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On the Effect of Transverse Shear Deformability on Stress Concentration Factors for Twisted and Sheared Shallow Spherical Shells

Explicit solutions are obtained, in terms of modified Bessel functions, for the problems of transverse twisting and of tangential shearing of transversely shear-deformable shallow spherical shells with a small circular hole. The relevant stress concentration factors are calculated for the entire range of a rise-to-thickness ratio parameter and a transverse shear deformability parameter. The modification of known results obtained previously by shear deformable plate theory, and by shallow shell theory without consideration of transverse shear deformation effects, is delineated.

Introduction

A recent reformulation of the differential equations of the linear theory of shear-deformable shallow two-dimensionally isotropic shells (Reissner and Wan, 1982) suggested the possibility of an explicit determination of the effect of transverse shear deformability on stress couple and stress resultant concentration factors for the problems of transverse twisting and tangential shearing of shallow spherical shells with a small circular hole as obtained in Reissner, (1980a, 1980b, 1981).

Given a shallow spherical shell with midsurface radius R, bending stiffness factor D, membrane flexibility factor B, transverse shear flexibility factor A, and a circular hole of radius a, it is found that the values of the stress couple concentration factors k_c and the stress resultant concentration factors k_r for both problems come out to be functions of just three dimensionless parameters. These are (1) a shell geometry parameter

$$\mu=\frac{a}{\sqrt[4]{DB}\,\sqrt{R}},$$

(2) a transverse shear deformability parameter

$$\lambda=\sqrt{\frac{2}{1-\nu}}\;\frac{a}{\sqrt{DA}},$$

and (3) Poisson's ratio ν. For a transversely homogeneous shell of wall thickness h, with a Young's modulus E for bending and membrane stresses, and a modulus of rigidity G for transverse shearing stress the quantities D, B and A are known to be of the form

$$D=\frac{Eh^3}{12(1-\nu^2)},\quad B=\frac{1}{Eh},\quad A=\frac{6}{5Gh},$$

and, accordingly

$$\mu=\sqrt[4]{12(1-\nu^2)}\,\frac{a}{\sqrt{Rh}},\quad \lambda=\sqrt{\frac{20(1+\nu)}{E/G}}\,\frac{a}{h}.$$

It is then evident that the limiting case $\lambda=\infty$ corresponds to configurations for which transverse shear deformation effects are absent or negligible. At the same time, the limiting case $\mu=0$ corresponds to configurations for which shell curvature effects are absent or negligible. Given the physical meaning of these two limiting cases it is possible to say that the quantitative results of the present analysis consist in the evaluation of four concentration factor functions $k(\lambda,\mu,\nu)$ in the range $0<\lambda\le\infty$, $0\le\mu<\infty$, in complementation of the results for $k(\lambda,\,0,\,\nu)$ in Reissner (1945) and for $k(\infty,\,\mu,\,\nu)$ in Reissner (1980a, 1980b, 1981).

Equations for Uniform Isotropic Shear-deformable Shallow Spherical Shells

Departing from a known system of equations for shear-deformable uniform isotropic shallow shells in cartesian coordinate form (Reissner and Wan, 1982) we have as polar coordinate expressions for stress resultants, stress couples, in generalization of an analogous statement for plates (Reissner, 1980c)

$$N_{rr}=\frac{K_{,r}}{r}+\frac{K_{,\theta\theta}}{r^2},\quad N_{\theta\theta}=\nabla^2K-N_{rr},\quad N_{r\theta}=\frac{K_{,\theta}}{r^2}-\frac{K_{,\theta r}}{r}\qquad(1a,b,c)$$

$$Q_r=-D(\nabla^2 v)_{,r}+\frac{\chi_{,\theta}}{r},\quad Q_\theta=-D\frac{(\nabla^2 v)_{,\theta}}{r}-\chi_{,r}\qquad(2a,b)$$

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$$M_{rr} = -D\left[\nabla^2 v - (1-\nu)\left(\frac{v_{,r}}{r} + \frac{v_{,\theta\theta}}{r^2}\right)\right]$$

$$+ (1-\nu)AD\left(\frac{\chi_{,\theta r}}{r} - \frac{\chi_{,\theta}}{r^2}\right), \quad (3a)$$

$$M_{\theta\theta} = -(1+\nu)D\nabla^2 v - M_{rr}, \quad (3b)$$

$$M_{r\theta} = (1-\nu)D\left(\frac{v_{,\theta}}{r^2} - \frac{v_{,\theta r}}{r}\right) + (1-\nu)AD\left(\frac{\chi_{,r}}{r} + \frac{\chi_{,\theta\theta}}{r^2}\right) - \chi. \quad (3c)$$

In these expressions the function v is given in terms of w and K in the form

$$v = w + AD(\nabla^2 w - AR^{-1}\nabla^2 K), \quad (4)$$

and K, w and χ are solutions of the tenth order system of differential equations

$$RD\nabla^2\nabla^2 w + (1 - AD\nabla^2)(\nabla^2 K + Rq) = 0, \quad (5)$$

$$RB\nabla^2\nabla^2 K - \nabla^2 w = 0, \qquad \frac{1}{2}(1-\nu)AD\nabla^2\chi = \chi. \quad (6a,b)$$

We note that equation (5) includes a transverse surface load term q which, in what follows, is set equal to zero.

Given the homogeneous eighth-order system of equations (5) and (6a) for w and K it is possible to express its solution in terms of two *harmonic* functions ϕ and ψ and in terms of a function ω, which is the solution of a fourth-order equation, as follows. We write

$$w = \phi + \omega, \qquad K = \psi + \kappa, \quad (7)$$

and deduce from equations (5) and (6a) that ω and κ must satisfy the fourth order system

$$RD\nabla^2\omega + \kappa - AD\nabla^2\kappa = 0, \qquad RB\nabla^2\kappa - \omega = 0. \quad (8a,b)$$

An introduction of $\nabla^2\kappa$ from equation (8b) into equation (8a) then gives an expression for κ in terms of ω and $\nabla^2\omega$

$$\kappa = \frac{AD}{RB}\omega - RD\nabla^2\omega, \quad (9)$$

and this introduced into equation (8b) gives as differential equation for ω

$$BDR^2\nabla^2\nabla^2\omega - AD\nabla^2\omega + \omega = 0. \quad (10)$$

Having equations (7), (9), (10), and (4), it is now possible to express the functions v and K in the defining relations (1) to (3) in terms of ϕ, ψ, and ω, as follows

$$K = \psi - RD\nabla^2\omega + \frac{AD}{RB}\omega, \quad (11a)$$

$$v = \phi + \left(1 - \frac{DA^2}{BR^2}\right)\omega + AD\nabla^2\omega. \quad (11b)$$

Equations (10) and (11a,b) reduce, as they should, to the corresponding relations for nonshear-deformable shells in Reissner (1980a), upon setting in them $A = 0$.

We note, for subsequent reference, that equations (11a, b), in conjunction with equation (10) imply the relations

$$\nabla^2 K = \frac{\omega}{RB}, \quad \nabla^2 v = \nabla^2\omega - \frac{A\omega}{BR^2}. \quad (12a,b)$$

The Boundary Value Problems

The conditions of vanishing stress resultants and couples at the edge of the circular hole are of the homogeneous form

$$r = a; \quad N_{rr} = N_{r\theta} = Q_r = 0, \quad M_{rr} = M_{r\theta} = 0. \quad (13)$$

The assumption of a *small* hole means the possibility of stipulating boundary conditions *at infinity*, consistent with the *uniform* states of stress which hold in the absence of the hole. For the problem of *transverse twisting* this uniform state of stress is such that $M_{xx} = M_{yy} = 0$, $M_{xy} = T/2$, and $Q_x = Q_y = N_{xx} = N_{yy} = N_{xy} = 0$. A transformation of this into polar-coordinate form then gives as the nonhomogeneous boundary conditions at infinity of the problem of transverse twisting

$$r = \infty; \quad N_{rr} = N_{r\theta} = Q_r = 0,$$

$$M_{rr} = \frac{1}{2}T\sin 2\theta, \qquad M_{r\theta} = \frac{1}{2}T\cos 2\theta. \quad (14)$$

For the problem of *tangential shearing* the corresponding uniform state is such that $M_{xx} = M_{yy} = M_{xy} = 0$, $N_{xx} = N_{yy} = Q_x = Q_y = 0$ and $N_{xy} = S$. The associated polar coordinate form of the boundary conditions at infinity is

$$r = \infty; \quad M_{rr} = M_{r\theta} = 0, \quad Q_r = 0,$$

$$N_{rr} = S\sin 2\theta,$$

$$N_{r\theta} = S\cos 2\theta. \quad (15)$$

A significant conclusion which follows from the boundary conditions at the edge of the hole, in conjunction with the defining relations (1b) and (3b), is that the expressions for the *edge* values of the resultant $N_{\theta\theta}$ and the couple $M_{\theta\theta}$ are

$$r = a; \quad N_{\theta\theta} = \nabla^2 K, \quad M_{\theta\theta} = -(1+\nu)D\nabla^2 v, \quad (16)$$

with $\nabla^2 K$ and $\nabla^2 v$ as in equations (12a,b).

Nondimensionalization

We introduce a nondimensional radial coordinate ρ and nondimensional solution functions Φ, Ψ, Ω, and X through the relations

$$r = a\rho, \quad \phi = \phi_o\Phi, \quad \psi = \psi_o\Psi, \quad \omega = \omega_o\Omega, \quad \chi = \chi_o X, \quad (17)$$

with a view towards expressing the four scale factors ϕ_o, ψ_o, ω_o and χ_o in terms of the load intensity measures T and S, and in terms of the geometrical and constitutive parameters a, R, A, B, and D.

We then rewrite the differential equations (6b) and (10) through the introduction of two dimensionless parameters

$$\lambda = \sqrt{\frac{2}{1-\nu}}\,\frac{a}{\sqrt{DA}}, \quad \mu = \frac{a}{\sqrt[4]{DB}\sqrt{R}}, \quad (18a,b)$$

in the form

$$\nabla^2 X = \lambda^2 X, \qquad \nabla^2\nabla^2\Omega = \frac{2}{1-\nu}\,\frac{\mu^4}{\lambda^2}\nabla^2\Omega - \mu^4\Omega, \quad (19a,b)$$

with the Laplace operator ∇^2 from now on considered to involve the variable ρ instead of r.

Given the defining relations in equations (17) and (18a,b), the functions K and v in equations (11a,b) now become

$$K = \psi_o\Psi - \omega_o\frac{RD}{a^2}\left[\nabla^2\Omega - \frac{2}{1-\nu}\,\frac{\mu^4}{\lambda^2}\Omega\right] \quad (20a)$$

$$v = \phi_o\Phi + \omega_o\left[\left(1 - \frac{4}{(1-\nu)^2}\,\frac{\mu^4}{\lambda^4}\right)\Omega + \frac{2}{1-\nu}\,\frac{1}{\lambda^2}\nabla^2\Omega\right]. \quad (20b)$$

As regards the fourth order differential equation (19b) we have the possibility of a factorization $(\nabla^2 - \alpha_+^2)(\nabla^2 - \alpha_-^2)\Omega = 0$ with real α_+ and α_-, as well as the possibility of a factorization $(\nabla^2 - \alpha^2)(\nabla^2 - \bar{\alpha}^2)\Omega = 0$ with conjugate complex α and $\bar{\alpha}$, in accordance with a characteristic equation

$$\alpha^4 - \frac{2}{1-\nu}\,\frac{\mu^4}{\lambda^2}\alpha^2 + \mu^4 = 0. \quad (21)$$

The roots of this quadratic equation for α^2 are

$$\begin{pmatrix}\alpha_+^2 \\ \alpha_-^2\end{pmatrix} = \frac{\mu^4/\lambda^2}{1-\nu}\left(1 \pm \sqrt{1 - (1-\nu)^2\frac{\lambda^4}{\mu^4}}\right), \quad \sqrt{1-\nu} < \frac{\mu}{\lambda}, \quad (22a)$$

$$\begin{pmatrix}\alpha^2 \\ \bar{\alpha}^2\end{pmatrix} = \frac{\mu^4/\lambda^2}{1-\nu}\left(1 \pm i\sqrt{(1-\nu)^2\frac{\lambda^4}{\mu^4} - 1}\right), \quad \frac{\mu}{\lambda} < \sqrt{1-\nu}. \quad (22b)$$

With these the general solution of equation (19*b*) will be of the form

$$\Omega=\Omega^{+}+\Omega^{-},\quad \nabla^2\Omega^{+}=\alpha_{+}^2\Omega^{+},\quad \nabla^2\Omega^{-}=\alpha_{-}^2\Omega^{-},\tag{23a}$$

in the parameter range $\lambda\sqrt{1-\nu}<\mu$, and of the form

$$\Omega=\Omega^{*}+\bar{\Omega}^{*},\quad \nabla^2\Omega^{*}=\alpha^2\Omega^{*},\quad \nabla^2\bar{\Omega}^{*}=\bar{\alpha}^2\bar{\Omega}^{*},\tag{23b}$$

in the parameter range $\mu<\lambda\sqrt{1-\nu}$.

With equations (17), (18*a*, *b*), and (23*b*) we may rewrite the expressions for K and v in equations (11*a*,*b*) in the form

$$K=\psi_o\Psi-\omega_o\frac{RD}{a^2}\left[\left(\alpha^2-\frac{2\mu^4}{1-\nu}\frac{1}{\lambda^2}\right)\Omega^{*}+\left(\bar{\alpha}^2-\frac{2\mu^4}{1-\nu}\frac{1}{\lambda^2}\right)\bar{\Omega}^{*}\right],\tag{24a}$$

$$v=\phi_o\Phi+\omega_o\left[\left(1-\frac{4}{(1-\nu)^2}\frac{\mu^4}{\lambda^4}+\frac{2}{1-\nu}\frac{\alpha^2}{\lambda^2}\right)\Omega^{*}+\left(1-\frac{4}{(1-\nu)^2}\frac{\mu^4}{\lambda^4}+\frac{2}{1-\nu}\frac{\bar{\alpha}^2}{\lambda^2}\right)\bar{\Omega}^{*}\right],\tag{24b}$$

with corresponding expressions when equation (23*a*) applies.

Choice of Particular Solutions

Given the form of the nonhomogeneous boundary conditions in equations (14) and (15) it is apparent that appropriate solution functions will be of the form

$$[\Phi,\Psi,\Omega]=[\Phi_2(\rho),\Psi_2(\rho),\Omega_2(\rho)]\sin 2\theta,\qquad X=X_2(\rho)\cos 2\theta.\tag{25}$$

The functions Φ_2 and Ψ_2 are

$$\Phi_2=C_1\rho^2+C_2\rho^{-2},\quad \Psi_2=C_3\rho^2+C_4\rho^{-2}.\tag{26a,b}$$

The function X_2 as determined by equation (19*a*), with appropriate behavior for large values of ρ, is

$$X_2=C_5K_2(\lambda\rho),\tag{27}$$

with K_2 being a modified Bessel function of order two.

The appropriate solution of equation (23*a*) is

$$\Omega_2=C_6K_2(\alpha_{+}\rho)+C_7K_2(\alpha_{-}\rho),\tag{28a}$$

and the appropriate solution of equation (23*b*) is

$$\Omega_2=C_{67}K_2(\alpha\rho)+\bar{C}_{67}K_2(\bar{\alpha}\rho),\tag{28b}$$

with $C_{67}\equiv C_6+iC_7$.

It remains to determine the seven constants of integration C_i with the help of the five boundary conditions in equation (13), and two of the five conditions in either equation (14) or (15), which determine C_1 and C_3. The other three conditions in (14) or (15) are satisfied automatically by the restricted choice of the functions X_2 and Ω_2 in equations (27) and (28*a*,*b*).

Satisfaction of Boundary Conditions

In satisfying the conditions in equations (14) and (15), we take account of the fact that X_2 and Ω_2 and all their derivatives vanish at infinity, with only the first terms in Φ_2 and Ψ_2 failing to do so. It is then a simple matter to deduce, on the basis of the expressions for $N_{r\theta}$ in equation (1*c*) and $M_{r\theta}$ in equation (3*c*), the two relations

$$(1-\nu)D\phi_oC_1=-\frac{1}{4}Ta^2,\quad \psi_oC_3=-\frac{1}{2}Sa^2.\tag{29a,b}$$

In satisfying the five conditions in equation (13) we begin by noting that the two conditions for N_{rr} and $N_{r\theta}$ are readily shown to be equivalent to two conditions of the form $K(a,\theta)=K_{,r}(a,\theta)=0$. Therewith, we obtain on the basis of equations (24*a*), (23*b*), (26*b*), and (28*b*)

$$\psi_o(C_3+C_4)-\omega_o\frac{RD}{a^2}\left[\left(\alpha^2-\frac{2\mu^4}{1-\nu}\frac{1}{\lambda^2}\right)K_2(\alpha)C_{67}+\left(\bar{\alpha}^2-\frac{2\mu^4}{1-\nu}\frac{1}{\lambda^2}\right)K_2(\bar{\alpha})\bar{C}_{67}\right]=0,\tag{30}$$

$$\psi_o(2C_3-2C_4)-\omega_o\frac{RD}{a^2}\left[\left(\alpha^2-\frac{2\mu^4}{1-\nu}\frac{1}{\lambda^2}\right)\alpha K_2'(\alpha)C_{67}+\left(\bar{\alpha}^2-\frac{2\mu^4}{1-\nu}\frac{1}{\lambda^2}\right)\bar{\alpha}K_2'(\bar{\alpha})\bar{C}_{67}\right]=0.\tag{31}$$

We note next that the condition $Q_r(a,\theta)=0$ becomes, with equations (2*a*), (12*b*), (17), (27), and (28*b*)

$$\chi_o2K_2(\lambda)C_5+\omega_o\frac{D}{a^2}\left[\left(\alpha^2-\frac{2\mu^4}{1-\nu}\frac{1}{\lambda^2}\right)\alpha K_2'(\alpha)C_{67}+\left(\bar{\alpha}^2-\frac{2\mu^4}{1-\nu}\frac{1}{\lambda^2}\right)\bar{\alpha}K_2'(\bar{\alpha})\bar{C}_{67}\right]=0,\tag{32}$$

and that a combination of equations (32) and (31) allows a replacement of equation (31) by the simpler relation

$$\psi_o(2C_3-2C_4)+2R\chi_oK_2(\lambda)C_5=0\tag{31*}$$

The condition $M_{rr}(a,\theta)=0$ becomes with equations (3*a*), (11*b*), (12*b*), (17), (24), (27), and (28*b*)

$$\omega_o\left\{\left[\left(\alpha^2-\frac{2\mu^4}{1-\nu}\frac{1}{\lambda^2}\right)K_2(\alpha)-\left(1-\nu-\frac{4\mu^4}{1-\nu}\frac{1}{\lambda^4}+2\frac{\alpha^2}{\lambda^2}\right)(\alpha K_2'(\alpha)-4K_2(\alpha))\right]C_{67}+\left[\left(\bar{\alpha}^2-\frac{2\mu^4}{1-\nu}\frac{1}{\lambda^2}\right)K_2(\bar{\alpha})-\left(1-\nu-\frac{4\mu^4}{1-\nu}\frac{1}{\lambda^4}+2\frac{\bar{\alpha}^2}{\lambda^2}\right)(\bar{\alpha}K_2'(\bar{\alpha})-4K_2(\bar{\alpha}))\right]\bar{C}_{67}\right\}+\chi_o2(1-\nu)A[\lambda K_2'(\lambda)-K_2(\lambda)]C_5+\phi_o2(1-\nu)(C_1+3C_2)=0.\tag{33}$$

In the same way the condition $M_{r\theta}(a,\theta)=0$ becomes

$$\omega_o\frac{2D}{a^2}\left\{\left[1-\nu-\frac{4\mu^4}{1-\nu}\frac{1}{\lambda^4}+2\frac{\alpha^2}{\lambda^2}\right][K_2(\alpha)-\alpha K_2'(\alpha)]C_{67}+\left[1-\nu-\frac{4\mu^4}{1-\nu}\frac{1}{\lambda^4}+2\frac{\bar{\alpha}^2}{\lambda^2}\right][K_2(\bar{\alpha})-\bar{\alpha}K_2'(\bar{\alpha})]\bar{C}_{67}\right\}-\chi_o\left\{K_2(\lambda)-\frac{2}{\lambda^2}[\lambda K_2'(\lambda)-4K_2(\lambda)]\right\}C_5-\phi_o2(1-\nu)\frac{D}{a^2}(C_1-3C_2)=0.\tag{34}$$

Insofar as the determination of stress concentration factors is concerned we obtain from the two relations in equation (16), in conjunction with equations (12*a*, *b*), (17), and (28*b*)

$$N_{\theta\theta}\left(a,\frac{\pi}{4}\right)=\frac{\omega_o}{RB}[C_{67}K_2(\alpha)+\bar{C}_{67}K_2(\bar{\alpha})]\tag{35}$$

and

$$M_{\theta\theta}\left(a,\frac{\pi}{4}\right)=-(1+\nu)\frac{\omega_oD}{a^2}\left[\left(\alpha^2-\frac{2\mu^4}{1-\nu}\frac{1}{\lambda^2}\right)C_{67}K_2(\alpha)+\left(\bar{\alpha}^2-\frac{2\mu^4}{1-\nu}\frac{1}{\lambda^2}\right)\bar{C}_{67}K_2(\bar{\alpha})\right].\tag{36}$$

Having equations (29*a*, *b*) to (36) it now remains to arrive at suitable dispositions of the four scale factors ϕ_o, ψ_o, ω_o and χ_o, and to determine the two constants of integration C_6 and C_7 through the solution of the system of equations (29*a*, *b*) to (34). Thereafter, the results must be introduced into the expressions for $N_{\theta\theta}(a,\pi/4)$ and $M_{\theta\theta}(a,\pi/4)$.

While equations (29*a*,*b*) to (36) have been deduced for the parameter value range $\mu<\lambda\sqrt{1-\nu}$, in accordance with equations (22*b*), (23*b*), and (28*b*), it is apparent that their use in the range $\lambda\sqrt{1-\nu}<\mu$ requires no other changes in equations (30)

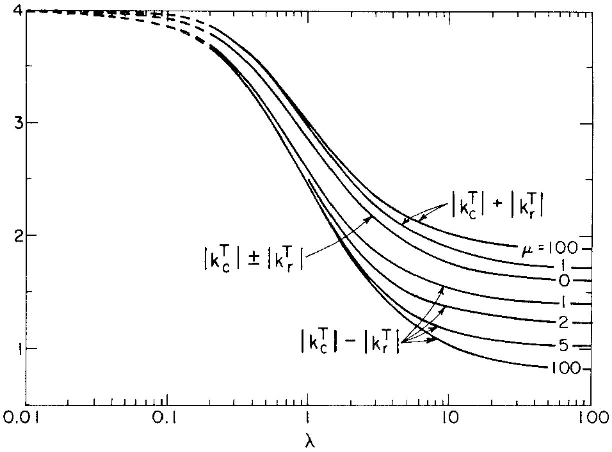

Fig. 1 Upper and lower values of stress concentration factors for the effect of a circular hole of radius *a*, in a shallow spherical shell of radius *R* for the problem of transverse twisting. The parameters μ and λ depend on *a* and *R*, on the bending stiffness factor *D*, on membrane and sheardeformability factors *B* and *A*, and on Poisson's ratio ν in accordance with equations (18a, b). The curves shown are for $\nu = 1/3$.

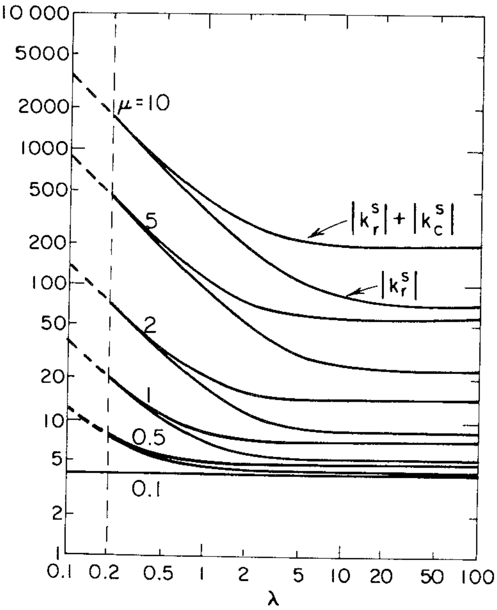

Fig. 2 Values of the membrane stress concentration factor k_r^S and of the maximum fiber stress concentration factor $|k_r^S| + |k_c^S|$ for the problem of tangential shearing, as a function of the parameters λ and μ, with $\nu = 1/3$

to (36) than to write α_+ and α_- as defined in equation (22*a*) in place of α and $\bar{\alpha}$, respectively, and to write C_6 and C_7 in place of C_{67} and $\bar{C}_{67}$, respectively. In either range the function K_2 and its derivative K_2' can be expressed as the linear combinations $K_2(z) = K_0(z) + 2z^{-1}K_1(z)$, $K_2'(z) = -2z^{-1}K_0(z) - (1+4z^{-2})K_1(z)$, with the functions K_0 and K_1 generated by appropriate special cases of the computer library functions *CBESK*(v,z,N) in Nicol (1973). Having the values of these functions equations (30) to (34) can then be solved for C_2. C_4, C_5, C_6, and C_7 in terms of C_1 and C_3, with λ, μ and ν as input parameters, by a typical simultaneous equation solver as in Nicol (1982). The values of these coefficients will be used for the calculation of the stress resultant and stress couple concentration factors, as defined in what follows.

Expressions for Stress Couple and Stress Resultant Concentration Factors

We now consider equations (29*a*, *b*) to (36) separately for the cases of transverse twisting, by setting $S=0$, and of tangential shearing, by setting $T=0$.

When $S=0$, we satisfy equations (29*a*, *b*) upon taking

$$(1-\nu)D\phi_o = -\frac{1}{4}Ta^2, \qquad C_1 = 1, \quad C_3 = 0. \tag{37}$$

Subsequent to this we simplify the remaining five simultaneous equations (30) to (34) by setting

$$\omega_o = (1-\nu)\phi_o, \quad a^2\psi_o = RD\omega_o, \quad a^2\chi_o = -2D\omega_o, \tag{38}$$

and we note that we have with these in equations (35) and (36)

$$\frac{\omega_o}{RB} = -\frac{T\mu^2}{4\sqrt{BD}}, \quad \frac{\omega_o D}{a^2} = -\frac{T}{4}. \tag{39a,b}$$

When $T=0$ we satisfy equations (29*a*, *b*) by setting

$$\psi_o = -\frac{1}{2}Sa^2, \quad C_3 = 1, \quad C_1 = 0, \tag{40}$$

and subsequent to this we simplify equations (30) to (34) by setting

$$RD\omega_o = a^2\psi_o, \quad R\chi_o = \psi_o, \quad (1-\nu)\phi_o = \omega_o, \tag{41}$$

and therewith in equations (35) and (36),

$$\frac{\omega_o}{RB} = -\frac{S\mu^4}{2}, \quad \frac{\omega_o D}{a^2} = -\frac{S\sqrt{BD}\,\mu^2}{2}. \tag{42a,b}$$

Given that for the problem of transverse twisting we will have $M_{\theta\theta}(\infty,\pi/4) = -M_{rr}(\infty,\pi/4) = -T/2$ we obtain from equations (36) and (39*b*) as expression for the stress couple concentration factor for the problem

$$\frac{M_{\theta\theta}(a,\pi/4)}{M_{\theta\theta}(\infty,\pi/4)} \equiv k_c^T(\lambda,\mu,\nu)$$

$$= \frac{1+\nu}{2}\left[\left(\alpha^2 - \frac{2\mu^4}{1-\nu}\frac{1}{\lambda^2}\right)C_{67}K_2(\alpha) + \left(\bar{\alpha}^2 - \frac{2\mu^4}{1-\nu}\frac{1}{\lambda^2}\right)\bar{C}_{67}K_2(\bar{\alpha})\right]. \tag{43}$$

Given that for the problem of tangential shearing we will have $N_{\theta\theta}(\infty, \pi/4) = -N_{rr}(\infty, \pi/4) = S$ we obtain from equations (35) and (42*a*)

$$\frac{N_{\theta\theta}(a,\pi/4)}{N_{\theta\theta}(\infty,\pi/4)} \equiv k_r^S(\lambda,\mu,\nu)$$

$$= -\frac{1}{2}\mu^4[C_{67}K_2(\alpha) + \bar{C}_{67}K_2(\bar{\alpha})]. \tag{44}$$

In order to deduce information on the associated concentration factors k_r^T and k_c^S it is necessary to have information concerning the nature of the dependence of the stress $\sigma_{\theta\theta}$ on perpendicular distance from the middle surface of the shell, with this involving the dependence of Young's modulus E on the thickness coordinate z. We here limit ourselves to a consideration of problems with z-independent E and, moreover, in this connection, stipulate that for present purposes $\sigma_{\theta\theta}$ is, to an acceptable degree of approximation, a linear function of z.

The expression for the stress resultant concentration factor for the problem of transverse twisting then is

$$k_r^T = \frac{N_{\theta\theta}(a,\pi/4)/h}{6M_{\theta\theta}(\infty,\pi/4)/h^2} = \frac{hN_{\theta\theta}(a,\pi/4)}{3T}, \tag{45}$$

and this becomes, with equations (16), (12*a*), (17), (28*b*), (39*a*), and with $(BD)^{-1/2} = [12(1-\nu^2)]^{1/2}h^{-1}$,

$$k_r^T = -\sqrt{\frac{1-\nu^2}{3}}\,\frac{\mu^2}{2}[C_{67}K_2(\alpha) + \bar{C}_{67}K_2(\bar{\alpha})]. \tag{46}$$

In the same way we obtain as expression for the stress couple concentration factor for the problem of tangential shearing

$$k_c^S = \frac{6M_{\theta\theta}(a,\pi/4)/h^2}{N_{\theta\theta}(\infty,\pi/4)/h} = \frac{6M_{\theta\theta}(a,\pi/4)}{Sh}, \quad (47)$$

and this becomes with equations (16), (12*b*), (17), and (42*b*)

$$k_c^S = -\sqrt{3\frac{1+\nu}{1-\nu}}\,\frac{\mu^2}{2}\left[\left(\alpha^2 - \frac{2\mu^4}{1-\nu}\,\frac{1}{\lambda^2}\right)C_{67}K_2(\alpha) + \left(\bar{\alpha}^2 - \frac{2\mu^4}{1-\nu}\,\frac{1}{\lambda^2}\right)\bar{C}_{67}K_2(\bar{\alpha})\right]. \quad (48)$$

The results for k_c^T in equation (43) and k_r^T in equation (46) obtained from our computer calculation as previously described are consistent with the explicit results in Reissner (1980a) for the case $\lambda=\infty$. The result for k_c^T in accordance with equation (43) is also consistent with the explicit result for $\mu=0$ in Reissner (1945). The results for k_r^S in equation (44) and k_c^S in equation (48) are consistent with Kirsch's result for $\mu=0$, and also with the results for $\lambda=\infty$ in E. Reissner (1980b) and in J. E. Reissner (1981). In order to gain an impression of the quantitative effect of transverse shear deformability for the stress concentration problems discussed in the foregoing we have calculated, in particular, the stress concentration magnitudes for outer and inner layers which result upon superimposing bending stresses and membrane stresses for these layers.

As regards these results we note that for the problem of transverse twisting, we have, in accordance with Fig. 1, increases in the magnitude of the stress concentration with increasing transverse shear deformability, similar to what had previously been known for the problem of the plate. The same as for the case of absent transverse shear deformability, the results for the shell are qualitatively similar to those for the limiting case of the plate.

For the problem of membrane shear, in accordance with Fig. 2, we find that transverse shear deformability aggravates the previously discovered extremely pronounced effect of shell curvature on the magnitude of the stress concentration due to a small hole. In connection with this discovery we recall that we had earlier ascribed this phenomena to the consequences of a not previously encountered necessity of reconciling the inextensional bending nature of the shell stress state as generated by the boundary conditions at the edge of the hole with the membrane nature of the shall stress state as generated by the loading conditions "at infinity." This being the case it is suggested that future work be concerned with the effect of an edge reinforcing ring on the stress concentration due to the effect of a small circular hole on the otherwise uniform state of membrane shear in a shallow spherical shell.

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