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The Interior Solution for Linear Problems of Elastic Plates

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Necessary conditions have been established recently for the prescribed data alon the cylindrical edge(s) of an elastic flat plate to induce only an exponentially deca ing elastostatic state. The present paper describes how these conditions may be us to determine the interior solution (or its various thin and thick plate theory approximations) of plate problems. The results in turn show that the necessary condition for a decaying state are also sufficient conditions. Boundary conditions for the in terior solution of circular plate problems with edgewise nonuniform boundary data are discussed in detail and then applied to two specific problems. One of them is concerned with a circular plate compressed by two equal and opposite point forces at the plate rim. The solution process for this problem illustrates for the first ti how the stretching action in the plate interior induced by transverse loads can properly analyzed.

1 Introduction

The exact solution of *linear* elastostatic problems for thin plates is known to consist of an *interior* component and *layer* components. The interior solution is significant throughout the plate, whereas a layer solution has only a localized effect in a region near an edge of the plate with a typical layer width of the order of the plate thickness. Boundary conditions at an edge of the plate are generally satisfied by a combination of the two types of solution components. The interior solution has been known since the work of M. Levy (1877) and confirmed more recently by Friedrichs and Dressler (1961), Gol'denveizer (1962), Gol'denveizer and Kolos (1965), Reiss (1962), Reiss and Locks (1961) and Reissner (1963). On the other hand, the exact layer solutions or their accurate approximations are difficult to obtain (see Friedrichs and Dressler (1961), for example). To the extent that the layer solution behavior may not be needed for some design purposes, there has been a continual effort over the years to determine the interior solution of Levy or its approximation (in the form of the Kirchhoff thin plate theory or some moderately thick plate solution) without any reference to the layer solutions (Reissner, 1985). For stress edge data, Saint-Venant's principle has been invoked to derive an appropriate set of stress boundary conditions for plate theory solutions (Love, 1944; Timoshenko and Goodier, 195 1). Strictly speaking, this principle does not apply to plate problems, as one linear dimension of the loaded area (namely, the circumference of the plate's midplane) or the characteristic length of the edge load there, i *not* small compared to a representative plate span. Another interesting approach which also makes use of Saint-Venant's principle has been proposed by Kolos (1965) but is not entirely satisfactory for reasons given previously (Gregory and Wan, 1984, 1985a, 1985b).

A new and completely different method was developed recently by Gregory and Wan (1984) for the determination of the "exact" or approximate interior solution without any knowledge of the layer solution components for the same problem. In contrast to previous attempts, all of which were based on some form of Saint-Venant's principle, the key to our method lies in a novel application of the Betti-Rayleigh reciprocal theorem. Necessary conditions have been deduced from this theorem for any admissible set of edge data to induce only an elastostatic state which decays exponentially at a distance away from the plate edge of the order of the plate thickness. These necessary conditions are to be translated into an appropriate set of boundary conditions for the Levy solution or some (approximating) plate theory solution. They effectively allow us to split the prescribed edge data correctly into two parts, one for the interior solution components and the other for the layer solution components. The special case of plates with straight edges in a state of plane strain induced by edgewise uniform data was first treated by this method (Gregory and Wan, 1984). The resulting edge conditions, for the interior solution component, made it possible for us to solve some specific displacement boundary-value problems in thin and thick plate theories which had not admitted a rational solution previously (Gregory and Wan, 1984).

Next, Gregory and Wan (1985a) treated the axisymmetric bending of circular plates. For stress edge data, our results show that indiscriminate applications of Saint-Venant's principle for this class of problems may lead to qualitatively and quantitatively incorrect solutions in the plate interior; they in

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effect delimit the range of applicability of the principle for axisymmetric bending problems. Correct boundary conditions for the Levy solution, or its approximations, were also deduced for edgewise nonuniform data along the straight edge of a semi-infinite plate in bending (Gregory and Wan, 1985a).

The development, for axisymmetric bending of a circular plate, is complemented by an analogous treatment for axisymmetric stretching (and torsion) in Gregory and Wan (1985b). The boundary conditions obtained there for the interior solution also delimit the applicability of Saint-Venant's principle in the case of stress edge data. Together, the two sets of results for axisymmetric stress distributions enabled us to rationally deduce the correct interior solution for a simply supported plate, subject to a point force at the center of its upper face (Gregory and Wan, 1985b).

For the plane-strain case, the necessary conditions for the edge data to induce only a decaying state were also shown to be sufficient conditions in Gregory and Wan (1984) because we have the explicit solutions for all Papkovich-Fadle eigenfunctions which are known to form a complete set (Gregory, 1980). For all other cases, the boundary conditions obtained for the interior solution component do not guarantee that the interior solution determined by these conditions is the correct solution in the plate interior up to exponentially small terms. In other words, we must still prove that the necessary conditions derived and used in Gregory and Wan (1985a and 1985b) are also sufficient conditions. In the present paper, we prove this sufficiency and thereby substantiate the claim made earlier in this introduction on the various "correct interior solutions." The sufficiency proof also offers a general method for determining the correct interior solution for general edge geometry and edgewise nonuniform loads. This general method, to be described in Section 3 of this paper, will be applied to a number of problems for a circular plate involving edgewise nonuniform data.

2 Decaying States in Elastic Plates

We consider in this paper flat plates bounded by two flat faces $z=\pm h$, and a cylindrical edge E_c, consisting of a midplane edge curve Γ and the generators $|z|\le h$ normal to Γ (Fig. 1). The plate material is homogeneous, isotropic and linearly elastic; some results for anisotropic plates in plane-strain or axisymmetric deformations have been obtained recently in Lin and Wan (1988). With no loss of generality, we take the plates to be free of body loads and the faces $z=\pm h$ to be free of surface tractions as these types of load terms can always be removed by substracting away a suitable equilibrium state of an infinite plate of the same material. The only loading is prescribed along the curved edge of the plate E_c and any set of edge data, consistent with the virtual work principle, is admissible. For the sake of definiteness, we will consider here only the case of stress data so that three appropriate combinations of the stress components σ_{ij} are prescribed on E_c in the form

$$\sigma_{nn}=\bar{\sigma}_{nn}(t,z), \quad \sigma_{nt}=\bar{\sigma}_{nt}(t,z), \quad \sigma_{nz}=\bar{\sigma}_{nz}(t,z) \tag{2.1}$$

where n, t, and z indicate directions normal, tangent, and transverse to the edge curve Γ, respectively, as shown in Fig. 1, and $\sigma_{n\alpha}=\sigma_{kj}n_k\alpha_j$ (summing in j and k from 1 to 3) with n_k (α_k) being the directional cosine of the direction n (α) and the direction x_k. Several other types of edge data were also considered in Gregory and Wan (1984, 1985a). The necessary modifications to the argument which follows are obvious for these and other sets of admissible edge data. So are the modifications required for a plate with more than one cylindrical edge.

The (fully three-dimensional) problem of determining the elastostatic state $\{\sigma, \mathbf{u}\}$ in the plate arising from the prescribed edge data (2.1) is generally intractable analytically. In this

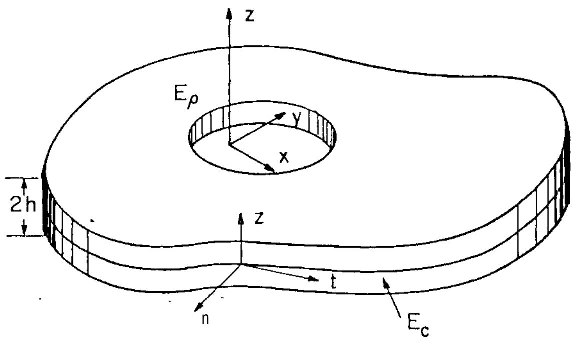

Fig. 1

paper, we shall be concerned with approximate solutions of this problem in the spirit of thin and thick plate theories; see also Gregory and Wan (1984). We will make this precise with the following definitions.

An *elastostatic state* of the plate consists of stress and displacement components $\{\sigma, \mathbf{u}\}$ which satisfy the conditions of equilibrium and compatibility with no body forces in the plate and no surface tractions on the two faces.

An elastostatic state in the plate is said to be *decaying state* if as $h\to 0$

$$\{\sigma,\mathbf{u}\}=0(Me^{-\gamma d/h}) \tag{2.2}$$

where d is the minimum distance of the observation point from the edge of the plate, M is the maximum modulus of the prescribed edge tractions and γ is a positive constant independent of the half-plate thickness h.

An elastostatic state is said to be a *symmetric* (or *stretching) interior state* of a flat plate (with traction-free faces) and denoted by a superscript E (for Extension) if it has the (quasi-two-dimensional) form given in Appendix I. The corresponding formulas for an *antisymmetric* (or *bending) interior state* are given in Appendix I of Gregory and Wan (1985a).

The fundamental thesis of plate theory is that the exact three-dimensional solution $\{\sigma, \mathbf{u}\}$ of the plate problem as described at the beginning of this section has a resolution of the form

$$\{\sigma,\mathbf{u}\}=\{\sigma^I,\mathbf{u}^I\}+\{\sigma^d,\mathbf{u}^d\} \tag{2.3}$$

where $()^I$ denotes an interior state (consisting of either symmetric or antisymmetric states, or both) and $()^d$ denotes a decaying state (with one or more components). Such a resolution has been proved rigorously for the special case of an infinite strip plate in plane-strain deformation (see Theorem (1) of Gregory and Wan (1984)). A similar argument would apply to a circular plate in axisymmetric deformation. To our knowledge, there is no fully rigorous proof for plates of general shape; however, the asymptotic results of the sixties leave little doubt that such a resolution does exist. Thus $\{\sigma^I, \mathbf{u}^I\}$ is an approximation to the exact solution $\{\sigma,\mathbf{u}\}$ of the problem with an error which is exponentially small as $h\to 0$ except near the edge E_c of the plate. Near E_c, there is a boundary layer (with a layer width of the order of the plate thickness) where the contribution from $\{\sigma^d, \mathbf{u}^d\}$ is not negligible.

The decaying state or edge zone solution $\{\sigma^d, \mathbf{u}^d\}$ is known to have a rather complex analytical structure (see Friedrichs and Dressler (1961) for example). It is therefore desirable to find the interior solution $\{\sigma^I, \mathbf{u}^I\}$ (or its plate theory approximations) without considering the edge zone solution $\{\sigma^d, \mathbf{u}^d\}$. The difficulty in doing this is now clear: The conditions (2.1) which determine $\{\sigma, \mathbf{u}\}$ are to be applied at the edge E_c where $\{\sigma^d, \mathbf{u}^d\}$ is generally not negligible compared to $\{\sigma^I, \mathbf{u}^I\}$; thus the conditions (2.1) cannot be applied to $\{\sigma^I, \mathbf{u}^I\}$ alone. In fact, $\{\sigma^I, \mathbf{u}^I\}$ is usually too restricted in its z-dependence and generally cannot satisfy (2.1) even if we try to do so (Love, 1944; Timoshenko and Goodier, 1951). These observations bring us to the central question in plate theory: What are the correct conditions to be applied to the interior solution at the edge of the plate E_c?

A procedure has been developed recently (Gregory and Wan, 1984, 1985a and 1985b), to answer this question of a proper splitting of the edge data for the two kinds of elastostatic state components. The central step in seeking the appropriate boundary conditions for the exact (Levy) or approximate (thin or thick plate) interior solution is to decide *what (necessary) conditions must the data in (2.1) (or any admissible set of data) satisfy in order that the resulting solution for the plate problem should be solely a decaying state.* It was shown (Gregory and Wan, 1984) that for the data (2.1) to induce only a decaying state, it is necessary to have

$$\int\int_{E_c}\{\bar{\sigma}_{nn}u_n^{(2)}+\bar{\sigma}_{nt}u_t^{(2)}+\bar{\sigma}_{nz}u_z^{(2)}\}dS=0 \tag{2.4}$$

up to exponentially small terms where $\{\sigma^{(2)},\mathbf{u}^{(2)}\}$ is any elastostatic state of the same plate satisfying the following conditions (in addition to the conditions of equilibrium and compatibility in the plate interior and traction-free conditions on the plate faces required by the definition of an elastostatic state of the plate):

(i) It has no singularity in some neighborhood of E_c well beyond the boundary layer adjacent to E_c.
(ii) It is traction-free also on E_c.
(iii) It has at worst an algebraic growth as $h\to 0$.

Note that in view of *(ii)* a nontrivial state (2) must have singularities somewhere in the plate unless it is a rigid-body displacement field. Different choices for the stipulated state (2) will lead to infinitely many independent necessary conditions in general.

For instance, a countably infinite number of suitable $\{\sigma^{(2)},\mathbf{u}^{(2)}\}$ states may be generated as follows: Suppose $\{\sigma^*,\mathbf{u}^*\}$ is any of the states given in Appendix II. These states are singular at $r=0$ (with the origin of the cylindrical coordinates chosen to be in the interior of plate) and are traction-free on $z=\pm h$, $r>0$. They are not, however, traction-free on E_c. We can now add to $\{\sigma^*,\mathbf{u}^*\}$ an elastostatic state which is regular throughout the plate and is chosen (uniquely to within a rigid motion) to annihilate any nonzero tractions on E_c due to state $\{\sigma^*,\mathbf{u}^*\}$. The resulting state is a possible $\{\sigma^{(2)},\mathbf{u}^{(2)}\}$ and will be denoted by $\{\Sigma^s,\ \mathbf{U}^s\}$. It can be characterized by the singularity at $r=0$ of, say, the radial stress component Σ^s_{rr}. For a fixed integer $n\geq 2$, there are four such singular states with an angular variation of $\cos(n\theta)$ or $\sin(n\theta)$. There are two such states for $n=1$ and only one for $n=0$ if we do not allow multivalued displacement fields. The set of all states $\{\Sigma^s,\mathbf{U}^s\}$ obtained by this construction will be denoted by S. Thus we can say that for the traction data (2.1) on E_c to induce a decaying state in the plate, it is necessary that

$$\int\int_{E_c}\{\bar{\sigma}_{nn}U_n^s+\bar{\sigma}_{nt}U_t^s+\bar{\sigma}_{nz}U_z^s\}dS=0 \tag{2.5}$$

for all states $\{\Sigma^s,\mathbf{U}^s\}$ in S, giving a countable infinity of necessary conditions. For special classes of the problems, symmetry requirements reduce this countably infinite set of necessary conditions to a finite number of conditions; they can be translated into an appropriate set of boundary conditions for thin plate theory, thick plate theories or the complete interior solution. This was in fact done for the axisymmetric stretching (and torsion) problems for circular plates (Gregory and Wan, 1985b).

The elastostatic states in S constructed from the singular states in Appendix II are associated with the stretching and shearing action of the plate with single-valued displacement fields. Evidently, there is another countable infinity of states having similar singularities and angular variations associated with the bending action of the plate. For the special case of axisymmetric bending of circular plates, the countably infinite necessary conditions also reduce to a finite number of conditions; they were then translated into an appropriate set of boundary conditions for the interior (or plate theory) solution of the problem (Gregory and Wan, 1983a). As bending and stretching are uncoupled in a linear theory, we will focus the subsequent development in this paper on the stretching problem for conciseness.

3 Sufficient Conditions for a Decaying State

The necessary conditions implied by (2.4) have been shown to be also sufficient for the edge data to induce only a decaying state for infinite strip plates under plane-strain deformation (Gregory and Wan, 1984). The proof, for that special class of problems, makes use of the completeness of the Papkovich-Fadle eigenfunctions (Gregory, 1980) and, therefore, does not extend to more general geometry and edge data. In this section, we will show how the reciprocal theorem may also be used to determine the interior stress state for general edge geometry and edge tractions. When we apply the result to edge tractions, which satisfy the necessary conditions (2.4), the interior solution will be found to be the trivial state, $\sigma^I_{ij}=u^I_j\equiv 0$. The necessary conditions (2.4) are therefore also sufficient for the edge data to induce only a decaying state.

Suppose the prescribed edge tractions induce only a symmetric (stretching) interior state. With Σ^s vanishing identically on E_c and with Σ^s and σ (the elastostatic state induced by $\bar{\sigma}_{nn}$, $\bar{\sigma}_{nt}$ and $\bar{\sigma}_{nz}$) both vanishing on $z=\pm h$, *we* have

$$\int\int_{E_c}\{\bar{\sigma}_{nn}U_n^s+\bar{\sigma}_{nt}U_t^s+\bar{\sigma}_{nz}U_z^s\}dS$$

$$=\int\int_{E_c}\{[\bar{\sigma}_{nn}U_n^s+\bar{\sigma}_{nt}U_t^s+\bar{\sigma}_{nz}U_z^s]-[u_n\Sigma^s_{nn}+u_t\Sigma^s_{nt}+u_z\Sigma^s_{nz}]\}dS$$

$$+\int\int_{z=h}\{[\sigma_{zn}U_n^s+\sigma_{zt}U_t^s+\sigma_{zz}U_z^s]-[u_n\Sigma^s_{zn}+u_t\Sigma^s_{zt}+u_z\Sigma^s_{zz}]\}dS$$

$$-\int\int_{z=-h}\{[\sigma_{zn}U_n^s+\ldots.]-[u_n\Sigma^s_{zn}+\ldots.]\}dS. \tag{3.1}$$

We now apply the reciprocal theorem over the plate with the cylinder $r<\rho$ removed, as there are no singularities of $\{\Sigma^s,\mathbf{U}^s\}$ in the interior of this volume. The result requires the right-hand side of (3.1) to be equal to a similar integral over only the cylindrical edge surface $E_\rho\equiv(r=\rho,|z|\leq h,\ 0\leq\theta\leq 2\pi)$ so that

$$\int\int_{E}\{\bar{\sigma}_{nn}U_n^s+\bar{\sigma}_{nt}U_t^s+\bar{\sigma}_{nz}U_z^s\}dS$$

$$=\int_{-h}^{h}\int_0^{2\pi}\{[\sigma_{rr}U_r^s+\sigma_{r\theta}U_\theta^s+\sigma_{rz}U_z^s]-[u_r\Sigma^s_{rr}+u_\theta\Sigma^s_{r\theta}+u_z\Sigma^s_{rz}]\}|_{r=\rho}\,\rho d\theta dz. \tag{3.2}$$

For any fixed ρ, sufficiently small and independent of h, *we* may replace $\{\Sigma^s,\mathbf{U}^s\}$ by the dominating singular state $\{\sigma^*,\mathbf{u}^*\}$ on E_ρ. As $h\to 0$, *we* have $\{\sigma,\mathbf{u}\}\sim\{\sigma^I,\mathbf{u}^I\}$ on E_ρ except for Exponentially Small Terms (E.S.T.) giving

$$\int\int_{E_c}\{\bar{\sigma}_{nn}U_n^s+\bar{\sigma}_{nt}U_t^s+\bar{\sigma}_{nz}U_z^s\}dS$$

$$=\int_{-h}^{h}\int_0^{2\pi}\{[\sigma^I_{rr}U_r^*+\sigma^I_{r\theta}U_\theta^*+\sigma^I_{rz}U_z^*]-[u^I_r\sigma^*_{rr}+u^I_\theta\sigma^*_{r\theta}+u^I_z\sigma^*_{rz}]\}|_{r=\rho}\,\rho d\theta dz \tag{3.3}$$

except for E.S.T.

Now $(\sigma^I,\mathbf{u}^I)$ is a stretching interior state $\{\sigma^E,\mathbf{u}^E\}$ (by our choice of edge data) and is derived from a biharmonic (stress) function F (*see* Appendix I). Any $F(r,\theta)$ which satisfies $\nabla^2\nabla^2 F=0$ and gives rise to a single-valued interior state regular in a neighborhood of $r=0$ has a local expansion of the form

$$F(r,\theta)=\sum_{m=2}^{\infty}[(a_m r^m+b_m r^{m+2})\cos(m\theta)+$$
$$+(c_m r^m+d_m r^{m+2})\sin(m\theta)]+b_1 r^3\cos\theta+d_1 r^3\sin\theta+b_0 r^2. \quad (3.4)$$

with the stress-free parts of F omitted from the expansion. The corresponding stress and displacement fields can be found in Appendix I. Let us substitute these expressions into (3.3) and also take $\{\Sigma^s,\mathbf{U}^s\}$ to be that member of S associated with the singular field derived from $F^*=r^{-k}\cos(k\theta)$ for some fixed integer k (*see* (II-la) and (II-2a) of Appendix II). We denote this particular singular field $\{\Sigma^s,\mathbf{U}^s\}$ by $\{\Sigma^k,\mathbf{U}^k\}$. Upon integrating term-by-term with respect to θ, all terms associated with a_m and b_m for $m\neq k$ give no contribution to the integral because of the orthogonality property of the trigonometric functions involved. For the same reason, neither do terms associated with c_m and d_m for all m. After some cancellations among the remaining terms, we are left with the rather remarkable result

$$\int\int_{E_c}[\bar{\sigma}_{nn}U_n^k+\bar{\sigma}_{nt}U_t^k+\bar{\sigma}_{nz}U_z^k]dS=\frac{16\pi h}{E}k(k+1)b_k \quad (k\geq 2) \quad (3.5)$$

except for E.S.T. (left out when we approximate $\{\sigma,\mathbf{u}\}$ by $\{\sigma^I,\mathbf{u}^I\}$ on E_ρ) as the right-hand side of (3.5) is independent of ρ. (The parameter E which appears in the denominator there is the Young's modulus of the material.)

By taking the singular field corresponding to $F^*=r^{-k+2}\cos(k\theta)$ instead, we obtain, except for E.S.T.,

$$\int\int_{E_c}[\bar{\sigma}_{nn}U_n^k+\bar{\sigma}_{nt}U_t^k+\bar{\sigma}_{nz}U_z^k]dS=-\frac{16\pi h}{E}k(k-1)a_k \quad (k\geq 2) \quad (3.6)$$

The right-hand side is again independent of the radius parameter ρ. Similar relations for c_k and $d_k\,(k\geq 2), b_1, d_1$ and b_0 can also be obtained by using other singular fields given in Appendix II. In all cases, U_n^s, U_t^s and U_z^s are the displacement components corresponding to the relevant state (2) in S for the particular singular field chosen.

Formulas (3.5) and (3.6), together with the corresponding formulas for b_1, d_1 and b_0, completely determine F (*see* (3.4)) and, hence, the interior stress state in terms of the stress edge data. In effect, they constitute a procedure for assigning an appropriate portion of the edge data to the interior solution to ensure its correct determination. Applications of this procedure to specific problems will be given in later sections of this paper. Here, we will use these results to show that the *necessary conditions (2.5) (which form a subset (2.4) are in fact sufficient conditions for the elastostatic state induced by the edge data to be a decaying state except possibly for a rigid body displacement field.*

The proof of sufficiency is now straightforward. If (2.5) is satisfied with $\{\Sigma^s,\mathbf{U}^s\}$ corresponding to the $F^*=r^{-k}\cos(k\theta)$ (and with $\bar{\sigma}$ generating only a stretching state), it follows from (3.5) that

$$b_k=0 \quad (k\geq 2) \quad (3.7)$$

Similarly, we have from (3.6) and the corresponding formulas for b_1, d_1 and b_0

$$a_k=0 \quad (k\geq 2),\quad b_1=d_1=b_0=0 \quad (3.8)$$

Consequently, the stress function F which generates the stretching interior state is necessarily zero (except for exponentially small terms). We have thus proved the following:

Theorem. The elastostatic state generated by the stress edge data is necessarily a decaying state (except possibly for a rigid-body displacement field) if the data satisfies the conditions (2.5).

Combining this theorem (and its counterpart for plate bending) with results previously obtained by Gregory and Wan (1984, 1985a, 1985b), we now have that the requirement (2.4) or (2.5) are necessary *and* sufficient conditions for the stress edge data to induce only a decaying state, possibly up to a rigid-body displacement field. With straightforward modifications, the analysis of this section applies also to other admissible edge data and the aforementioned theorem can thereby be extended to include these edge data as well.

4 Boundary Conditions for Some Interior Solutions

The deviation of the interior solution from the exact solution, $\{\sigma-\sigma^I,\mathbf{u}-\mathbf{u}^I\}$, must be a decaying state. It follows that this difference must satisfy the necessary and sufficient conditions (2.5) in the case of prescribed stress edge data (2.1). As an immediate consequence, $\{\sigma^I,\mathbf{u}^I\}$ must satisfy the condition

$$\int\int_{E_c}\{\sigma_{nn}^I U_n^s+\sigma_{nt}^I U_t^s+\sigma_{nz}^I U_z^s\}dS$$
$$=\int\int_{E_c}\{\bar{\sigma}_{nn}U_n^s+\bar{\sigma}_{nt}U_t^s+\bar{\sigma}_{nz}U_z^s\}dS \quad (4.1)$$

for all $\{\Sigma^s,\mathbf{U}^s\}$ in S. It is effectively these (countably infinite) conditions which completely determine the interior state up to a rigid motion in the last section. (Analogous conditions can also be formulated for the determination of the interior state induced by displacement or mixed edge data.) In the form (4.1), these conditions also conveniently provide the appropriate boundary conditions for thin and thick plate theories. Different symmetry restrictions often reduce the infinite number of necessary and sufficient conditions (4.1) to a finite number of conditions in particular cases as seen from our previous results. The actual boundary conditions for the interior solution of some special cases for circular plates will be worked out below for applications in later sections.

(a) **Axially Symmetric Stretching of a Circular Plate.** This case has already been worked out (Gregory and Wan 1985b) and the results are summarized here for the sake of completeness. When the stress data and the resulting deformation are axisymmetric, the conditions (4.1) for the circular plate $\{r\leq a, |z|\leq h\}$, are satisfied identically for all $\{\Sigma^s,\mathbf{U}^s\}$ having an angular variation of the form $\cos(k\theta)$ or $\sin(k\theta), k\geq 1$. The one axially symmetric $\{\Sigma^s,\mathbf{U}^s\}\equiv\{\Sigma^0,\mathbf{U}^0\}$ in S is found to be

$$\Sigma_{rr}^0=E\left(1-\frac{a^2}{r^2}\right),\quad \Sigma_{\theta\theta}^0=E\left(1+\frac{a^2}{r^2}\right),$$
$$\Sigma_{r\theta}^0=\Sigma_{rz}^0=\Sigma_{\theta z}^0=\Sigma_{zz}^0=0,\quad U_\theta^0=0 \quad (4.2)$$
$$U_r^0=(1-\nu)r+(1+\nu)\frac{a^2}{r},\quad U_z^0=-2\nu z.$$

It follows that at $r=a$, the axisymmetric interior solution must satisfy the condition

$$\int_{-h}^{h}\left[\sigma_{rr}^I-\frac{\nu}{a}z\sigma_{rz}^I\right]_{r=a}dz=\int_{-h}^{h}\left\{\bar{\sigma}_{rr}-\frac{\nu}{a}z\bar{\sigma}_{rz}\right\}dz \quad (4.3)$$

or, in terms of the Airy stress function F (*see* Appendix I) in cylindrical coordinates

$$\left[\frac{1}{r}\frac{\partial F}{\partial r}\right]_{r=a} = \frac{1}{2h}\int_{-h}^{h}\left\{\bar{\sigma}_{rr} - \frac{\nu}{a} z\bar{\sigma}_{rz}\right\}dz \tag{4.4}$$

In either form, the stress boundary condition for the axisymmetric stretching of the circular plate is not identical to the condition obtained by the application of the (unproved) Saint-Venant principle

$$\left[\frac{1}{r}\frac{\partial F}{\partial r}\right]_{r=a} = \frac{1}{2h}\int_{-h}^{h}\bar{\sigma}_{rr}dz \tag{4.5}$$

Some consequences of this difference have already been discussed (Gregory and Wan, 1985a and 1985b). Given that the edge data are applied over the entire circular edge of the plate, $\{r=a, |z|\le h\}$, Saint-Venant's principle is in fact inappropriate for this problem as the surface tractions are not localized. The incorrect condition (4.5) is nevertheless appropriate for the leading term (thin plate) approximation in powers of h/a for plate stretching (or the theory of generalized plane stress) as long as $\bar{\sigma}_{rr}$ is not $O(h\bar{\sigma}_{rz}/a)$ or smaller.

(b) Lateral Deformation of a Circular Plate. Suppose that the circular plate with traction-free faces, $z=\pm h$, is in equilibrium under traction along its edge $r=a$ with an angular variation of the form $\sin\theta$ or $\cos\theta$

$$\bar{\sigma}_{rr}(\theta,z) = \hat{\sigma}_{rr}(z)\{\cos\theta, \sin\theta\},\ \bar{\sigma}_{r\theta}(\theta,z) = \hat{\sigma}_{r\theta}(z)\{\sin\theta, -\cos\theta\},$$
$$\bar{\sigma}_{rz}(\theta,z) = \hat{\sigma}_{rz}(z)\{\cos\theta, \sin\theta\} \tag{4.6}$$

where $\hat{\sigma}_{rr}$ and $\hat{\sigma}_{r\theta}$ are even functions of z while $\hat{\sigma}_{rz}$ is an odd function of z. Overall force equilibrium in the x and y-direction requires

$$\int_{-h}^{h}[\hat{\sigma}_{rr}(z) - \hat{\sigma}_{r\theta}(z)]dz = 0 \tag{4.7}$$

For the edge data (4.6), all those states, $\{\Sigma^s, \mathbf{U}^s\}$, having an angular variation of the form $\cos(n\theta)$ or sin(&) with $n\neq 1$, do not contribute to the right-hand side of the condition (4.1). It follows from the analysis at the end of Section 3 that the stress and displacement components of the plate depend on 8 in the form $\cos\theta$ or $\sin\theta$. The corresponding singular field $\{\Sigma^s, \mathbf{U}^s\} \equiv \{\Sigma^1, \mathbf{U}^1\}$ can be constructed according to the procedure outlined in Section 3 to be

$$\Sigma^1_{rr} = 2E\left[\frac{r}{a} - \frac{a^3}{r^3}\right]\{\cos\theta, \sin\theta\},$$

$$\Sigma^1_{r\theta} = 2E\left[\frac{r}{a} - \frac{a^3}{r^3}\right]\{\sin\theta, -\cos\theta\}$$

$$\Sigma^1_{\theta\theta} = 2E\left[3\frac{r}{a} + \frac{a^3}{r^3}\right]\{\cos\theta, \sin\theta\},$$

$$\Sigma^1_{rz} = \Sigma^1_{\theta z} = \Sigma^1_{zz} = 0,$$

$$U^1_r = \left[(1-3u)\frac{r^2}{a} + (1+\nu)\frac{a^3}{r^2} - \frac{4\nu}{3a}(h^2-3z^2)\right]\{\cos\theta, \sin\theta\} \tag{4.8}$$

$$U^1_\theta = \left[(5+\nu)\frac{r^2}{a} + (1+\nu)\frac{a^5}{r^2} + \frac{4\nu}{3a}(h^2-3z^2)\right] \times \{\text{sine}, -\cos\theta\}$$

$$U^1_z = -\frac{8\nu r}{a}z\{\cos\theta, \sin\theta\}.$$

If we now insert U^1_r, U^1_θ and U^1_z from (4.8) into (4.1), we get, with $\sigma^I_{rr} = \sigma^E_{rr} \equiv s_{rr}(r,z)\{\cos\theta, \sin\theta\}$, $\sigma^I_{r\theta} = \sigma^E_{r\theta} \equiv s_{r\theta}(r,z)\{\sin\theta, -\cos\theta\}$ and $\sigma^I_{rz} = \sigma^E_{rz} = 0$

$$\int_{-h}^{h}\left\{s_{rr}(a,z)\left[2(1-\nu) - \frac{4\nu}{3a^2}(h^2-3z^2)\right] + s_{r\theta}(a,z)\left[2(3+\nu) + \frac{4\nu}{3a^2}(h^2-3z^2)\right]\right\}dz$$

$$= \int_{-h}^{h}\left\{\hat{\sigma}_{rr}(z)\left[2(1-\nu) - \frac{4\nu}{3a^2}(h^2-3z^2)\right] + \hat{\sigma}_{rz}\left[-8\nu\frac{z}{a}\right] + \hat{\sigma}_{r\theta}(z)\left[2(3+\nu) + \frac{4\nu}{3a^2}(h^2-3z^2)\right]\right\}dz \tag{4.9}$$

As we have $F(r,\theta) = \hat{F}(r)\{\cos\theta, \sin\theta\}$ and therefore $s_{rr}(r,z) = s_{r\theta}(r,z) = d(r^{-1}\hat{F})/dr$ (see (I-21*a*, *b*)), the relation (4.9) may be written as

$$\frac{d}{dr}\left(\frac{\hat{F}}{r}\right)\Big|_{r=a} = \frac{1}{2h}\int_{-h}^{h}\hat{\sigma}_{rr}dz - \frac{\nu}{2ha}\int_{-h}^{h}\hat{\sigma}_{rz}z\,dz + \frac{\nu}{4a^2h}\int_{-h}^{h}(\hat{\sigma}_{rr} - \hat{\sigma}_{r\theta})z^2dz \tag{4.10}$$

where we have made use of (4.7) to simplify the right-hand side. Given that $F(r,\theta) = r^3\{b_1\cos\theta, d_1\sin\theta\}$, the condition (4.10) completely determines the stress function F and, hence, the interior stress state of the plate.

It is of some interest to note that for the present class of problems, possible candidates for state (2) for (2.4) include the rigid-body displacement fields $\{u_x = 1, u_y = u_z = 0\}$ and $\{u_y = 1, u_x = u_z = 0\}$. Upon inserting these into (2.4), applied to the residual edge data $\{\boldsymbol{\sigma} - \boldsymbol{\sigma}^I, \mathbf{u} - \mathbf{u}^I\}$, we get for both cases

$$\int_{-h}^{h}\{\hat{\sigma}_{rr} - \hat{\sigma}_{r\theta}\}dz = \int_{-h}^{h}\{s_{rr} - s_{r\theta}\}dz \tag{4.11}$$

With $s_{rr} - s_{r\theta} = 0$ (and independent of z), we have again the overall force equilibrium requirement (4.7), now as a consequence of the reciprocal theorem.

(c) Higher Harmonics for a Circular Plate. For stress data with angular variation of the form $\cos(n\theta)$ or $\sin(n\theta)$, $n\ge 2$, it does not seem possible to obtain the needed singular fields $\{\Sigma^s, \mathbf{U}^s\}$ in simple explicit form. In the case $n=2$, for example, we would need to seek a regular elastostatic field which satisfies the stress boundary conditions

$$\sigma_{rr}(a,\theta,z) = \{\cos(2\theta), \sin(2\theta)\},$$
$$\sigma_{r\theta}(a,\theta,z) = \{\sin(2\theta), -\cos(2\theta)\},\quad \sigma_{rz}(a,\theta,z)\ 0 \tag{4.12}$$

to cancel the contribution to the edge tractions by the singular field associated with $F^* = -(a^4/6r^2)\{\cos(2\theta), \sin(2\theta)\}$ and another regular elastostatic field which satisfies the stress boundary conditions

$$\sigma_{rr}(a,\theta,z) = \left\{1 - \frac{\nu(h^2-3z^2)}{(1+\nu)a^2}\right\}\{\cos(2\theta), \sin(2\theta)\}$$

$$\sigma_{r\theta}(a,\theta,z) = \left\{\frac{1}{2} - \frac{\nu(h^2-3z^2)}{(1+\nu)a^2}\right\}\{\sin(2\theta), -\cos(2\theta)\}$$

$$\sigma_{rz}(a,\theta,z) = 0 \tag{4.13}$$

to cancel the effect of $F^* = (a^2/4)\{\cos(2\theta), \sin(2\theta)\}$. (Actually, we need only the displacement fields $\mathbf{U}^s$ associated with the foregoing two singular fields to be used in the boundary conditions (4.1) for the interior solution.)

The two boundary-value problems for two three-dimensional regular elastostatic fields with the stress boundary conditions (4.12) and (4.13), respectively, [and the corresponding problems for $n>2$] are ***canonical*** problems in that they do not depend on the actual stress data and can be solved once and for all, numerically or otherwise. The solutions for these problems may be used in (3.5) and (3.6) for the same circular plate with any admissible set of stress data. The situation is similar to that of displacement boundary-value problems for a semi-infinite plate in plane-strain deformation (Gregory and Wan, 1984). Three canonical displacement boundary-value problems were solved numerically there, and the results were then used to get the correct displacement boundary conditions of the same semi-infinite strip plate for a number of different problems.

5 Lateral Deformation of a Vertical Circular Plate Under Its Own Weight

Let the circular plate $\{r\leq a, |z|\leq h\}$ of uniform density ρ be so-positioned that the direction of gravity is in the positive x-direction. The plate is supported vertically at its center $r=0$ by means of a force $P=2\pi h a^2 \rho g$ in the negative x-direction. The plate is thus traction-free at the two faces and at its only edge r= a. The gravity load in the plate interior is in equilibrium with only one nonvanishing stress component in a Cartesian coordinate system; namely, $\sigma^g_{xx}=-\rho g x$. In terms of cylindrical coordinates (r,θ,z), we have correspondingly

$$\sigma^g_{rr}=-\frac{1}{4}\rho g r[3\cos(\theta)+\cos(3\theta)],$$

$$\sigma^g_{r\theta}=\frac{1}{4}\rho g r[\sin(\theta)+\sin(3\theta)]$$

$$\sigma^g_{\theta\theta}=-\frac{1}{4}\rho g r[\cos(\theta)-\cos(3\theta)],$$

$$\sigma^g_{rz}=\sigma^g_{\theta z}=\sigma^g_{zz}=0. \tag{5.1}$$

On the other hand, the side force at the plate center is known to be associated with a multivalued stress function

$$F_P=-\frac{1}{4}\rho g a^2[(1-\nu)r\ln(r)\cos(\theta)-2r\theta\sin(\theta)] \tag{5.2}$$

giving rise to a stress field with components

$$\sigma^P_{rr}=\frac{\rho g a^2}{4r}\left[(3+\nu)-\frac{2\nu}{3r^2}(h^2-3z^2)\right]\cos(\theta),$$

$$\sigma^P_{r\theta}=-\frac{\rho g a^2}{4r}\left[(1-\nu)+\frac{2\nu}{3r^2}(h^2-3z^2)\right]\sin(\theta),$$

$$\sigma^P_{\theta\theta}=-\frac{\rho g a^2}{4r}\left[(1-\nu)-\frac{2\nu}{3r^2}(h^2-3z^2)\right]\cos(\theta),$$

$$\sigma^P_{rz}=\sigma^P_{\theta z}=\sigma^P_{zz}=0. \tag{5.3}$$

For the solution of the original problem, a third stress field must be added to the two fields σ^g and σ^P so that the total stress state is in equilibrium with the external loads, vanishes at the two faces $z=\pm h$ and vanishes at the edge $r=a$. The third stress field σ^c is therefore in equilibrium with no external load in the plate interior and vanishes at the two faces $z=\pm h$. At the edge $r=a$, it must be the negative of the sum of σ^g and σ^P so that

$$\sigma^c_{rr}|_{r=a}=-\frac{1}{4}\rho g a\left\{\left[\nu-\frac{2\nu}{3a^2}(h^2-3z^2)\right]\cos(\theta)-\cos(3\theta)\right\}\equiv\bar{\sigma}_{rr}$$

$$\sigma^c_{r\theta}|_{r=a}=-\frac{1}{4}\rho g a\left\{\left[\nu-\frac{2\nu}{3a^2}(h^2-3z^2)\right]\sin(\theta)+\sin(3\theta)\right\}\equiv\bar{\sigma}_{r\theta}$$

$$\sigma^c_{rz}|_{r=a}=0. \tag{5.4}$$

Note that the edge data (5.4) satisfies the overall equilibrium requirement (4.7).

It would appear from the form of (5.4) that we could not apply the results of Sections 3 and 4 to obtain the interior solution (given by a stress function F_c) induced by the data $\bar{\sigma}_{rr}$ and $\bar{\sigma}_{r\theta}$, without first obtaining a numerical solution for the singular field $\{\Sigma^s,\mathbf{U}^s\}$ associated with the $\cos(3\theta)$ and $\sin(3\theta)$ term in the data. However, we may set

$$F_c(r,\theta)=F_1(r,\theta)-\frac{1}{24}\rho g r^3\cos(3\theta) \tag{5.5}$$

with $\nabla^4 F_1=0$ implying $\nabla^4 F_c=0$. In terms of the stress components corresponding to F_1, the boundary conditions (5.4) become

$$\sigma^c_{rr}|_{r=a}=-\frac{1}{4}\rho g a\left[\nu-\frac{2\nu}{3a^2}(h^2-3z^2)\right]\cos(\theta)\equiv\hat{\sigma}^c_{rr}\cos(\theta)$$

$$\sigma^c_{r\theta}|_{r=a}=-\frac{1}{4}\rho g a\left[\nu-\frac{2\nu}{3a^2}(h^2-3z^2)\right]\sin(\theta)\equiv\hat{\sigma}^c_{r\theta}\sin(\theta)$$

$$\sigma^c_{rz}|_{r=a}=0. \tag{5.6}$$

Evidently, the interior solution induced by the stress data (5.6) is necessarily the stretching state corresponding to $F_1=b_1r^3\cos\theta$ by the results of Section 3. In that case, the condition (4.10) with $\hat{\sigma}^c_{r\theta}=\hat{\sigma}^c_{rr}$ gives

$$2ab_1=\frac{1}{2h}\int_{-h}^{h}\hat{\sigma}^c_{rr}dz=-\frac{\nu}{4}\rho g a \quad \text{or} \quad b_1=-\frac{\nu}{8}\rho g \tag{5.7}$$

Combining the various stress fields, we have for the actual interior (stretching) state

$$\sigma^I_{rr}=-\frac{1}{4}\rho g a\left[(1+\nu)\frac{r}{a}-(3+\nu)\frac{a}{r}+\frac{2\nu a}{3r^3}(h^2-3z^2)\right]\cos(\theta)$$

$$\sigma^I_{r\theta}=-\frac{1}{4}\rho g a\left[(1-\nu)\frac{r}{a}-(1-\nu)\frac{a}{r}-\frac{2\nu a}{3r^3}(h^2-3z^2)\right]\sin(\theta)$$

$$\sigma^I_{\theta\theta}=-\frac{1}{4}\rho g a\left[(1+3\nu)\frac{r}{a}+(1-\nu)\frac{a}{r}-\frac{2\nu a}{3r^3}(h^2-3z^2)\right]\cos(\theta)$$

$$\sigma^I_{rz}=\sigma^I_{\theta z}=\sigma^I_{zz}=0. \tag{5.8}$$

Given that $\hat{\sigma}^c_{rr}\equiv\hat{\sigma}^c_{r\theta}$ for our problem so that there is no contribution from the second integral in (4.10) to the interior solution, the same results can be obtained by applying Saint-Venant's principle to the solution corresponding to F_1. This will not always be the case for other stress data.

We have been deliberately vague about how the vertical force P is applied at the plate center except that the resulting stress fields in the plate should behave as given by (5.3) near $r=0$. In general, there may be another layer phenomenon centered at $r=0$ similar to that for the axisymmetric case worked out in Gregory and Wan (1985b). While we could do so, we will not enter into a discussion of this layer phenomenon; it would obscure the development pertaining to the main thrust of this paper.

6 Circular Disk Compressed by Equal and Opposite Point Forces at the Rim

Consider a circular plate $\{r \le a, |z| \le h\}$ under equal and opposite point loads of magnitude P at the points $r=a, \theta=0$ and $z=\pm h$. The stress boundary conditions at $r=a$ may be compactly expressed in terms of the Dirac delta function $\delta(\theta)$ and $\delta(z \pm h)$

$$\sigma_{rr}|_{r=a} = \bar{\sigma}_{rr} \equiv 0, \quad \sigma_{r\theta}|_{r=a} = \bar{\sigma}_{r\theta} \equiv 0,$$

$$\sigma_{rz}|_{r=a} = \bar{\sigma}_{rz} \equiv -\frac{P}{a}\,\delta(\theta)[\delta(z-h)-\delta(z+h)]. \tag{6.1}$$

For our solution process, however, it is more convenient to resolve $\bar{\sigma}_{rz}$ into its Fourier components with respect to θ to get

$$\bar{\sigma}_{rz} = -\frac{P}{a}\,[\delta(z-h)-\delta(z+h)]\left[\frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\cos(n\theta)\right] \tag{6.2}$$

In the spirit of the development in the previous sections, we will examine the individual Fourier components of this loading separately.

(a) The $n=0$ Component. This corresponds to an axisymmetric loading of the form

$$\bar{\sigma}_{rr}^{(0)} = \bar{\sigma}_{r\theta}^{(0)} = 0, \quad \bar{\sigma}_{rz}^{(0)} = -\frac{P}{2\pi a}\,[\delta(z-h)-\delta(z+h)] \tag{6.3}$$

The interior solution induced by the set of stress data may be determined by the condition (4.4) which, with (6.3) for edge data, becomes

$$\frac{1}{r}\frac{\partial F_0}{\partial r}\Big|_{r=a} = \frac{1}{2h}\int_{-h}^{h}\left[\bar{\sigma}_{rr}^{(0)} - \nu\frac{z}{a}\bar{\sigma}_{rz}^{(0)}\right]dz = \frac{\nu P}{2\pi a^2} \tag{6.4}$$

Given $F_0 = b_0 r^2$ for axisymmetric problems, we have from (6.4) $b_0 = \nu P/4\pi a^2$ so that $F_0 = \nu P r^2/4\pi a^2$. The corresponding nonvanishing stress and displacement components, denoted by a superscript (0), are

$$\sigma_{rr}^{(0)} = \sigma_{\theta\theta}^{(0)} = \frac{\nu P}{2\pi a^2},$$

$$Eu_r^{(0)} = \nu(1-\nu)\,\frac{Pr}{2\pi a^2}, \quad Eu_z^{(0)} = -\frac{\nu^2 P z}{\pi a^2} \tag{6.5}$$

***(b)* The $n=1$ Component.** This corresponds to a lateral loading of the form

$$\bar{\sigma}_{rr}^{(1)} = \bar{\sigma}_{r\theta}^{(1)} = 0,$$

$$\bar{\sigma}_{rz}^{(1)} = -\frac{P}{\pi a}\,[\delta(z-h)-\delta(z+h)]\cos(\theta) \equiv \hat{\sigma}_{rz}^{(1)}\cos(\theta) \tag{6.6}$$

which satisfies the overall equilibrium requirement (4.7). The interior solution induced by this set of edge data corresponds to the stress function $F_1 = b_1 r^3 \cos(\theta)$. The unknown constant b_1 is determined by the boundary condition (4.10) which, with (6.6) for edge data, becomes

$$2ab_1 = -\frac{\nu}{2ah}\int_{-h}^{h}\hat{\sigma}_{rz}^{(1)} z\,dz = \frac{VP}{\pi a^2} \quad \text{or} \quad b_1 = \frac{VP}{2\pi a^3} \tag{6.7}$$

so that $F_1 = \nu P r^3 \cos(\theta)/2\pi a^3$. The corresponding nonvanishing stress and displacements components, denoted by superscript (1) are

$$\sigma_{rr}^{(1)} = \frac{\nu P}{\pi a^3}\,r\cos(\theta), \quad \sigma_{\theta\theta}^{(1)} = 3\sigma_{rr}^{(1)}, \quad \sigma_{r\theta}^{(1)} = \frac{\nu P}{\pi a^3}\,r\sin(\theta)$$

$$Eu_z^{(1)} = -\frac{4\nu^2 P}{\pi a^3}zr\cos(\theta)$$

$$Eu_r^{(1)} = \frac{\nu P}{2\pi a^3}\left\{(1-3\nu)r^2 - \frac{4\nu}{3}(h^2-3z^2)\right\}\cos(\theta) \tag{6.8}$$

$$Eu_\theta^{(1)} = \frac{UP}{2\pi a^3}\left\{(5+\nu)r^2 + \frac{4\nu}{3}(h^2-3z^2)\right\}\sin(\theta)$$

In principle, we can continue the process to determine the contribution of the higher harmonics in the edge load (6.2) to the interior solution. To do so, we would need the solution for $\{\Sigma^s, U^s\}$ for each of the higher harmonics as described in Section 3. In general, these solutions will have to be obtained numerically as noted in Section 4 and will not be pursued in this paper. If we should determine the collection of $\{\Sigma^s, U^s\}$ for the higher harmonic components of stress function, these canonical solutions are independent of the actual loading; hence, they can be used for other stress boundary-value problems for the same plate.

It is often asserted in the literature for linearly elastic plates that, when the external load is entirely normal to the midplane of the plate, then the interior of the plate is in transverse bending action. Even without calculating the contributions from the higher harmonics of the load term to the interior solution, we see from the results of (6.5) and (6.8) for the present problem that this is not so in general. The example of this section shows for the first time how the stretching interior state induced by transvere loads can be properly analyzed.

7 Concluding Remarks

The results in Sections 2 and 3 of this paper effectively complete the development initiated in Gregory and Wan (1984) and continued in Gregory and Wan (1985a, 1985b). Together, they form the theoretical foundation for a correct formulation of the appropriate boundary conditions for the interior solution (or its various approximations) for plate problems. The method outlined herein for stress data in plate extension problems also applies to plate bending problems as well as other types of admissible edge data. In the spirit of plate theories, our results allow the determination of the correct solution for plate problems accurate in the plate interior to any prescribed order of $(h/d)^n$ without the solution of the related boundary-layer problems.

The separate consideration of the interior and boundary-layer solution components for plate problems made possible by the results of this paper also has another interpretation when viewed in the context of the present day computing power. To attempt a numerical solution (by a finite-element method or otherwise) for a three-dimensional elasticity problem, it would seem attractive to be able to compute separately solution components with different scale lengths and thereby avoid the need for the use of meshes or elements, which differ in size by orders of magnitude. Moreover, with the interior solution being computed separately, the numerical solution for the boundary-layer solution components may now be restricted to a narrow region near the plate edge(s) in which these solution components are expected to be important.

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APPENDIX I

Interior Solution for Extension and Torsion of Plates With Traction-Free Faces

For an isotropic and homogeneous plate with traction-free faces at $z=\pm h$, the in-plane shear and extension portion of the interior solution may be expressed in terms of a biharmonic (Airy stress) function F by

$$\sigma^E_{xx}=\frac{\partial^2}{\partial y^2}\left[1+\frac{\nu(h^2-3z^2)}{6(1+\nu)}\nabla^2\right]F,\qquad \sigma^E_{yy}=\ldots \tag{I-1,2}$$

$$\sigma^E_{xy}=-\frac{\partial^2}{\partial x\partial y}\left[1+\frac{\nu(h^2-3z^2)}{6(1+\nu)}\nabla^2\right]F,$$

$$\sigma^E_{xz}=\sigma^E_{yz}=\sigma^E_{zz}=0 \tag{I-3,4}$$

$$Eu^E_x=E\tilde{u}_x-\frac{\nu}{6}(h^2-3z^2)\frac{\partial}{\partial x}(\nabla^2F),$$

$$Eu^E_y=\ldots\ldots,\qquad Eu^E_z=-\nu z\nabla^2F \tag{I-5,6,7}$$

where $\nabla^2(\nabla^2F)=0$ and

$$E\frac{\partial\tilde{u}_x}{\partial x}=\nabla^2F-(1+\nu)\frac{\partial^2F}{\partial x^2},\quad E\frac{\partial\tilde{u}_y}{\partial y}=\ldots\ldots \tag{I-8,9}$$

$$E\left(\frac{\partial\tilde{u}_y}{\partial x}+\frac{\partial\tilde{u}_x}{\partial y}\right)=-2(1+\nu)\frac{\partial^2F}{\partial x\partial y}, \tag{I-10}$$

∇^2 being the two-dimensional Laplacian in x and y. The formulas for σ^E_{yy}, u^E_y and $\tilde{u}_y$ are obtained from σ^E_{xx}, u^E_x and $\tilde{u}_x$ by interchanging x and y. It should be noted that these formulas are ***not*** identical to the formulas of generalized plane stress. The additional terms in our formulas ensure that the interior state is a genuine three-dimensional elastostatic state.

In cylindrical coordinates, the corresponding formula for the interior solution of plate extension and torsion are

$$\{\sigma^E_{\theta\theta},\sigma^E_{rr}\}=\left\{\frac{\partial^2}{\partial r^2},\ \nabla^2-\frac{\partial^2}{\partial r^2}\right\}\left[1+\frac{\nu(h^2-3z^2)}{6(1+\nu)}\nabla^2\right]F \tag{I-11}$$

$$\sigma^E_{r\theta}=-\frac{\partial^2}{\partial r\partial\theta}\left\{\frac{1}{r}\left[1+\frac{\nu(h^2-3z^2)}{6(1+\nu)}\nabla^2\right]F\right\},$$

$$\sigma^E_{rz}=\sigma^E_{\theta z}=\sigma^E_{zz}=0 \tag{I-12,13}$$

$$Eu^E_r=E\tilde{u}_r-\frac{\nu}{6}(h^2-3z^2)\frac{\partial}{\partial r}\nabla^2F \tag{I-14}$$

$$Eu^E_\theta=E\tilde{u}_\theta-\frac{\nu}{6}(h^2-3z^2)\frac{1}{r}\frac{\partial}{\partial\theta}\nabla^2F,$$

$$Eu^E_z=-\nu z\nabla^2F \tag{I-15,16}$$

where $\nabla^2\nabla^2F=0$ with ∇^2 being the two-dimensional Laplacian in polar coordinates and where

$$E\frac{\partial\tilde{u}_r}{\partial r}=\nabla^2F-(1+\nu)\frac{\partial^2F}{\partial r^2} \tag{I-17}$$

$$E\left(\frac{1}{r}\frac{\partial\tilde{u}_\theta}{\partial\theta}+\frac{1}{r}\tilde{u}_r\right)=(1+\nu)\frac{\partial^2F}{\partial r^2}-\nu\nabla^2F \tag{I-18}$$

$$E\left(\frac{\partial\tilde{u}_\theta}{\partial r}-\frac{1}{r}\tilde{u}_\theta+\frac{1}{r}\frac{\partial\tilde{u}_r}{\partial\theta}\right)=-2(1+\nu)\frac{\partial^2}{\partial r\partial\theta}\left(\frac{1}{r}F\right) \tag{I-19}$$

The following stress function fields and the associated regular symmetric elastostatic states are useful in the development of Section 3 of this paper.

$$(1)\quad F=r^{m+2}\{\cos(m\theta),\ \sin(m\theta)\},\ (m\geq 1) \tag{I-20a,b}$$

$$\sigma^E_{\theta\theta}=(m+1)\left[(m+2)r^m+\frac{2\nu(h^2-3z^2)}{3(1+\nu)}m(m^2-1)r^{m-2}\right]\{\cos(m\theta),\sin(m\theta)\}$$

$$\sigma^E_{rr}=-(m+1)\left[(m-2)r^m+\frac{2\nu(h^2-3z^2)}{3(1+\nu)}m(m^2-1)r^{m-2}\right]\{\cos(m\theta),\sin(m\theta)\}$$

$$\sigma^E_{r\theta}=m(m+1)\left[r^m+\frac{2\nu(h^2-3z^2)}{3(1+\nu)}(m^2-1)r^{m-2}\right]\{\sin(m\theta),-\cos(m\theta)\}$$

$$\sigma^E_{rz}=\sigma^E_{\theta z}=\sigma^E_{zz}=0 \tag{I-21a,b}$$

$$Eu^E_r=\left\{[2(1-\nu)-m(1+\nu)]r^{m+1}-\frac{2\nu}{3}(h^2-3z^2)m(m+1)r^{m-1}\right\}\{\cos(m\theta),\sin(m\theta)\}$$

$$Eu^E_\theta=\left\{[2(1-\nu)+(m+2)(1+\nu)]r^{m+1}+\frac{2\nu}{3}(h^2-3z^2)m(m+1)r^{m-1}\right\}\{\sin(m\theta),-\cos(m\theta)\}$$

$$Eu^E_z=-4\nu z(m+1)r^m\{\cos(m\theta),\sin(m\theta)\}.$$

$$(2)\quad F=r^m\{\cos(m\theta),\sin(m\theta)\},\ (m\geq 2) \tag{I-22a,b}$$

$$\sigma^E_{\theta\theta}=m(m-1)r^{m-2}\{\cos(m\theta),\sin(m\theta)\}=-\sigma^E_{rr}$$

$$\sigma^E_{r\theta}=m(m-1)r^{m-1}\{\sin(m\theta),-\cos(m\theta)\}$$

$$\sigma^E_{rz}=\sigma^E_{\theta z}=\sigma^E_{zz}=0,\quad Eu^E_z=0 \tag{I-23a,b}$$

$$Eu^E_r=-(1+\nu)mr^{m-1}\{\cos(m\theta),\sin(m\theta)\}$$

$$Eu^E_\theta=(1+\nu)mr^{m-1}\{\sin(m\theta),-\cos(m\theta)\}.$$

$$(3)\quad F=r^2 \tag{I-24}$$

$$\sigma_{\theta\theta}^E=\sigma_{rr}^E=2,\ \sigma_{r\theta}^E=\sigma_{rz}^E=\sigma_{\theta z}^E=\sigma_{zz}^E=0$$

$$Eu_r^E=2(1-\nu)r,\ Eu_z^E=-4\nu z,\ Eu_\theta^E=0. \qquad \text{(I-25)}$$

APPENDIX II

Singular States for Plates With Traction-Free Faces

For an isotropic, homogeneous and linearly elastic plate with traction-free faces at X= $\pm h$, we have the following singular elastostatic states derived from different biharmonic functions F (with a singularity at $r=0$) by way of the formulas in Appendix I. Evidently these singular states are associated with (single-valued) in-plane extension and shear of the plate.

(1) $F^*=r^{-n}\{\cos(n\theta),\sin(n\theta)\},\ (n\geq 2)$ (II-la,b)

$$\sigma_{rr}^*=-n(n+1)r^{-n-2}\{\cos(n\theta),\sin(n\theta)\}=-\sigma_{\theta\theta}^*$$

$$\sigma_{r\theta}^*=-n(n+1)r^{-n-2}\{\sin(n\theta),-\cos(n\theta)\} \qquad \text{(II-2a,b)}$$

$$\sigma_{rz}^*=\sigma_{\theta z}^*=\sigma_{zz}^*=0,\ Eu_z^*=0$$

$$Eu_r^*=(1+\nu)nr^{-n-1}\{\cos(n\theta),\sin(n\theta)\}$$

$$Eu_r^*=(1+\nu)nr^{-n-1}\{\sin(n\theta),-\cos(n\theta)\}.$$

(2) $F^*=r^{-n+2}\{\cos(n\theta),\sin(n\theta)\},(n\geq 2)$ (II-3a,b)

$$\sigma_{rr}^*=-(n-1)\Big\{(n+2)r^{-n}-\frac{2\nu(h^2-3z^2)}{3(1+\text{v})}n(n+1)r^{-n-2}\Big\}\{\cos(n\theta),\sin(n\theta)\}$$

$$\sigma_{\theta\theta}^*=(n-1)\Big\{(n-2)r^{-n}-\frac{2\nu(h^2-3z^2)}{3(1+\nu)}n(n+1)r^{-n-2}\Big\}\{\cos(n\theta),\sin(n\theta)\}$$

$$\sigma_{r\theta}^*=-n(n-1)\Big\{r^{-n}-\frac{2\nu(h^2-3z^2)}{3(1+\text{v})}(n+1)r^{-n-2}\Big\}\{\sin(n\theta),-\cos(n\theta)\}$$

(II-4a,b)

$$\sigma_{rz}^*=\sigma_{\theta z}^*=\sigma_{zz}^*=0$$

$$Eu_r^*=\Big\{[2(1-\nu)+n(1+\nu)]r^{-n+1}-\frac{2\nu}{3}(h^2-3z^2)n(n-1)r^{-n-1}\Big\}\{\cos(n\theta),\sin(n\theta)\}$$

$$Eu_\theta^*=\Big\{[(1+\nu)(n-2)-2(1-\nu)]r^{-n+1}-\frac{2\nu}{3}(h^2-3z^2)n(n-1)r^{-n-1}\Big\}\{\sin(n\theta),-\cos(n\theta)\}$$

$$Eu_u^*=4\nu(n-1)zr^{-n}\{\cos(n\theta),\sin(n\theta)\}.$$

(3) $F^*=r^{-1}\{\cos\theta,\sin\theta\}$ (II-5a,b)

$$\sigma_{\theta\theta}^*=\frac{2}{r^3}\{\cos\theta,\sin\theta\}=-\sigma_{rr}^*,\ \sigma_{r\theta}^*=-\frac{2}{r^3}\{\sin\theta,-\cos\theta\},$$

$$\sigma_{rz}^*=\sigma_{\theta z}^*=\sigma_{zz}^*=0,\ Eu_z^*=0 \qquad \text{(II-6a,b)}$$

$$Eu_r^*=(1+\nu)r^{-2}\{\cos\theta,\sin\theta\},$$

$$Eu_\theta^*=-(1+\nu)r^{-2}\{\sin\theta,-\cos\theta\}$$

(4) $F^*=\ln(r)$ (11-7)

$$\sigma_{\theta\theta}^*=-\frac{1}{r^2}=-\sigma_{rr}^*,\qquad \sigma_{r\theta}^*=\sigma_{rz}^*=\sigma_{\theta z}^*=\sigma_{zz}^*=0,$$

$$Eu_r^*=-(1+\nu)r^{-2},\qquad Eu_\theta^*=Eu_z^*=0.$$

It should be noted that the singular fields $r\ln(r)\{\cos\theta,\sin\theta\}$ and $r^2\ln(r)$ all give rise to a multivalued displacement field and will not be considered herein. Also, singular states, for the transverse midplane displacement $w(r,\theta)$, similar to those given by (II-l), (II-3), (11-5) and (11-7) are available for transverse bending problems.