrt{12(1-\nu^2)}/a^2$. In contrast to the shallow shell problem, the axisymmetric buckling problem here is of the bifurcation type. The objective then is to determine the postbuckling solution branches, as in the problem of the circular plate under radial thrust. However, this task is much more difficult in the spherical shell problem for reasons to be discussed presently.

The governing equations are obtained upon specialization of (6.14) to spherical geometry, $r = a\sin\xi, z = -a\cos\xi, 0 \leqslant \xi \leqslant \pi$, assuming small finite deflections for loads near the buckling load. These equations can be scaled to the following dimensionless form [67]

$$\begin{aligned}\delta(\beta_s'' + \beta_s'\cot\xi - \beta_s\cot^2\xi) + 2\lambda\beta_s + \psi_s &= \beta_s\psi_s\cot\xi,\\ \delta(\psi_s'' + \psi_s'\cot\xi - \psi_s\cot^2\xi) - \beta_s &= -\tfrac{1}{2}\beta_s^2\cot\xi\end{aligned} \tag{8.9}$$

where $\delta = (h/a)[12(1-\nu^2)]^{-1/2}$ and $\lambda = p/p_c$. Solution regularity requires $\beta_s = \psi_s = 0$ at $\xi = 0$ and $\xi = \pi$. The uniformly contracted state corresponds to $\beta_s = \psi_s = 0$ for $0 \leqslant \xi \leqslant \pi$. Bifurcation points are given by the eigenvalues of the linearized problem (8.9), that is,

$$\lambda = \lambda_n(\delta) = \frac{1}{2}\left(\delta\mu_n + \frac{1}{\delta\mu_n}\right), \quad \mu_n := n(n+1) - 1, \quad n = 1, 2, \ldots .$$

The eigenvalues are discrete and either simple or double (if $\delta^2 = 1/\mu_n\mu_m$). Hence the classical methods of analyzing the branches in the neighbourhood of λ_n, such as the Lyapunov–Schmidt method are applicable. However, the branching problem is singular in the sense that the eigenvalues are closely spaced for small δ. In fact, the difference of two eigenvalues near the critical value $\lambda_c = 1$ satisfies $\lambda_m - \lambda_n = O(\delta)$ as $\delta \to 0$. This property is typical for buckling problems of nonshallow shells. It is responsible for the limited practical applicability of the results of the standard perturbation theory. In fact, its range of validity is restricted by $\lambda - \lambda_n = O(\delta)$, as $\delta \to 0$. This was discussed by Koiter [63], who applied his general theory of stability and imperfection sensitivity to study the problem [58, 60].

A new method for obtaining solutions with a much larger region of validity was developed by Lange and Kriegsmann, using techniques of singular perturbation theory [66, 67]. In experiments, the advanced buckled state usually observed is an inward dimple. By means of the new method [67], the postbuckling branch can be continued from the immediate vicinity of the bifurcation point into the region where the buckled shape is essentially an inward dimple at the poles. Numerical calculations in [67] have confirmed the validity of the boundary layer solution. Similar calculations have been carried out by Gräff et al. [37]. According to these results, the axisymmetric postbuckling branch sharply decreases in a load deflection diagram and therefore presents unstable states of equilibrium.

The importance of an initial postbuckling analysis lies in the fact that it provides useful information about the behavior of an imperfect structure. In the case of an unstable descending postbuckling path for the perfect structure, the load deflection curve of the imperfect structure has a limit point for a load smaller than the bifurcation load. The amount of this drop depends on the size of the imperfection and on the form of the postbuckling path of the perfect structure. An indication of the latter is provided by its initial slope at the branch point, which can always be obtained by a local analysis. The postbuckling branch calculated in [67] permits a more complete assessment of the strong imperfection-sensitivity of the complete spherical shell. Although the analysis is restricted to axisymmetric imperfections, it is not unrealistic, because the production process of spherical shells often favors axisymmetric imperfections in shape and thickness (see also [22, 37]).

9. Asymptotic methods

9.1. The flat membrane as the limiting case of a plate

The equations for axisymmetric deformation of membranes of revolution result from the shell equations (6.7), (6.8) or their simplified versions by setting $D = 0$. Thus the fourth order problem (6.7), (6.8) is reduced to a second order problem whose solution generally cannot satisfy the four boundary conditions prescribed for the shell problem. This situation is typical for a singular perturbation problem. Asymptotic methods have been developed in recent years for this type of linear boundary value problems (BVP) and certain classes of nonlinear BVPs.

In dimensionless form, the equations of shell theory usually contain one or more small parameters when the shell is sufficiently thin. In most cases, boundary or interior layers with rapidly changing solution behavior occur, and an accurate numerical solution of the relevant BVP is difficult. Fortunately, the state of stress is usually given by the membrane theory or the inextensional bending theory (corresponding to the limiting case of $A = 0$) throughout the shell except in the layers. The possibility of layer phenomena in linear shell theory was recognized at the turn of the century [9, 76]. Boundary layer theory has been applied to nonlinear membrane and plate problems in [5, 14, 28] and in numerous recent papers on shells. We restrict ourselves to some selected examples of nonlinear problems, where the use of boundary layer theory has been essential.

Physical reasoning suggests that a circular plate under uniform normal pressure (Section 7.4) should behave like a membrane if h is so small that bending stiffness becomes negligible or if the load intensity becomes extremely large. In either case the parameter $\gamma = m^6 a^4 p/4Eh^4$ in (7.20) is large. Setting $\varepsilon = \gamma^{-1/3}$ and changing the variables f, g to $I = \varepsilon f$, $J = \varepsilon^2 g$, equations (7.20) for $Q = 1$ become

$$\varepsilon^2 L[I] = IJ + 2, \quad L[J] = -\tfrac{1}{2}I^2. \tag{9.1}$$

The limit case $\varepsilon = 0$ is precisely the circular membrane problem discussed in Section 7.1, for which $I_0 J_0 + 2 = 0$ and $L[J_0] = -2/J_0^2$. An asymptotic analysis for small ε of (9.1) has been given by Bromberg [4]. The solution of (9.1) is assumed in the form of a power series in ε called the "outer solution"; its leading term is the membrane solution I_0, J_0. The outer solution can only satisfy the boundary condition for $g(x)$. It must be supplemented by an "inner solution" valid in the boundary layer, satisfying the relevant boundary conditions of the problem and matching smoothly, to any order of ε, with the outer solution. The technique of constructing inner and outer expansions and matching the two in an appropriate overlap domain has become a standard tool in applied mathematics.

An important result of Bromberg's analysis is that the circular plate bending problem can be solved accurately for all positive values of the single parameter γ, by numerical methods for $0 < \gamma < \gamma_N$, say, and by the asymptotic method for $\gamma > \gamma_A$ where $\gamma_A < \gamma_N$. This type of program has been carried out for numerous other and more difficult shell problems. In the present problem, for $\varepsilon \ll 1$, bending moments take on significant values only in a narrow strip of width ε near the edge $\gamma = a$.

9.2. Boundary layer problems for shallow shells

The shallow shell equations (8.1) contain two parameters. When the rise-to-thickness parameter (Reissner's number) $\mu^2 = \sqrt{12(1 - \nu^2)}H/h$ is of order unity, the shell behaves in a plate-like manner and the foregoing analysis for $\gamma \to \infty$ can be extended to such shells. For a fixed limited range of γ, boundary layer behavior also occurs if the Reissner number μ^2 becomes large [89, 90]. From the definition of μ^2, the shell becomes very thin as $\mu^2 \to \infty$; but the opening angle may be such that the shallow shell approximation still applies. Again, the state of stress is essentially given by shallow membrane theory or inextensional bending theory if $\mu^2 \to \infty$, except in narrow layers where the effect of "(edge) bending" is significant. The calculation of the symmetric and asymmetric buckling loads for spherical caps for sufficiently large μ^2 is also simplified when the boundary layer solution is used.

The nonshallow shell equations contain a third geometric (shallowness) parameter in addition to the thickness parameter μ and the load parameter γ. Furthermore, there may be more than one loading parameter, as in problems of steadily spinning shells of revolution under normal pressure [72]. Hence, a variety of different scalings of the dependent variables can be found which transform Reissner's equations into dimensionless form. For an appropriate asymptotic analysis it is important to arrive at the "right scaling", often achieved by physical reasoning. A systematic method to find all distinct sets of dimensionless equations was recently given for a clamped spherical shell under internal pressure [115]. This asymptotic analysis was carried out for the finite rotation equations, including the special cases for shallow shells. Seventeen distinct sets of simplified equations were found for the various limit cases of this three-parameter problem.

When the surface load is directed inward near the apex of a shell of revolution, but outward in the adjacent region, "polar dimpling" is possible, that is, the shell exhibits a state of inextensional bending in a dimple region centered at the pole. Away from the (inward) dimple, either inextensional bending or nonlinear membrane action is encountered depending on the size of the given load parameters. The solutions in the dimple region and outside the dimple region are connected by interior layer solutions. Thus one has a singular perturbation problem where the inextensional bending and the layer solutions correspond to outer and inner expansions, respectively, of the exact solution of the BVP. The dimple base radius can be determined asymptotically by a simple formula without reference to the layer solution [85, 131–133].

Another recent example where solutions by asymptotic methods cover almost all ranges of parameters of interest is the problem of a steadily spinning shallow elastic shell under a uniform pressure distribution [72], originally formulated in [100]. As in the problem described in Section 9.1, it turns out that in the range of large pressure, the Föppl membrane solution is the leading term outer expansion of the exact solution. Numerical solutions have been obtained to confirm the results of the asymptotic analyses in the various parameter ranges [72].

An interesting recent example for a regular perturbation solution is the circular plate problem under a concentrated load as treated in [26]. The first term of a perturbation solution here corresponds to linear plate theory. Although the range of convergence of the perturbation series appears to be very small, the range of applicability of the solution can be extended dramatically by applying the Aitken–Shanks transformation.

9.3. Boundary layer behaviour near small holes

Asymptotic methods may also be used to describe deformation of shells with a small circular hole. For a hole placed at the apex of a shell of revolution subject to normal pressure, the axisymmetric bending theory applies. Unlike the cases discussed so far, the parameter ε is not related to the thickness of the shell but is the ratio of the radius of the small hole to the radius of the outer edge of the shell, thus resulting in a different type of boundary layer behavior. For smoothly varying surface load the solution for the shell without hole will generally be slowly varying throughout the shell except possibly near the outer edge. However, if the hole is assumed free of radial edge traction and bending moment, the radial stress and the bending moment will generally change sharply near the hole, while away from the hole the solution will be close to the solution without hole. This behavior is confirmed by numerical solutions which become increasingly difficult (and costly) as ε approaches zero. The problem can be analyzed by singular perturbation techniques, which yield asymptotic solutions for small ε with little computing effort. The only nonlinear problem to be solved numerically is the BVP for the shell without hole [135, 146].

The annular membrane problem was formulated in Section 7 (see (7.10), (7.11)). It can be shown that the limit problem with $\varepsilon = 0$ and $Q(x) = R(x, 0)$ does not have a (classical) solution. The leading term outer solution turns out to be identical with the solution without hole, satisfying the boundary condition at the outer edge. This solution is matched to an inner solution valid near the edge $x = \varepsilon$. Introducing the stretched variable $s = (x/\varepsilon) - 1$, the resulting ODE in terms of s then suggests an inner expansion $y(s; \varepsilon) = z_0(s) + \varepsilon z_1(s) + \varepsilon^2 z_2(s) + \cdots$. The functions $z_i(s)$ can all be found in closed form. They must satisfy $z_i(0) = 0$, and they contain constants of integration C_i to be determined by matching $y(s; \varepsilon)$ with the outer solution. A quantity of particular interest is the stress concentration factor defined in (7.15). From the inner expansion the following formula is easily deduced [135]

$$S_c(\varepsilon) = 2\left[1 + \frac{\varepsilon^2}{C_0^3}\left(C_0^2 C_2 - \tfrac{1}{2}\right) + O(\varepsilon^4)\right], \quad \text{as } \varepsilon \to 0, \tag{9.2}$$

which confirms Schwerin's formal result $S_c(\varepsilon) \to 2$ as $\varepsilon \to 0$. In contrast, the rigorous result mentioned in Section 7 [36] does not provide information on how S_c depends on ε, for small ε. The method described here for the Föppl membrane theory carries over with little change to the Reissner finite rotation theory.

The nature of the asymptotic expansion is slightly more complicated when bending stiffness is included. The shallow shell with a small hole has been treated in the sub-buckling range of γ (see equations (7.20)) in [146]. The relevant boundary conditions are (7.26) with $s = m = 0$. The inner expansion for both f and g is here of the form $f(s, \varepsilon) = f_0(s) + \varepsilon^2(\log \varepsilon) f_1(s) + \varepsilon^2 f_2(s) + \varepsilon^4(\log \varepsilon)^2 f_3(s) + \cdots$. The conditions supplied by matching these inner solutions with outer solutions determine the appropriate boundary condition for the outer solution for $x = 0$, as well as the constants of integration for the inner solution. Again, the leading term outer solution is given by the nonlinear BVP for the shell without a hole. All other functions that occur in the asymptotic expansion can be calculated by solving standard linear initial value problems. The stress concentration factor for the circumferential stress resultant corresponding to (9.2) is now

$$S_N(\varepsilon) = 2\left[1 + \tfrac{1}{2}(\varepsilon^2 \log \varepsilon)\, \frac{B_0^2}{C_0}\, \frac{\nu - 1}{\nu + 1} + O(\varepsilon^2)\right], \quad \text{as } \varepsilon \to 0, \tag{9.3}$$

where the constants B_0 and C_0 are determined from the leading term outer solution, that is, from the numerical solution of the BVP without hole.

The method described here can obviously be applied to nonshallow shells and to the Reissner finite rotation equations. For buckling problems, we may consider a similar approach to obtain the buckling load of a shell with a small hole from the corresponding buckling load for the shell without the hole, modified by small correction terms due to the presence of the hole determined by an asymptotic analysis.

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