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On lateral buckling of end-loaded cantilevers, including the effect of warping stiffness

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Abstract. We investigate the numerical consequences of the presence of certain non-linear terms in the expressions for the components of transverse shearing strain which occur in the derivation of one-dimensional equations for small finite deflections of straight beams from three-dimensional finite elasticity through use of the principle of minimum potential energy. While particular emphasis is placed on the effect of warping stiffness, the paper also includes results of interest in connection with the classical Michell-Prandtl-analysis of lateral buckling of endloaded cantilevers. Comprehensive numerical results are obtained for the entire range of the relevant dimensionless parameters, using power series, asymptotic expansion and modern numerical methods procedures.

1 Introduction

In what follows we extend earlier considerations of the problem of lateral buckling of end-loaded cantilever beams in Reissner (1983, 1984) which were based on a system of one-dimensional beam equations with *seven* equilibrium differential equations, in several ways.

In the first place, we complement two distinct derivations of one-dimensional theories from a three-dimensional formulation through use of the principle of minimum potential energy by a more general derivation which reproduces the earlier results as two limiting cases.

In the second place, we complement the *analytical* results for the two limiting cases by *numerical* results, in the entire range of values of the relevant warping stiffness parameter. The principal consequence of our simultaneous consideration of these two distinct cases turns out to be an explicit recognition of the importance of taking into account certain non-linear terms in the expressions for transverse shearing strains, within the framework of three-dimensional finite elasticity, when experience with comparable problems in plate theory, as well as numerical results for lateral buckling without the effect of warping stiffness, might have suggested that these terms would not be important.

As regards the problem of lateral buckling of beams without warping stiffness we further obtain the remarkable result that the insensitivity of the values of the buckling load to the difference in the assumptions which characterize the two limiting cases does not persist in the *transition region* of parameter values leading from one to the other of the limiting cases. While this numerical experiment is carried out through the use of ad hoc power series expansions, it turns out that the halfway in-between case, for which the numerical result differs the most from the classical Michell-Prandtl-result, allows an explicit solution in terms of tabulated functions.

2 Derivation of differential equations

We obtain one-dimensional equations for small finite deflections of prismatical beams with axial coordinate z and cross sectional coordinates x, y from three-dimensional elasticity through use of the variational equation

$$\delta \iiint (E\varepsilon_z^2 + G\gamma_x^2 + G\gamma_y^2)\,dx\,dy\,dz = 0 \ , \quad (1)$$

with E and G as given functions, in conjunction with strain component approximations

$$\varepsilon_z = \tilde{w}_{,z} + (1/2)\tilde{u}_{,z}^2 + (1/2)\tilde{v}_{,z}^2 \ , \quad (2)$$

$$\gamma_x = \tilde{u}_{,z} + \tilde{w}_{,z} + \eta\tilde{v}_{,z}\tilde{v}_{,x} \ , \qquad \gamma_y = \tilde{v}_{,z} + \tilde{w}_{,y} + \eta\tilde{u}_{,z}\tilde{u}_{,y} \ , \quad (3)$$

and with displacement approximations

$$\tilde{w} = w + \alpha x + \beta y + \lambda\phi \ , \quad (4)$$

$$\tilde{u} = u - \theta y - (1/2)\theta^2 x \ , \qquad \tilde{v} = v + \theta x - (1/2)\theta^2 y \ . \quad (5)^1$$

In this the constant parameter η had ealier been assumed to have the value zero, Reissner (1983), or one, Reissner (1984), the quantities $w, \alpha, \beta, \lambda, u, v, \theta$ are seven functions of z for which differential equations are to be obtained through use of Eq. (1), and $\phi(x, y)$ is St. Venant's warping function of torsion for the cross section of the beam. Our earlier analysis had given some indication that the results with $\eta = 1$ represented an improvement over the results with $\eta = 0$, except that for the limiting case of negligible warping stiffness the same Michell-Prandtl-buckling load was obtained for both cases.

With the relations $\sigma_z = E\varepsilon_z$, $\tau_x = G\gamma_x$, $\tau_y = G\gamma_y$ for components of stress, and with the defining relations

$$(F, M_x, M_y, R, N) = \iint (1, x, y, \phi, x^2 + y^2)\sigma_z\,dx\,dy \ , \quad (6)$$

$$(Q_x, Q_y, T, S) = \iint (\tau_x, \tau_y, x\tau_y - y\tau_x, \phi_{,x}\tau_x + \phi_{,y}\tau_y)\,dx\,dy \ , \quad (7)$$

for stress resultants, Eq. (1) now leads to seven one-dimensional equilibrium differential equations

$$F' = 0 \ , \qquad (T + \theta' N + v' M_x - u' M_y)' + \eta(u' Q_y - v' Q_x) = 0 \ , \quad (8)$$

$$(Q_x + u' F - \theta' M_y - \eta\theta Q_y)' = 0 \ , \qquad (Q_y + v' F + \theta' M_x + \eta\theta Q_x)' = 0 \ , \quad (9)$$

$$M_x' - Q_x = 0 \ , \qquad M_y' - Q_y = 0 \ , \qquad R' - S = 0 \ . \quad (10)$$

These seven equilibrium equations are associated altogether with nine constitutive equations for the nine stress resultants in (6) and (7). For the case that the beam cross-section is doubly symmetric, to which attention is here restricted, these constitutive equations come out to be

$$F = A_E\varepsilon_F + I_{pE}\varepsilon_N \ , \qquad N = I_{pE}\varepsilon_F + I_{pp}\varepsilon_N \ , \quad (11)$$

$$M_x = I_x\kappa_x \ , \qquad M_y = I_y\kappa_y \ , \qquad R = \Gamma\kappa_R \ , \quad (12)$$

$$Q_x = A_G\gamma_x^{(1)} \ , \qquad Q_y = A_G\gamma_y^{(1)} \ , \quad (13)$$

$$T = I_{pG}\kappa_T - J\kappa_S \ , \qquad S = -J\kappa_T + J\kappa_S \ , \quad (14)$$

where

$$\varepsilon_F = w' + (1/2)(u')^2 + (1/2)(v')^2 \ , \qquad \varepsilon_N = (1/2)(\theta')^2 \ , \quad (15)$$

$$\kappa_x = \alpha' + v'\theta' \ , \qquad \kappa_y = \beta' - u'\theta' \ , \qquad \kappa_R = \lambda' \ , \quad (16)$$

$$\gamma_x^{(1)} = u' + \alpha + \eta\theta v' \ , \qquad \gamma_y^{(1)} = v' + \beta - \eta\theta u' \ , \quad (17)$$

$$\kappa_T = \theta' \ , \qquad \kappa_S = \lambda \ , \quad (18)$$

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1 Incorporation of the θ^2-terms simplifies some of the ensuing analysis. A physical motivation for these terms comes from an observation of the finite rotation formulas $\tilde{u} = u - y\sin\theta - x(1 - \cos\theta)$ and $\tilde{v} = v + x\sin\theta - y(1 - \cos\theta)$

and

$$(A_E, I_x, I_y, I_{pE} I_{pp}, \Gamma) = \iint [1, x^2, y^2, x^2+y^2, (x^2+y^2)^2, \phi^2] E \mathrm{d}x \mathrm{d}y \ , \tag{19}$$

$$(A_G, I_{pG}, J) = \iint (1, x^2+y^2, \phi_{,x}^2 + \phi_{,y}^2) G \mathrm{d}x \mathrm{d}y \ . \tag{20}$$

We note for what follows that the terms with η in the equilibrium Eqs. (8) and (9) turn out to be significant, while the associated η-terms in the strain displacement relations (17) do not.

In regard to the formulation of boundary conditions for the system (8) to (18) we are here limiting ourselves to the problem of a cantilever which is acted upon by forces F_z, F_x, F_y at the end $z=0$, and which is built-in at the end $z=L$. For this problem we have, consistent with (4, 5, 8, 9) and (10)

$$w(L) = \alpha(L) = \beta(L) = \lambda(L) = u(L) = v(L) = \theta(L) = 0 \ , \tag{21}$$

$$M_z(0) = M_y(0) = T(0) + \theta'(0) N(0) = R(0) = 0 \ , \tag{22}$$

$$F(0) = F_z \ , \quad (Q_x + u'F - \eta\theta Q_y)_0 = F_x \ , \quad (Q_y - v'F + \eta\theta Q_x)_0 = F_y \ . \tag{23}$$

3 Equations for buckling

We assume an unbuckled state with end load components $F_z = F_x = 0$ and $F_y = P$, and with end support conditions as in (21). For this unbuckled state the only non-vanishing stress and displacement measures are Q_y, M_y, v and β where then, in accordance with (9, 10, 12, 13, 16) and (17)

$$Q_y = P = A_G(v' + \beta) \ , \quad M_y = Pz = I_y \beta' \ . \tag{24}$$

We further assume, for present purposes, the negligibility of prebuckling *deformations*, by setting $\beta = 0$ and $v = 0$ in the equations for buckling. With this, and with the omission of quantities which are non-linear in terms of buckling stress resultants and displacements we then obtain from (8) to (18) as buckling differential equations

$$(T - u'Pz)' + \eta u'P = 0 \ , \quad M_x' - Q_x = 0 \ , \tag{25}$$

$$(Q_x - \theta' Pz - \eta\theta P)' = 0 \ , \quad R' - S = 0 \ , \tag{26}$$

$$M_x = I_x \alpha' \ , \quad R = \Gamma \lambda' \ , \quad Q_x = A_G(u' + \alpha) \ , \tag{27}$$

$$T = I_{pG}\theta' - J\lambda \ , \quad S = J\lambda - J\theta' \tag{28}$$

and from (21) to (22) as buckling boundary conditions

$$\alpha(L) = \lambda(L) = u(L) = \theta(L) = 0 \ , \tag{29}$$

$$M_x(0) = T(0) = R(0) = Q_x(0) - \eta\theta(0) P = 0 \ . \tag{30}$$

The eighth order problem in (25) to (30) is such as to allow the analysis of the effect of transverse shear deformability, as well as the effect of warping stiffness. We here limit consideration to the latter effect, by stipulating the geometrical constraint conditions

$$u' + \alpha = 0 \ , \quad \lambda - \theta' = 0 \ , \tag{31}$$

with this making reactive quantities out of Q_x and S, with $T = (I_{pG} - J)\theta' - S$.

Elimination of u' in (25) and integration of the first relation in (26) in conjunction with the last relation in (30) reduces the eighth order problem to a sixth order problem which can be written in the form of two equilibrium differential equations

$$(T + \alpha Pz)' = \eta\alpha P \ , \quad M_x' = (\theta' z + \eta\theta) P \ . \tag{32}$$

The two relations in (32) hold in conjunction with two constitutive equations, which follow from (27, 28, 26) and (31) in the form

$$M_x = I_x \alpha' \ , \quad T = D_t \theta' - \Gamma \theta''' \ , \tag{33}$$

where $D_t = I_{pG} - J$. The introduction of (33) into (32) gives as a system of differential equations for the displacement variables θ and α,

$$D_t\theta'' - \Gamma\theta^{IV} = -[z\alpha' + (1-\eta)\alpha]P \ , \quad I_x\alpha'' = (z\theta' + \eta\theta)P \ . \tag{34}$$

The boundary conditions for this sixth order system are three conditions of support which follow from (29) and (31) as

$$\alpha(L) = \theta'(L) = \theta(L) = 0 \ , \tag{35}$$

and three loading conditions which follow from (30), (27), and (31) in the form

$$I_x\alpha'(0) = D_t\theta'(0) - \Gamma\theta'''(0) = \Gamma\theta''(0) = 0 \ . \tag{36}$$

The above problem can be reduced, as previously shown in a somewhat less systematic manner in Reissner (1983), to one of fourth order, for both limiting cases, $\eta = 0$ and $\eta = 1$. Remarkably, when $\Gamma = 0$ then both fourth order problems reduce to the *same* second order problem, albeit not for the same dependent variable. Furthermore, when $\Gamma = 0$ for a general η, the sixth order problem, reduces, in general, no further than to a fourth order problem, with the additional reduction to the second order depending on choosing $\eta = 0$ or $\eta = 1$, or, as shown in what follows, $\eta = 1/2$.

3.1 Non-dimensionalization

An introduction of dimensionless variables ζ, f, g, in accordance with

$$z = L\zeta \ , \quad \theta = f(\zeta) \ , \quad \alpha = \sqrt{D_t/I_x}\, g(\zeta) \ , \tag{37}$$

in conjunction with the introduction of two dimensionless parameters

$$\varepsilon^2 = \Gamma/D_t L^2 \ , \quad \sigma = PL^2/\sqrt{D_t I_x} \ , \tag{38}$$

and with dots indicating differentiation with respect to ζ leaves as a sixth order characteristic value problem the system of differential equations

$$\ddot{f} - \varepsilon^2 f^{\cdot\cdot\cdot\cdot} = -\sigma[\zeta\dot{g} + (1-\eta)g] \ , \quad \ddot{g} = \sigma[\zeta\dot{f} + \eta f] \ , \tag{39}$$

with the boundary conditions

$$\dot{f}(0) - \varepsilon^2 \dddot{f}(0) = \ddot{f}(0) = \dot{g}(0) = g(1) = f(1) = \dot{f}(1) = 0 \ , \tag{40}$$

for the determination of the smallest positive $\sigma = \sigma(\varepsilon, \eta)$.

3.2 Reduction of order when $\eta = 0$ or $\eta = 1$

When $\eta = 0$ the *first* relations in (39) and (40) imply the first integral relation $\dot{f} - \varepsilon^2\dddot{f} = -\sigma\zeta g$. With this, and upon setting $\dot{f} = h$, the first and second relation in (39) are, in effect, a fourth order system

$$h - \varepsilon^2\ddot{h} = -\sigma\zeta g \ , \quad \ddot{g} = \sigma\zeta h \ , \tag{41}$$

with the four boundary conditions

$$\dot{h}(0) = \dot{g}(0) = h(1) = g(1) = 0 \ . \tag{42}$$

When $\eta = 1$ it is the *second* relation in (39), in conjunction with the *third* relation in (40) which implies a first integral relation, $\dot{g} = \sigma\zeta f$. With this the first relation in (39) becomes a differential equation for just f, of the form

$$\ddot{f} - \varepsilon^2 f^{\cdot\cdot\cdot\cdot} + \sigma^2\zeta^2 f = 0 \ , \tag{43}$$

with the four boundary conditions for f being, on the basis of (40),

$$\dot{f}(0) - \varepsilon^2\dddot{f}(0) = \ddot{f}(0) = f(1) = \dot{f}(1) = 0 \ . \tag{44}$$

When $\varepsilon=0$ and $\eta=0$, it is the system (41) and (42), written in terms of the lateral bending variable g, which reduces to the classical Michell-Prandtl-system. When $\varepsilon=0$ and $\eta=1$, it is the system (43) and (44) for the torsional variable f which reduces in this fashion. While this reduction to one and the same system of equations, for $\eta=0$ *and* $\eta=1$, seems to suggest that when $\varepsilon=0$ the critical value of σ might come out to be independent of η we show in what follows that this is not the case.

4 Determination of buckling loads without warping restraint

When $\varepsilon=0$ it is convenient, for general values of η, to introduce an independent variable ξ, and a parameter μ by means of the relations

$$\sqrt{\sigma}\,\zeta=\xi\ , \quad \eta=(1/2)+\mu\ , \tag{45}$$

and, now using primes to indicate differentiation with respect to ξ, to write on the basis of (39) and (40)

$$f''+\xi g'+[(1/2)-\mu]g=0\ , \quad g''-\xi f'-[(1/2)+\mu]f=0\ . \tag{46}$$

$$f'(0)=g'(0)=f(\sqrt{\sigma})=g(\sqrt{\sigma})=0\ . \tag{47}$$

The solutions of (46) which satisfy the conditions for $\xi=0$ in (47) at the outset may be taken in the form

$$f=c_1 f_1+c_2 f_2\ , \quad g=c_1 g_1+c_2 g_2\ , \tag{48}$$

with constants of integration c_1 and c_2, where

$$f_1=1+a_4\xi^4+a_8\xi^8+\dots\ , \quad g_1=b_2\xi^2+b_6\xi^4+\dots\ , \tag{49}$$

$$f_2=a_2\xi^2+a_6\xi^6+\dots\ , \quad g_2=1+b_4\xi^4+\dots\ , \tag{50}$$

and with recurrence relations for the coefficients a_i and b_i following from (46).

The introduction of (48) into the conditions for $\xi=\sqrt{\sigma}$ in (47) then gives the critical value of $\sqrt{\sigma}$ as the smallest positive root of the determinantal equation

$$f_1(\sqrt{\sigma})g_2(\sqrt{\sigma})=f_2(\sqrt{\sigma})g_1(\sqrt{\sigma})\ . \tag{51}$$

It is readily shown that $\sigma(\mu)=\sigma(-\mu)$ so that it is sufficient to obtain values of σ in the range $0\le\mu\le 1/2$.

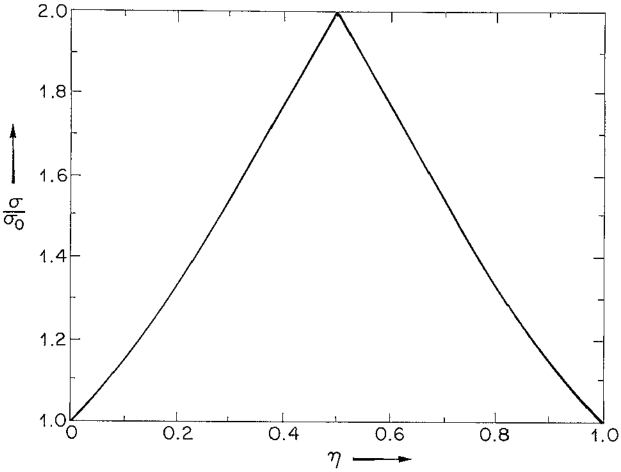
The fact that the fourth order problem (46) and (47) can be reduced, when $\mu=1/2$, to the second order problem $f''+x^2 f=0$ with $f'(0)=f(\sqrt{\sigma})=0$ has been known previously. We now show that a similar, less evident, reduction is also possible when $\mu=0$ and therewith $\eta=1/2$. When $\mu=0$ then the introduction of a function $h=f+\mathrm{i}g$ permits the deduction of a second order *complex* boundary value problem from (46) and (47), of the form

$$h''-\mathrm{i}[\xi h'+(1/2)h]=0\ , \quad h'(0)=h(\sqrt{\sigma})=0\ . \tag{52}$$

The general solution of the differential equation for h is

$$h=\xi^{1/2}\exp(\mathrm{i}\xi^2/4)\,[c_1 J_{1/4}(\xi^2/4)+c_2 J_{-1/4}(\xi^2/4)]\ . \tag{53}$$

The condition $h'(0)=0$ makes $c_1=0$. The condition $h(\sqrt{\sigma})=0$ is then equivalent to the condition $J_{-1/4}(\sigma/4)=0$. This, in turn, means that when $\eta=1/2$ the critical value of σ is exactly twice the Michell-Prandtl value $\sigma_0=4.013\dots$ which applies when $\eta=0$ or $\eta=1$. Numerical values for σ as a function of η are shown in Fig. 1. What is remarkable about these values is that the same Michell-Prandtl critical load values result upon full retention or complete omission of the non-linear term in the three-dimensional transverse shear stress components, while a *partial* retention of these non-linear terms gives significantly different results, with the critical load being up to twice as high as the Michell-Prandtl load.



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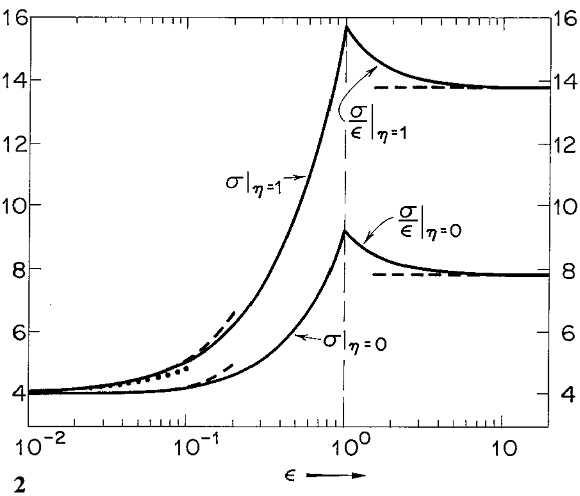

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Fig. 1. Values of lateral cantilever buckling load parameter $\sigma = PL^2 \sqrt{D_t I_x}$ as a function of the non-linear transverse shearing strain parameter η in Eq. (3) in relation to the Michell-Prandtl value $\sigma_0 = 4.013\ldots$

Fig. 2. Values of the lateral buckling load parameter σ as a function of the warping stiffness parameter $\varepsilon = \Gamma D_t / L^2$ for the two limiting values zero and one of the transverse shearing strain parameter η in Eq. 3

5 Effect of warping restraint without non-linear terms in the three-dimensional transverse shear strain components

In order to determine the dependence of σ on ε in accordance with Eqs. (41) and (42) we introduce a modified dimensionless axial coordinate ξ and a modified dimensionless warping deflection function k through the relations

$$\lambda^{1/3}\zeta = \xi \;, \qquad h(\zeta) = \varrho \lambda^{1/3} k(\xi) \;, \tag{54}$$

in terms of dimensionless parameters λ (unrelated to the quantity λ in Sections 2 and 3 which is eliminated through Eq. 31) and ϱ. We furthermore set

$$\varepsilon = 1/\varrho\lambda^{1/3} \;, \qquad \sigma = \lambda^{2/3}/\varrho \;. \tag{55}$$

With primes now indicating differentiation with respect to ξ, this transforms Eq. (41) into

$$k'' - \varrho^2 k = \xi g \;, \qquad g'' = \xi k \;, \tag{56}$$

and Eq. (42) takes on the form

$$k'(0) = g'(0) = k(\lambda^{1/3}) = g(\lambda^{1/3}) \;. \tag{57}$$

We solve (56) and (57), in analogy to the procedure used in connection with (46) and (47), by writing

$$g = c_1 g_e + c_2 g_0 \;, \qquad k = c_1 k_0 + c_2 k_e \;, \tag{58}$$

where $g_e = 1 + a_2\xi^2 + \ldots$, $g_0 = a_3\xi^3 + a_5\xi^5 + \ldots$, $k_e = 1 + b_2\xi^2 + \ldots$, $k_0 = b_3\xi^3 + \ldots$, with (58) being such that the boundary conditions for $\xi = 0$ are satisfied at the outset, and with the boundary conditions for $\xi = \lambda^{1/3}$ giving as equation for the determination of $\lambda = \lambda(\varrho)$

$$g_e(\lambda^{1/3}) k_e(\lambda^{1/3}) = g_0(\lambda^{1/3}) k_0(\lambda^{1/3}) \;. \tag{59}$$

Values of σ as a function of ε follow subsequently through use of the expressions for ε and σ in Eq. (55).

The above numerical calculations give reliable values of σ as a function of ε in the range $0.1 < \varepsilon$. For sufficiently large values of ε for which $\varrho = 0$, effectively, we have in addition the possibility of an explicit determination of $\lambda(0)$, upon recognizing that (56) and (57) imply the second order problem

$(k-g)''+\xi(k-g)=0$, with $(k-g)'(0)=0$ and $(k-g)(\lambda^{1/3})=0$. The solution of this problem determines $\lambda(0)$ as the smallest positive root of the transcendental equation $J_{-1/3}(2\lambda^{1/2}/3)=0$. This makes $\lambda(0)\approx 7.84$ and therewith $\sigma\approx 7.84\varepsilon$, consistent with the consequences of the numerical power series solution.

For the remaining range $0\leq\varepsilon\leq 0.1$ we have previously shown, in Reissner (1984), the possibility of an asymptotic solution of the fourth order problem (41) and (42), upon separating h and g into interior and edge zone contributions, $h=h_i+h_e$, $g=g_i+g_e$, with the application of a simple singular perturbation procedure leading to a *second order* interior problem

$$g_i^{\cdot\cdot}+\sigma^2\zeta^2 g_i+\varepsilon^2\sigma^2\zeta(\zeta g_i)^{\cdot\cdot}=0\ , \qquad g_i(0)=g_i(1)=0\ . \tag{60}$$

The use of regular perturbation expansions $g_i=g_{i0}+\varepsilon^2 g_{i1}+\ldots$ and $\sigma^2=\sigma_0^2(1+c_2\varepsilon^2+\ldots)$, with $g_{i0}=\zeta^{1/2}J_{-1/4}(\sigma_0\zeta^2/2)$ and $\sigma_0=4.013\ldots$, then gives as expression for the coefficient c_2,

$$c_2=\frac{\int_0^1(1+\sigma_0^2\zeta^4)g_{i0}^2\,d\zeta}{2\int_0^1\zeta^2 g_{i0}^2\,d\zeta}=\frac{\sigma_0}{4}\,\frac{\int_0^{\sigma_0/2}(1+4\xi^2)[J_{-1/4}(\xi)]^2\,d\xi}{\int_0^{\sigma_0/2}\xi[J_{-1/4}(\xi)]^2\,d\xi}=\frac{4.013}{4}\,\frac{1.806+4\times 0.483}{0.650}\approx 5.76\ , \tag{61}$$

so that, for sufficiently small values of ε,

$$\sigma\approx\sigma_0(1+5.76\varepsilon^2)\ . \tag{62}$$

6 Effect of warping restraint including non-linear terms in the three-dimensional transverse shear strain components

The introduction of λ, ξ and ϱ from (54) and (55) into Eq. (43) changes the differential equation of the problem into

$$f''''-\varrho^2 f''-\xi^2 f=0\ , \tag{63}$$

while the boundary conditions (44) take on the form

$$f''(0)=f'''(0)-\varrho^2 f'(0)=f(\lambda^{1/3})=f'(\lambda^{1/3})=0\ . \tag{64}$$

We now write $f=c_1 f_e+c_2 f_0$ where $f_e=1+a_4\xi^4+\ldots$ and $f_0=\xi+\varrho^2\xi^3/6+b_5\xi^5+\ldots$ where f_0 and f_e satisfy the boundary conditions for $\xi=0$ and where the conditions for $\xi=\lambda^{1/3}$ give the relation $f_e(\lambda^{1/3})f_0'(\lambda^{1/3})=f_0(\lambda^{1/3})f_e'(\lambda^{1/3})$ for the determination of λ as a function of ϱ. We then again find σ as a function of ε through use of the parametric representations $\varepsilon=1/\varrho\lambda^{1/3}$, $\sigma=\lambda^{2/3}/\varrho$, and we again find that this procedure becomes impractical in the range $0\leq\varepsilon\leq 0.1$.

As ϱ tends to zero we do not have now an explicit solution in terms of tabulated functions. However, our numerical calculations lead to the conclusion that for the present problem we have $\sigma\approx 13.81\varepsilon$ when $1\ll\varepsilon$, in place of the result $\sigma\approx 7.84\varepsilon$ without consideration of the non-linear transverse shear strain terms.

In the range $0\leq\varepsilon\leq 0.1$ we have supplemented the above power series results by asymptotic considerations and by calculations using the numerical boundary value problem solver COLSYS in Asher, Christiansen and Russell (1979, 1981), with this computer code remaining usable for *all* finite values of ε.

In order to make use of COLSYS, we return to the eigenvalue problem in its original form, (43) and (44), and transform it into an inhomogeneous boundary value problem. This is done by supplementing the differential equation (43) with two first order ODE's

$$\sigma^{\cdot}=0\ , \qquad F^{\cdot}=\zeta^2 f^2 \tag{65a, b}$$

and the boundary conditions (44) with two boundary conditions

$$F(0)=0\ , \qquad F(1)=1\ . \tag{66a, b}$$

Equation (65a) is just another way of saying that σ is a constant while (65b) and (66a) imply

$$F(\zeta)=\int_0^\zeta x^2 f^2(x)\,\mathrm{d}x \ , \tag{67}$$

with (66b) then giving

$$\int_0^1 \zeta^2 f^2(\zeta)\,\mathrm{d}\zeta=1 \ , \tag{68}$$

as a normalization condition for the eigenfunctions of (43) and (44). The sixth order system (43) and (65a, b) and the six boundary conditions in (44) and (66a, b) define the inhomogeneous two point BVP which is to be solved by COLSYS. For an outline of the nature of the COLSYS procedure see Appendix B.

In addition to the COLSYS results in the range of small values of ε we again have the possibility of an asymptotic solution of Eq. (43) with the boundary conditions (44). Application of a singular perturbation procedure, in connection with the stipulation $f=f_i+f_e$, now gives, as shown previously for the identical mathematical problem in connection with its appearance in a different physical context in Reissner (1982), as a second order problem for the interior solution contribution

$$f_i^{\cdot\cdot}+\sigma^2\zeta^2 f_i=0 \ , \quad f_i^{\cdot}(0)=f_i(1)-\varepsilon f_i^{\cdot}(1)=0 \ . \tag{69}$$

With $f_i=\zeta^{1/2}J_{-1/4}(\sigma\zeta^2/2)$, as a consequence of the differential equation and the first boundary condition in (69), the second boundary condition in (69) gives, upon setting $\sigma=\sigma_0+\sigma_1\varepsilon+\ldots$,

$$\sigma\approx\sigma_0(1+2\varepsilon) \ . \tag{70a}$$

The above first-order asymptotic result coincides with the corresponding COLSYS result to within about one percent, as long as $\varepsilon<0.05$.

A more systematic asymptotic expansion procedure, as outlined in Appendix C, permits an extension of the first order result, so as to read $\sigma/\sigma_0=1+2\varepsilon+5.981\ldots\varepsilon^2+\mathrm{O}(\varepsilon^3)$. The approximation

$$\sigma\approx\sigma_0(1+2\varepsilon+5.981\,\varepsilon^2) \ , \tag{70b}$$

(as well as an approximation in which the factor 5.981 is replaced by a factor 6), coincides with the corresponding COLSYS result to within about one percent, as long as $\varepsilon<0.1$.

Appendix A: Determination of the dimensionless parameters ε^2 and σ for thin walled *I*-section beams

Without an assumption of thinness for flanges and web, the determination of the twisting stiffness factor $D_t=I_{pG}-J$ as well as of the warping stiffness factor Γ requires the determination of the St. Venant-warping function ϕ on the basis of the differential equation

$$[G(\phi_{,x}-y)]_{,x}+[G(\phi_{,y}+x)]_{,y}=0 \tag{A.1}$$

subject to the boundary condition

$$(\phi_{,x}-y)\,\mathrm{d}y-(\phi_{,y}+x)\,\mathrm{d}x=0 \ . \tag{A.2}$$

We here limit ourselves to an appropriate consideration of the problem for a symmetrical thin-walled *I*-cross section beam with uniform G, with flange and web thicknesses t_f and t_w, with flange width w_f and with web height h_w. For this case, the torsional stiffness factor D_t is known to be given, effectively, by the sum of its values for the two narrow rectangular flange sections and the one narrow rectangular web section, in the form

$$D_t=\frac{1}{3}\,G(h_w t_w^3+2w_f t_f^3) \ . \tag{A.3}$$

The associated value of Γ, in accordance with the defining relation in (19), is obtained upon using the approximate warping function $\phi=xy$. With the web contribution to the value of Γ assumed to be negligible and with E assumed to be constant, we then have

$$\Gamma = \frac{E}{24} t_f w_f^3 h_w^2 \ . \tag{A.4}$$

Equations (A.3) and (A.4) give as an expression for the dimensionless warping stiffness parameter ε^2, as defined in (38),

$$\varepsilon^2 = \frac{1}{8} \frac{E}{G} \frac{h_w^2}{L^2} \frac{t_f w_f^3}{h_w t_w^3 + 2 w_f t_f^3} \ . \tag{A.5}$$

It is of interest to note that for the special case $w_f = h_w = h$ and $t_f = t_w = t$, the above expression reduces to the simple form

$$\varepsilon^2 = \frac{E}{24G} \frac{h^4}{t^2 L^2} \ . \tag{A.6}$$

The associated lateral bending stiffness factor I_x is, in accordance with (19),

$$I_x = \frac{E}{12} (h_w t_w^3 + 2 t_f w_f^3) \tag{A.7}$$

and therewith, in accordance with (38),

$$\sigma = \frac{6PL^2}{\sqrt{EG(h_w t_w^3 + 2 w_f t_f^3)(h_w t_w^3 + 2 t_f w_f^3)}} \ . \tag{A.8}$$

Appendix B: Outline of COLSYS

The general purpose BVP solver COLSYS [1,2] is based on spline collocation at Gaussian points and is capable of handling mixed order systems of non-linear multi-point BVPs. An indication of what is done follows from a consideration of the two-point BVP for a first order system,

$$\mathbf{z}' = \mathbf{f}(x, \mathbf{z}) \quad x \in (a, b) \ , \tag{B.1}$$

$$\mathbf{g}^1(z(a)) = 0 \quad \mathbf{g}^2(z(b)) = 0 \ , \tag{B.2}$$

where $\mathbf{z}$ and $\mathbf{f}$ are vector functions of order m, $\mathbf{g}^1$ is of order m_1 and $\mathbf{g}^2$ is of order $m_2 = m - m_1$.

In COLSYS, the problem (B.1) and (B.2) is solved on a sequence of meshes, until user-specified error tolerances are satisfied. For a specific mesh $a = x_0 < x_1 < \ldots < x_N = b$, with $h_i = x_i - x_{i-1}$, $h = \max\{h_i, 1 < i < N\}$ and an integer $n > 1$, the collocation solution $\mathbf{v}(x) = (v_1, \ldots, v_m)$ is a piecewise polynomial vector function: for each j, $1 < j < m$, $v_j \in C[a, b]$ is a polynomial of degree $\leq n$ on each element (x_{i-1}, x_i), $i = 1, \ldots, N$. The piecewise polynomial solutions are represented in terms of a B-spline basis. The approximate solution is determined by requiring that it satisfy (B.2) and the differential equation (B.1) at the images of the n zeros of the appropriate Legendre-polynomial in each element. Under sufficient smoothness conditions, the error in $\mathbf{v}$ for $x \in (x_1, x_{i+1})$ is given by

$$z_j(x) - v_j(x) = K z_j^{(n+1)}(x_i) h_i^{n+1} + O(h^{n+2}) \ , \tag{B.3}$$

where K is a known bounded function of x, and at the mesh point x_i

$$z_j(x_i) - v_j(x_i) = O(h^{2n}) \ , \quad j = 1, \ldots, m \ ; \quad i = 0, 1, \ldots, N \ . \tag{B.4}$$

Expression (B.3) is used both for estimating the error accurately via mesh halving and for automatic new mesh selection. For our sixth order non-linear problem, the damped Newton's method is used for the first mesh to find the collocation solution, and modified Newton iterations with a fixed Jacobian are performed for subsequent refined meshes.

Appendix C: A formal asymptotic expansion for Eqs. (43) and (44)

In order to obtain the result in equation (70b), we take the solution of the eigenvalue problem defined by (43) and (44) in the form

$$\sigma = \sigma_0 + \sigma_1 \varepsilon + \sigma_2 \varepsilon^2 + \dots \tag{C.1}$$

and

$$f = F_i(\zeta;\varepsilon) + F_0(\zeta;\varepsilon) e^{-\zeta/\varepsilon} + F_1(\zeta;\varepsilon) e^{-(1-\zeta)/\varepsilon} , \tag{C.2}$$

with

$$\{F_i(\zeta;\varepsilon),\ F_0(\zeta;\varepsilon),\ F_1(\zeta;\varepsilon)\} = \sum_{n=0}^{\infty} \{F_{in}(\zeta),\ F_{0n}(\zeta),\ F_{1n}(\zeta)\} \varepsilon^n , \tag{C.3}$$

and we stipulate that differentiation with respect to ζ does not change orders of magnitude. We then substitute (C.1)–(C.3) into (43) and (44) and require that (43) and (44) be satisfied identically in ε. This gives for the O(1)-term coefficients

$$F_{00}(\zeta) \equiv 0 , \quad F_{10}(\zeta) \equiv 0 , \tag{C.4}$$

and

$$\ddot{F}_{i0} + \sigma_0^2 \zeta^2 F_{i0} = 0 , \quad \dot{F}_{i0}(0) = F_{i0}(1) = 0 . \tag{C.5}$$

The lowest eigenvalue of (C.5) is known to be $\sigma_0 = 4.012599 \dots$ with $\sigma_0/2$ being the first zero of $J_{-1/4}(x)$. The corresponding eigenfunction is

$$F_{i0}(\zeta) = a_0 \sqrt{\zeta}\, J_{-1/4}(\sigma_0 \zeta^2/2) , \tag{C.6}$$

where a_0 is introduced to normalize F_{i0} so that

$$\int_0^1 [F_{i0}(\zeta)]^2 \zeta^2 \, d\zeta = a_0^2 \int_0^1 \zeta^3 [J_{-1/4}(\sigma_0 \zeta^2/2)]^2 \, d\zeta = 1 . \tag{C.7}$$

The corresponding relations for the coefficients of the O(ε)-terms in (C.3) are

$$F_{01}(\zeta) \equiv 0 , \quad F_{11}(\zeta) = -\dot{F}_{i0}(1) , \tag{C.8}$$

$$\ddot{F}_{i1} + \sigma_0^2 \zeta^2 F_{i1} = -2\sigma_0 \sigma_1 \zeta^2 F_{i0} , \quad \dot{F}_{i1}(0) = 0 , \quad F_{i1}(1) = \dot{F}_{i0}(1) . \tag{C.9}$$

For the inhomogeneous problem (C.9) to have a solution, we must have

$$\frac{\sigma_1}{\sigma_0} = \frac{1}{2\sigma_0^2} [\dot{F}_{i0}(1)]^2 = 2 , \tag{C.10}$$

with the solvability condition $[\dot{F}_{i0}(1)]^2 = 4\sigma_0^2$ following from

$$\int_0^1 F_{i0} [\ddot{F}_{i0} + \sigma_0^2 \zeta^2 F_{i0}] \, d\zeta = -\int_0^1 [\dot{F}_{i0}(\zeta)]^2 \, d\zeta + \sigma_0^2 = 0 , \tag{C.11}$$

and

$$0 = \int_0^1 \zeta \dot{F}_{i0} [\ddot{F}_{i0} + \sigma_0^2 \zeta^2 F_{i0}] \, d\zeta = \frac{1}{2} [\dot{F}_{i0}(1)]^2 - \frac{1}{2} \int_0^1 [\dot{F}_{i0}]^2 \, d\zeta - \frac{3}{2} \sigma_0^2 \int_0^1 [F_{i0}]^2 \zeta^2 \, d\zeta , \tag{C.12}$$

in conjunction with (C.7).

To determine σ_2, we consider the O(ε^2)-terms in (C.3), which give

$$F_{02}(\zeta) \equiv 0 , \quad F_{12}(\zeta) = (\sigma_0^2/6) \dot{F}_{i0}(1) (1 - \zeta^3) - \dot{F}_{i1}(1) , \tag{C.13}$$

and

$$\ddot{F}_{i2} + \sigma_0^2 \zeta^2 F_{i2} = F_{i0} [\sigma_0^2 \zeta^4 - 2\sigma_0^2 - (\sigma_1^2 + 2\sigma_2 \sigma_0) \zeta^2] - 4\sigma_0^2 \zeta \dot{F}_{i0} - 2\sigma_0 \sigma_1 \zeta^2 F_{i1} , \tag{C.14a}$$

$$F^{\cdot}_{i2}(0)=0 \ , \quad F_{i2}(1)=F^{\cdot}_{i1}(1) \ . \tag{C.14b}$$

The solvability condition for the problem in (C.14a, b) is

$$\frac{\sigma_2}{\sigma_0}=-2+\frac{1}{2\sigma_0^2}\,F^{\cdot}_{i0}(1)F^{\cdot}_{i1}(1)-2\int_0^1 F_{i1}F_{i0}\zeta^2\,d\zeta+\frac{1}{2}\,\sigma_0^2\int_0^1 [F_{i0}]^2\zeta^4\,d\zeta \ . \tag{C.15}$$

Equation (C.15) is obtained by multiplying both sides of (C.14a) by F_{i0} and then integrating over the interval [0, 1]. With (C.15) satisfied, the solution of (C.14) can be determined by the method of variation of parameters. A numerical evaluation of the two integrals in (C.15) then gives

$$\sigma_2=5.981\ldots\sigma_0 \ . \tag{C.16}$$

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