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EDGE EFFECTS IN THE STRETCHING OF PLATES*

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ABSTRACT

The stretching of flat plates is investigated by methods first introduced by the authors in the context of plate bending. The elastic reciprocal theorem is used to generate necessary conditions which the prescribed data at the edge of the plate must satisfy in order that it should generate a decaying state within the plate; these decaying state conditions are obtained explicitly for the case of axisymmetric stretching (and torsion) of a circular plate when stress or mixed conditions are prescribed at the plate edge. The conditions which any interior solution must satisfy at the plate edge are then deduced. As an example we obtain the complete interior solution (correct to within exponentially small error) for the problem of a simply supported thick circular plate under a concentrated load. It is shown conclusively that applications of Saint-Venant's principle lead to wrong corrections to the Kirchhoff thin plate theory.

1. INTRODUCTION

The exact solution of linear elastostatic problems for flat plates is known to consist of an <u>interior</u> component and <u>layer</u> components. The interior solution is significant throughout the plate, while a layer solution has only a localized effect in a region with typical linear dimensions of the order of the plate thickness. Boundary conditions at an edge of the plate are generally satisfied only by a combination of the two types of solution components. However, the layer solutions are difficult to obtain and the solution behavior near the plate edges is often not needed for design purposes. Hence, there has been a continual effort over the years to determine the interior solution of plate problems without any reference to the layer solutions. Saint-Venant's principle [1] has been used for this purpose (see [2] and [3] for examples). As it is well known, the principle makes it possible to derive an appropriate set of stress boundary conditions for the interior solution. Strictly speak-

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ing, Saint-Venant's principle does not apply to plate problems as one linear dimension of the loaded area, namely the circumference of the plate's midplane, is not small compared to a representative plate span. Another interesting approach to obtaining the interior solution which also invokes (the unproven) Saint-Venant's principle was proposed in [4] but was not entirely satisfactory for reasons previously discussed in [5] and [6].

In [5] and [6], we developed a new and completely different method determining the interior solution without any reference to the layer solutions. The method effectively obtains also an appropriate set of boundary conditions for the interior solution (which includes both the classical and related thick plate theories) corresponding to the given set of admissible stress, displacement or mixed edge data for the three-dimensional elasticity problem. In contrast to all previous attempts, the key to our method lies in a novel application of the Betti-Raleigh reciprocal theorem. The special case of a semi-infinite plate in a state of plane strain induced by edgewise uniform edge data was first analyzed by this method in [5]. For stress edge data, the result obtained there rigorously justifies the application of Saint-Venant's principle. More importantly, correct boundary conditions for the interior solution for plates with displacement and mixed edge data were obtained for the first time. Next, we obtained in [6] the corresponding results for axisymmetric bending of a circular plate. For the stress edge data, the results show that indiscriminate use of Saint-Venant's principle for plate bending problems may lead to quantitatively and qualitatively incorrect solutions even in the plate interior; they in effect delimit the range of applicability of Saint-Venant's principle for axisymmetric bending problems. Results were also obtained in [6] for edgewise nonuniform data along the straight edge of a semi-infinite plate in bending.

In this paper, we complement the discussion in [6] for circular plates with an analysis of the problem of axisymmetric stretching and torsion. The stress boundary conditions obtained for the interior solution again delimit the applicability of Saint-Venant's principle in the case of stress edge data. The stress boundary conditions for axisymmetric extension along with the corresponding conditions for axisymmetric bending from [6] enable us to obtain the first correct interior solution for a circular plate subject to a point force at the center of its upper face and "simply-supported" at its edge. A number of results for this problem are of general interest, including the following:

(1) It is demonstrated conclusively that the results in [2], [7] and [8] for the same problem, obtained by incorrect applications of Saint-Venant or other ad hoc methods, are in significant error for thick plates and may be inferior to the Kirchhoff solution.

(2) The correction to the classical Kirchhoff thin plate solution for the "exact" interior solution is in fact singular at the point of load application as predicted by Reissner's plate theory [9] (see also Appendix IV of the present paper).

(3) There is generally an extensional component in the interior solution of the same order of magnitude as the corrections to the Kirchhoff solution in the bending part of the problem, a fact not known previously.

(4) The effect in the interior solution of the different interpretation of a "concentrated load" and a "simply supported edge" in a three-dimensional setting are of the same order as the corrections to the Kirchhoff solution.

From these results for the point load problem and other results in [6], a conclusion which is of considerable importance in engineering analysis and design may be inferred. In the framework of a thick plate theory, it is not possible to determine the interior solution when only stress resultants and stress couples are prescribed at the edge.

2. DECAYING STATES IN A CIRCULAR PLATE

The central step in our method of approach to obtaining the interior solution is to seek the answer to the following question: What (necessary) conditions must an admissible set of edge data satisfy in order that the resulting solution in the plate should be a decaying elasto-static state? The notion of a decaying (elasto-static) state for a plate of general shape is defined in [6]. Briefly, the stress and displacement components, σ_{ij} and u_j, of a decay-state decay exponentially away from the plate edge:

$$\{u_j,\sigma_{ij}\} = O(e^{-\gamma d/h}) \qquad \text{as} \quad h \to 0$$

where $2h$ is the plate thickness, d is the distance of the observation point from the plate edge and γ is a positive constant. Once we have a sufficient number of the necessary conditions sought above, they can be translated into boundary conditions for the interior solution of plate problems.

Necessary conditions for various types of admissible edge data have been derived in [6] for plates of general shape by way of the elastic reciprocal theorem. In cylindrical coordinates (r,θ,z) a necessary condition for the stress data $\bar{\sigma}^d_r$, $\bar{\sigma}^d_{r\theta}$ and $\bar{\sigma}^d_{rz}$ (prescribed at the edge of a circular plate of radius a and thickness $2h$) to induce only a decaying state in the plate was found to be

$$\int_{-h}^{h}\int_{0}^{2\pi} \{\bar{\sigma}^d_{rr}u^{(2)}_r + \bar{\sigma}^d_{r\theta}u^{(2)}_\theta + \bar{\sigma}^d_{rz}u^{(2)}_z\}_{r=a}\, a\,d\theta\,dz = 0 \qquad (2.1)$$

for any suffix (2) elasto-static state $\{u_j^{(2)}, \sigma_{jk}^{(2)}\}$ which satisfies the stress-free conditions on the faces $z = \pm h$ and along the edge $r = a$, and has at worst an algebraic growth as $h \to 0$*. With different suffix (2) states, the condition (2.1) gives a whole class of necessary conditions for the stress data to induce a decaying elastostatic state.

It was also shown in [6] that the corresponding necessary condition for displacement edge data $\bar{u}_r^d$, $\bar{u}_\theta^d$, and $\bar{u}_z^d$ to induce only a decaying elastostatic state is

$$\int_{-h}^{h}\int_{0}^{2\pi} \{u_r^d\sigma_{rr}^{(2)} + \bar{u}_\theta^d\sigma_{r\theta}^{(2)} + \bar{u}_z^d\sigma_{rz}^{(2)}\}_{r=a} a\,d\theta\,dz = 0 \tag{2.2}$$

for any regular suffix state (2) which satisfies the traction-free condition on the faces $z = \pm h$ and the conditions of no displacement at $r = a$. Similarly, a necessary condition for the typical mixed edge data $\bar{u}_r^d$, $\bar{u}_\theta^d$ and $\bar{\sigma}_{rz}^d$ was found to be

$$\int_{-h}^{h}\int_{0}^{2\pi} \{u_r^d\sigma_{rr}^{(2)} + \bar{u}_\theta^d\sigma_{r\theta}^{(2)} - \bar{\sigma}_{rz}^d u_z^{(2)}\}_{r=a} a\,d\theta\,dz = 0 \tag{2.3}$$

where the regular suffix (2) state satisfies the traction-free conditions on $z = \pm h$ and the homogeneous edge condition $u_r^{(2)} = 0$ and $u_\theta^{(2)} = 0$ and $\sigma_{rz}^{(2)} = 0$ on $r = a$. (In all cases, the edge conditions for the suffix (2) state are the homogeneous counterparts of the actual edge conditions for the decaying state.) The practical difficulty in deriving from (2.1)-(2.3) the desired boundary conditions for the interior solution lies in the determination of suitable suffix (2) states which satisfy the required boundary conditions. This task simplifies considerably for problems with axisymmetry.

For axisymmetric edge data and an axisymmetric suffix (2) state, the integration with respect to θ can be carried out, leaving us with a condition involving only integration across the plate thickness. For example, the condition for stress data becomes

$$\int_{-h}^{h} \{\bar{\sigma}_{rr}^d(z)u_r^{(2)}(a,z) + \bar{\sigma}_{r\theta}^d(z)u_\theta^{(2)}(a,z) + \bar{\sigma}_{rz}^d(z)u_z^{(2)}(a,z)\}dz = 0 \tag{2.4}$$

where we have retained the torsion action by allowing for a nonvanishing $u_\theta^{(2)}$. For axisymmetric bending, one suffix (2) state for stress data is a rigid body translation in the z direction. With $u_r^{(2)} \equiv 0$, $u_\theta^{(2)} \equiv 0$, and $u_z^{(2)} = 1$, the condition (2.4) becomes

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* An elastostatic state $\{u_j, \sigma_{jk}\}$ which has at worst an algebraic growth as $h \to 0$ is called a regular (elastostatic) state.

$$\int_{-h}^{h} \bar{\sigma}^{d}_{rz} dz = 0 \ . \tag{2.5}$$

A second suffix (2) state was found to be

$$u_r^{(2)} = [(1+\nu)\frac{a}{r} + (1-\nu)\frac{r}{a}]z \ , \qquad u_\theta^{(2)} \equiv 0$$

$$u_z^{(2)} = -(1+\nu)a \,\ell n(\frac{r}{a}) - (1-\nu)\frac{r^2}{2a} - \nu\frac{z^2}{a} \tag{2.6}$$

for which (2.4) becomes

$$\int_{-h}^{h} \{\bar{\sigma}^{d}_{rr} - \frac{\nu}{2a} z^2 \bar{\sigma}^{d}_{rz}\} dz = 0 \tag{2.7}$$

While (2.5) is the expected requirement of no resultant axial force, the condition (2.7) is not the usual condition ([2], [3])

$$\int_{-h}^{h} z\bar{\sigma}^{d}_{rr} dz = 0 \tag{2.8}$$

resulting from a direct application of Saint-Venant's principle to plate problems with prescribed edge tractions. A thorough discussion of this difference can be found in [6] and will not be repeated here.

We now take $\bar{\sigma}^{d}_{rz} = \sigma^{I}_{rz}(a,z) - \bar{\sigma}_{rz}(z)$ and $\bar{\sigma}^{d}_{rr} = \sigma^{I}_{rr}(a,z) - \bar{\sigma}_{rr}(z)$ where the state with a superscript I is the interior solution of the plate problem and $\bar{\sigma}_{rr}$ and $\bar{\sigma}_{rz}$ are the actual edge traction of the plate. Then (2.5) and (2.7) give two conditions for the determination of the two unknown constants in σ^{I}_{rz} and σ^{I}_{rr}, and hence the interior solution (see examples in [6] and later sections of this paper).

3. AXISYMMETRIC EXTENSION AND TORSION OF A CIRCULAR PLATE

Additional suitable suffix (2) states have to be found to obtain boundary conditions for the generalized plane stress problem which determines the in-plane extension and torsion of the plate. As in the axisymmetric bending problem, simple suffix (2) states can be found for the stress data case. For the stress data to induce only a decaying state, they must satisfy the necessary condition (2.1) for any admissible suffix (2) state. One admissible suffix (2) state is the rigid body displacement field

$$u_r^{(2)} \equiv u_z^{(2)} \equiv 0 \ , \quad u_\theta^{(2)} = r \tag{3.1}$$

for which (2.1) or (2.4) becomes

$$\int_{-h}^{h} \bar{\sigma}^{d}_{r\theta}(z)dz = 0 \tag{3.2}$$

A second admissible suffix (2) state is

$$u^{(2)}_r = (1-\nu)r + (1+\nu)\frac{a^2}{r}, \quad u^{(2)}_z = -2\nu z, \quad u^{(2)}_\theta \equiv 0$$
$$\sigma^{(2)}_{rr} = E(1 - \frac{a^2}{r^2}), \quad \sigma^{(2)}_{rz} \equiv \sigma^{(2)}_{zz} \equiv \sigma^{(2)}_{z\theta} \equiv \sigma^{(2)}_{r\theta} \equiv 0 \tag{3.3}$$

for which (2.4) becomes

$$\int_{-h}^{h} [\bar{\sigma}^{d}_{rr} - \frac{\nu}{a} z\sigma^{d}_{rz}]dz = 0 \tag{3.4}$$

The interior solution for axisymmetric in-plane stress components, even in z, is expressed in terms of a biharmonic stress function F by (see Appendix I)

$$\sigma^{I}_{rr}(r,z) = \frac{1}{r}[F_{,r} + \frac{\nu(h^2-3z^2)}{6(1+\nu)}(\nabla^2F)_{,r}] \tag{3.5a}$$

$$\sigma^{I}_{r\theta}(r,z) = -\{\frac{1}{r}[F_{,\theta} + \frac{\nu(h^2-3z^2)}{6(1+\nu)}(\nabla^2F)_{,\theta}]\}_{r} . \tag{3.5b}$$

With $\bar{\sigma}^{d}_{ij} = \sigma^{I}_{ij}(a,z) - \bar{\sigma}_{ij}$, the necessary conditions (3.2) and (3.4) may be written as

$$-2h[(r^{-1}F_{,\theta})_{,r}]_{r=a} = \int_{-h}^{h} \bar{\sigma}_{r\theta}dz , \quad 2h[r^{-1}F_{,r}]_{r=a} = \int_{-h}^{h} [\bar{\sigma}_{rr} - \frac{\nu z}{a}\bar{\sigma}_{rz}]dz \tag{3.2',4'}$$

For axisymmetric extension and torsion, we have

$$F(r,\theta) = c_0\theta + [B'\ell n(r) + C'r^2 + A'r^2\ell nr + d'] \equiv F_T(\theta) + F_E(r) \tag{3.5d}$$

It is evident from (3.2'), (3.4'), and (3.5) that there is no coupling between torsion and extension in axisymmetric problems. The condition (3.2') determines c_0 and therefore the solution $F_T(\theta)$ for the torsion problem in terms of the actual edge data $\bar{\sigma}_{r\theta}(z)$:

$$2h\frac{c_0}{a^2} = \int_{-h}^{h} \bar{\sigma}_{r\theta}(z)dz . \tag{3.6}$$

For in-plane extension, we set $d' = 0$ as this term gives rise to no stress fields. We must set $A' = 0$ in order to have single valued displacement fields (see p. 68 of [3]). For a circular plate with only one edge at $r = a$, B' may be set equal to zero for bounded stress fields or determined by some other prescribed condition at the origin as we shall see later (in Appendix III). Application of the condition (3.4'), which now takes the form

$$2h\left[\frac{1}{r} F_{E,r}\right]_{r=a} = \int_{-h}^{h} [\bar{\sigma}_{rr} - \frac{\nu}{a} z\bar{\sigma}_{rz}\, dz], \tag{3.7}$$

determines the only remaining C' and therefore the solution $F_E(r)$ of the problem of axisymmetric extension in terms of the edge data $\bar{\sigma}_{rr}(z)$ and $\bar{\sigma}_{rz}(z)$. For an annular disc, the condition (3.4') applied to both edges gives two conditions for B' and C'; the interior solution for the axi-symmetric problem is again completely determined.

The condition (3.4') (or its immediate consequence (3.7)) agrees with the approximate condition (correct to order h/a) obtained by Kolos [9] for a plate of general shape (except for ν missing in Kolos' formula, presumably a slip or a typographic error). As a necessary condition for a decaying state of extension (3.4) (or (3.4')) is exact (up to exponentially small terms) for circular plates.

For mixed data in which u_r, σ_{rz}, $\sigma_{r\theta}$ are prescribed on $r = a$, the rigid body displacement (3.1) once again leads to the decaying state condition (3.2). A second admissible suffix (2) state is

$$\sigma_{rr}^{(2)} = (1+\nu) + (1-\nu)\frac{a^2}{r^2}, \quad \sigma_{\theta\theta}^{(2)} = (1+\nu) - (1-\nu)\frac{a^2}{r^2},$$

$$\sigma_{r\theta}^{(2)} = \sigma_{rz}^{(2)} = \sigma_{\theta z}^{(2)} = \sigma_{zz}^{(2)} = 0, \tag{3.8}$$

$$Eu_r^{(2)} = (1-\nu^2)(r - \frac{a^2}{r}), \quad Eu_z^{(2)} = -2\nu(1+\nu)z, \quad Eu_\theta^{(2)} = 0,$$

for which the decaying state condition (2.3) becomes

$$\int_{-h}^{h} \left[E\bar{u}_r^d + \nu(1+\nu)z\bar{\sigma}_{rz}^d\right] dz = 0. \tag{3.9}$$

It follows from (3.2), (3.9) that the conditions which should be applied to the stress function F at $r = a$ when the edge data $\bar{u}_r(z)$, $\bar{\sigma}_{rz}(z)$, $\bar{\sigma}_{r\theta}(z)$ are prescribed there are (3.2') and

$$2h\left[rF_{,rr} - \nu F_{,r}\right]_{r=a} = \int_{-h}^{h} \left[E\bar{u}_r + \nu(1+\nu)z\bar{\sigma}_{rz}\right] dz. \tag{3.10}$$

We have not found elementary suffix (2) states for the case of pure displacement edge data. However, canonical boundary value problems may be formulated for numerical determination of appropriate suffix (2) states which, upon insertion in (2.1) - (2.3) (or their axisymmetric counterparts), yield the necessary conditions for a decaying state to be satisfied by the axisymmetric edge data for problems of extension and torsion. The details of this procedure and the derivation of displacement boundary conditions for plate theories are similar to the developments in [5] for displacement edge data in the case of a semi-infinite plate in a state of plane strain. Further discussion of these details for axisymmetric extension and torsion problem will not be pursued here. We do note however an important consequence of the necessary conditions for decaying residual states such as (3.7) or (3.9). It is evident from these conditions that an interior state of axisymmetric extension may be the consequence of a transverse shear edge stress $\bar{\sigma}_{rz}(z)$ (or a transverse displacement $\bar{u}_z(z)$) which are odd functions of z. Furthermore, depending on the relative magnitude of the edge data, the interior state associated with the transverse shear stress (or transverse displacement) may be dominant compared to that associated with the in-plane radial stress $\bar{\sigma}_{rr}$ (or displacement $\bar{u}_r$). In the next three sections, we apply the stress boundary conditions (3.7) obtained above and the conditions (2.5) and (2.7) obtained in [6] to determine the correct interior solution for a plate under a point load. The solution process will require additional developments involving decaying states.

4. A CIRCULAR PLATE UNDER A CONCENTRATED FORCE AT THE CENTER OF ITS UPPER FACE

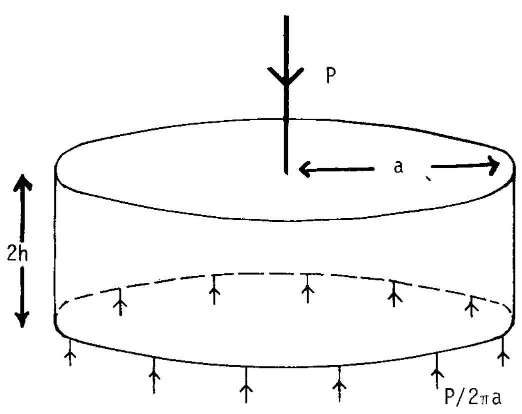

Fig. 1. The plate, the loading and the support

Consider a homogeneous, isotropic and linear elastic circular plate occupying the region $\{r \leq a, |z| \leq h\}$. As shown in Figure 1, the plate is under a concentrated load P acting at the center of the upper face $(r = 0, z = h)$, and is "simply supported" around its lower edge $(r = a, z = -h)^*$. We will

* As we shall see later, it is important to specify the precise nature of this support; other forms of simple support do not lead to the same solution even in the interior of the plate.

regard this support as supplying a uniform vertical line load distribution of magnitude $P/2\pi a$ (with resultant force P) around the lower edge of the cylindrical edge surface so that the plate is in overall equilibrium. Thus, the relevant elastostatic boundary value problem is one of prescribed surface tractions.

The exact (three-dimensional elastostatic) solution $\{u_j, \sigma_{ij}\}$ of this boundary value problem is quite intractable. However, in the spirit of plate theory, we will determine the interior component $\{u_j^I, \sigma_{ij}^I\}$ of the exact solution whose difference from the exact solution is exponentially small as $(h/a) \to 0$ in the region $\eta_0 < r < a - \eta_a, |z| \leq h$ for any fixed positive η_0 and η_a. The fact that this difference is exponentially small as $(h/a) \to 0$ is of great significance; it means that our theory is the best possible "thick plate" theory in the sense that it is accurate to $O([h/a]^N)$ for arbitrarily large N. It will also turn out that earlier "thick plate" solutions of this problem, e.g., [2](see p. 475), which claims to be correct to $O(h^2/a^2)$ are not correct to this order and are even inferior to the Kirchhoff "thin plate" solution.

Layer phenomena associated with the exponentially decaying components of the exact solution are known to occur adjacent to the plate edges. For the present point load problem, there is also (as we shall see) a layer phenomenon around the central axis, $r = 0$, of the circular plate where the point load is applied. Both this interior layer and the boundary layer adjacent to the edge $r = a$ extend radially a distance of the order of the plate thickness. We wish to determine the interior solution which is accurate to within exponentially small error in the annular region of the plate excluding the two layer regions. To do so, it is necessary to determine the form of the singularity at $r = 0$ imparted to the interior solution by the point load P. This is one of the interesting features of the present problem; it does not occur in the other problems treated in [5] and [6] and has not been treated correctly in the existing literature.

With the applied forces all in a direction normal to the plate's mid-plane, it is customary to expect the extensional part of the problem to involve only layer phenomena (and its interior solution component to be identically zero). But this is not so; in fact, it will be seen from our results of sections 6 and 7 that the extensional effects are of the same order as the corrections to the Kirchhoff solution in the bending part of the problem. This second interesting feature of the point load problem is a direct consequence of the elastic reciprocal theorem. In what follows, the solution of the bending portion of the solution for the point load problem will be discussed in section 5. The stretching portion of the solution will be obtained in section 6. The full interior solution and comparisons with previous results will be given in section 7.

5. THE BENDING PORTION OF THE SOLUTION FOR THE POINT FORCE PROBLEM

In this section only, all the fields which appear refer to the bending part of the elastostatic boundary value problem and will be denoted by $\{u_j^B, \sigma_{ij}^B\}$. For plate bending with traction-free faces $(|z| = h)$, the interior solution may be derived (see Appendix 1 of [6]) from the midplane transverse displacement $w(r,\theta) = u_z^B(r,\theta,0)$ which satisfies the two-dimensional biharmonic equation, $\nabla^2\nabla^2 w = 0$. The corresponding interior solution is given by equations (I.8) - (I.16) of [6]. These formulae are similar but not identical to the formulae of the Kirchhoff thin plate theory. The additional terms ensure that $\{u_j^B, \sigma_{ij}^B\}$ is a genuine three-dimensional elastostatic state.

Because of the obvious axisymmetry of the present problem about $r = 0$, it follows that w must have the form

$$W = Ar^2 \ln r + B \ln r + Cr^2 + d \; , \tag{5.1}$$

where the constants A, B, C are yet to be determined. The constant d corresponds to rigid body vertical translation and will be chosen so that $w(a) = 0$.

Theorem 1 The constants A and B, in the expression (5.1) for the midplane transverse displacement, are related to the load magnitude P by

$$A = -\frac{P}{8\pi D} \; , \quad B = \frac{2}{5}\,\frac{8-3\nu}{1-\nu}\,h^2\,\frac{P}{8\pi D} \tag{5.2,3}$$

where the flexural rigidity D is given in terms of Young's modulus E, Poisson's ratio ν (of the homogeneous and isotropic plate medium) and plate thickness $2h$ by $2Eh^3/3(1-\nu^2)$.

The proof of Theorem 1, given in Appendix II, is also by the reciprocal theorem.

We observe that the value of A as given by (5.2) is identical to that obtained in the Kirchhoff thin plate theory. The coefficient B is taken to be zero in the Kirchhoff theory, on the grounds that "$w(r)$ must be bounded at $r = 0$" and other authors (e.g. Love [2] p. 475) have applied the same reasoning when $w(r)$ is the midplane deflection in the interior of a thick plate. We now see that this is not so, as $B \neq 0$ for thick plates. At the same time with $B = O(h^2 A)$ it is indeed correct to take $B = 0$ in the Kirchhoff thin plate limit. It may seem strange at first that the interior midplane deflection $w = u_z(r,0)$ should be unbounded at $r = 0$. However, it should be remembered that the interior solution only approximates the exact solution (as $h/a \to 0$) in regions bounded away from $r = 0$ and $r = a$; thus the singularity at $r = 0$ in (5.1) need not be present in the exact solution.

The assumption that "$w(r)$ must be bounded at $r = 0$" is not made in a solution by Reissner's plate theory [9] obtained in Appendix IV. This solution gives the same value of B as (5.3) except for a multiplicative factor 8 instead of $8-3\nu$. The numerically small difference may be attributed to the omission of transverse normal stress effects in our use of Reissner's theory. In contrast, an argument given in [8] gives $B = Ph^2/8\pi D(1+\nu)$.

It remains to determine the coefficient C in (5.1); this step has also been performed incorrectly in previous solutions (e.g., Love [2] and Lur'e [7] Chap. 4, pp. 226-230). In these solutions it is assumed that the fields $\{\sigma^d_{ij}, u^d_j\}$, which are the difference between the exact bending solution and the interior bending solution, must satisfy the 'Saint-Venant type' conditions

$$\int_{-h}^{h} [\sigma^d_{rz}]_{r=a}\, dz = 0 , \qquad \int_{-h}^{h} z[\sigma^d_{rr}]_{r=a}\, dz = 0 \tag{5.4,5}$$

on the outer edge $r = a$. They are equivalent to requiring the interior bending solution to have the same transverse shear resultant and bending (stress) couple at $r = a$ as the prescribed data. However we have previously shown in Section 4 of [6] that, while the condition (5.4) is appropriate for the problem, the condition (5.5) is not and should in fact be replaced by the condition

$$\int_{-h}^{h} [z\sigma^d_{rr} - \frac{\nu}{2a} z^2\sigma^d_{rz}]_{r=a}\, dz = 0 . \tag{5.6}$$

The additional term $-\nu z^2\sigma^d_{rz}/2a$ represents a $O(h/a)$ correction to (5.5), which turns out to be the same order as the error in the Kirchhoff solution; thus to improve on the Kirchhoff solution it is essential to use (5.6) and not (5.5). In our problem

$$[\sigma^d_{rr}]_{r=a} = -\frac{Ez}{1-\nu^2}\,[A\{2(1+\nu)\ln a + (3+\nu) - \frac{4}{a^2}(h^2 - \frac{2-\nu}{6}z^2)\} - B(1-\nu)\frac{h^2}{a^2} + 2C(1+\nu)] , \tag{5.7}$$

$$[\sigma^d_{rz}]_{r=a} = -\frac{2E}{a(1-\nu^2)}(h^2-z^2)A - \frac{P}{4\pi a}\{\delta(z-h) + \delta(z+h) , \tag{5.8}$$

where $\delta(z)$ is the Dirac delta function. (The delta functions in (5.8) arise from the bending part of the applied line load around $z = -h$, $r = a$.) With the value of A given by (5.2), we see that (5.4) is satisfied identically. Upon substituting (5.7), (5.8) into (5.6) and using (5.2), (5.3), the value of C is found to be

$$C = \frac{P}{8\pi D}\,[\ln a + \frac{3+\nu}{2(1+\nu)}] . \tag{5.9}$$

We now substitute (5.2), (5.3), (5.9) into (5.1) to get

$$w = \frac{P}{8\pi D}\left[r^2 \ell n\left(\frac{a}{r}\right) - \frac{2(8-3\nu)}{5(1-\nu)} h^2 \ell n\left(\frac{a}{r}\right) - \frac{3+\nu}{2(1+\nu)}(a^2-r^2)\right] , \tag{5.10}$$

where we have adjusted the constant d so that $w(a) = 0$ to permit easy comparison with earlier solutions.

It should be noted that since the extensional part of the solution (i.e. the part which is symmetrical about $z = 0$) leaves the midplane $z = 0$ undeflected, the expression (5.10) also represents the midplane deflection in the full interior solution $u_z^I(r,0)$.

The interior bending fields $\{u_j^B, \sigma_{ij}^B\}$ can now be calculated from (5.20) by way of the formulae in Appendix 1 of [6]. In particular the important radial tensile stress on the lower face $z = -h$ is given by

$$\sigma_{rr}^B(r,-h) = \frac{3P}{8\pi h^2}\left[(1+\nu)\ell n\left(\frac{a}{r}\right) - \frac{2(2-7\nu)}{15}\frac{h^2}{r^2}\right] \tag{5.11}$$

The leading terms of (5.10) and (5.11) are identical to the customary expressions predicted by the Kirchhoff thin plate theory. The additional terms of $O(h^2)$ in our expressions represent the true correction to the Kirchhoff theory in the interior of the plate. It should be noted that there are no higher order corrections of $O(h^N)$, $N \geqslant 3$; the expressions (5.10), (5.11) are complete as they stand.

6 THE STRETCHING PORTION OF THE SOLUTION FOR THE POINT FORCE PROBLEM

In this section only, all the fields which appear refer to the extensional part of the problem. The extensional part of the interior solution will be denoted by $\{u_j^E, \sigma_{ij}^E\}$. Since the loading corresponding to the extensional part of the problem is symmetrical about the midplane $z = 0$ and perpendicular to that plane, it is often supposed that the extensional part of such a problem makes no contribution to the interior solution. We shall see, however, that the extensional part also makes a correction to the Kirchhoff solution of $O(h^2)$.

For the case of extension (in regions of the plate where the faces $|z| = h$ are traction free) the interior solution may be derived from a scalar function $F(r,\theta)$, which is related to the Airy stress function (see Appendix I). F satisfies the plane bi-harmonic equation

$$\nabla^2(\nabla^2 F) = 0 , \tag{6.1}$$

and the corresponding interior solution is given by equations (I.11)-(I.16) of

Appendix I. These formulae are similar to, but not identical with, the formulae of generalized plane stress. The additional terms which appear ensure that $\{u_j^E, \sigma_{ij}^E\}$ is a genuine three-dimensional elastostatic state. From the axisymmetry of the problem it follows from (6.1) that F must have the form

$$F = A'r^2\ln r + B'\ln r + C'r^2 + d' \ . \tag{6.2}$$

Now $A' = 0$, for otherwise the displacement field in the plate would not be single-valued; also d' makes no contribution to the fields and may be taken to be zero. Thus it remains to determine the constants B', C'. As in the case of bending, B' (the coefficient of the singular term $\ln r$) is determined solely by the point loads.

Theorem 2. In (6.2), the constant B' is given by

$$B' = -\frac{\nu P}{4\pi} \ . \tag{6.3}$$

The proof of Theorem 2 is given in Appendix III. [The interior solution corresponding to $F = -\nu P\ln r/4\pi$ is given by

$$\begin{aligned} \sigma_{rr}^I &= -\frac{\nu P}{4\pi}\frac{1}{r^2} , \quad \sigma_{rz}^I = \sigma_{zz}^I = 0 , \\ Eu_r^I &= \frac{\nu(1+\nu)P}{4\pi}\frac{1}{r} , \quad Eu_z^I = 0 , \end{aligned} \tag{6.4}$$

from which it is clear that the extensional part of the problem must contribute to the interior solution and must also be unbounded at $r = 0$.]

It now remains to determine the constant C' from the prescribed data on the edge $r = a$. We found in section 4 that for the extensional part of the solution, the condition corresponding to (5.4), (5.6) is (3.7), or

$$\int_{-h}^{h} [\sigma_{rr}^d - \frac{\nu z}{a}\sigma_{rz}^d]_{r=a} dz = 0 \tag{6.5}$$

and not the commonly assumed 'Saint-Venant' condition

$$\int_{-h}^{h} [\sigma_{rr}^d]_{r=a} dz = 0 \ . \tag{6.6}$$

Upon applying (6.5) to (6.2) and using (6.3) we obtain

$$C' = \frac{\nu P}{4\pi a^2} \ . \tag{6.7}$$

The corresponding extensional interior solution can now be found from (I.11)-(I.16). In particular, we have

$$\sigma_{rr}^{E}(r,z) = \frac{\nu P}{4\pi a^2}\,[2 - \frac{a^2}{r^2}]\ , \qquad u_z^{E}(r,0) = 0 \tag{6.8,9}$$

7. THE INTERIOR SOLUTION

The full interior solution, denoted by $\{u_j^I,\ \sigma_{ij}^I\}$, is now just given by

$$\{u_j^I,\ \sigma_{ij}^I)\} = \{u_j^B,\ \sigma_{ij}^B\} + \{u_j^E,\ \sigma_{ij}^E\}\ . \tag{7.1}$$

In particular it follows from (5.10), (6.9) that the interior midplane deflection is given by

$$u_z^I(r,0) = \frac{P}{8\pi D}\,[r^2 \ln(\tfrac{a}{r}) - \frac{2(8-3\nu)}{5(1-\nu)}\,h^2 \ln(\tfrac{a}{r}) - \frac{3+\nu}{2(1+\nu)}\,(a^2-r^2)]\ . \tag{7.2}$$

It similarly follows from (5.11), (6.8) that the radial tensile stress on the lower face is given by

$$\sigma_{rr}^I(r,-h) = \frac{P}{8\pi h^2}\,[3(1+\nu)\ln(\tfrac{a}{r}) - \frac{4}{5}\,(1-\nu)\,\frac{h^2}{r^2} + 4\nu\,\frac{h^2}{a^2}]\ . \tag{7.3}$$

The expressions (7.2), (7.3) differ from the exact values by an exponentially small error (as $h/a \to 0$) except near $r = 0$ and $r = a$, where there are 'boundary layers' of width $O(h)$.

In the limit as $h/a \to 0$, (7.2), (7.3) tend to the values predicted by the Kirchhoff thin plate theory, that is

$$u_z^k(r,0) = \frac{P}{8\pi D}\,[r^2 \ln(\tfrac{a}{r}) - \frac{3+\nu}{2(1+\nu)}\,(a^2-r^2)]\ , \tag{7.4}$$

$$\sigma_{rr}^k(r,-h) = \frac{3(1+\nu)P}{8\pi h^2}\,\ln(\tfrac{a}{r}). \tag{7.5}$$

For thick plates the difference between u_z^I and u_z^k may be quite substantial. For instance

$$\frac{u_z^I(r,0)-u_z^k(r,0)}{u_z^k(0,0)} = \frac{4(1+\nu)(8-3\nu)}{5(1-\nu)(3+\nu)}\,\frac{h^2}{a^2}\,\ln(\tfrac{a}{r}) \doteq -4.46\,\frac{h^2}{a^2}\,\ln(\tfrac{a}{r}) \tag{7.6}$$

when $\nu = 1/2$. For $h = 0.2a$, (7.6) has the value 0.29 .... at $r = h$; thus in this case the Kirchhoff theory underestimates the deflection of the plate by about 30%. Here we are assuming that the boundary layer around the point load has a negligible influence at the point $r = h$. See [6], section 5 for a discussion.

The expression (7.2) is quite different from the expression derived by Love [2], p. 475, for this same problem. Both expressions reduce to (7.4) as $h/a \to 0$, but the other terms in Love's expression are incorrect, for the reasons explained earlier. In the work of Lur'e [6] the concentrated load is represented exactly as a series of Bessel functions. While this approach has an advantage in that it allows the fields to be calculated near the load, its complicated form does obscure the essential simplicity of the interior solution, which (as a result) is never discovered. Lur'e did not treat the precise problem of the present paper; however it is clear from his treatment of the corresponding problem with uniform pressure applied to the upper face that the stress boundary conditions at $r = a$ are applied incorrectly, as was also done by Love [2], and Timoshenko and Goodier [8] p. 351. In all these works, the condition (5.5) is applied instead of the true condition (5.6).

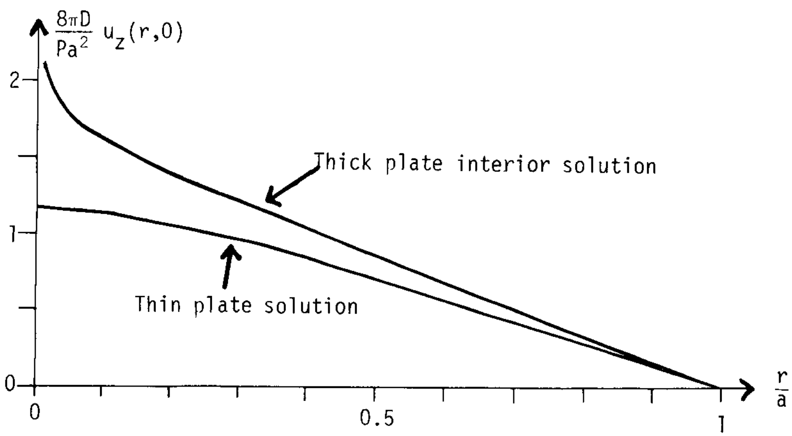

Figure 2. The transverse deflection of the mid-plane of the plate as predicted by (i) the thick plate interior solution (ii) the thin plate Kirchhoff solution, in the case $h/a = 0.2$, $\nu = 0.5$

In Figure 2 we plot $u_z^I(r,0)$, the transverse deflection of the midplane $z = 0$ (as predicted by the interior solution), against r for the case $\nu = 0.5$, $h/a = 0.2$. The difference between u_z^I and the Kirchhoff thin plate solution (shown also) is substantial. u_z^I may be expected to be a good approximation to u_z, the exact deflection, for $0.2 < r/a < 0.8$; in this range we note that the plate profile is quite straight (giving a cone-like shape), unlike the "parabolic" shape predicted by the thin plate theory.

8. DIFFERENT FORMS OF LOADING AND OF SIMPLE SUPPORT

The results (7.2), (7.3) are influenced by the precise method of loading and support employed. If the plate were simply supported by a vertical line load around its upper edge (or around its midplane circle $r = a$, $z = 0$), then this would affect the solution, even in the interior of the plate, by terms of order $O(h^2/a^2)$. The same is true if the concentrated load were applied at $r = 0$, $z = -h$ instead of $r = 0$, $z = h$.

Thus in thick plate theory it is not meaningful to refer to "a concentrated load" or "simple support'; the nature of the loading and support must be specified precisely, for otherwise errors will be introduced which are of the same order as the thick plate corrections being determined. In particular, it is not possible to determine the interior solution when only the stress resultants and couples are prescribed at the edge.

APPENDIX I INTERIOR SOLUTION FOR EXTENSION AND TORSION OF PLATES WITH TRACTION-FREE FACES

For an isotropic and homogeneous plate with traction-free faces at $z = \pm h$, the interior solution for in-plane extension and torsion portion of its elastostatic fields may be expressed in terms of a biharmonic (Airy stress) function F by

$$\sigma^E_{xx} = \frac{\partial^2}{\partial y^2}\left[1 + \frac{\nu(h^2-3z^2)}{6(1+\nu)}\nabla^2\right]F\ , \quad \sigma^E_{yy} = \ldots \tag{I.1,2}$$

$$\sigma^E_{xy} = -\frac{\partial^2}{\partial x\partial y}\left[1 + \frac{\nu(h^2-3z^2)}{6(1+\nu)}\nabla^2\right]F\ , \quad \sigma^E_{xz} = \sigma^E_{yz} = \sigma^E_{zz} = 0 \tag{I.3,4}$$

$$Eu^E_x = E\tilde{u}_x - \frac{\nu}{6}(h^2-3z^2)\frac{\partial}{\partial x}(\nabla^2 F) \tag{I.5}$$

$$Eu^E_y = \ldots\ldots\ , \quad Eu^E_z = -\nu z^2\nabla^2 F \tag{I.6,7}$$

where $\nabla^2(\nabla^2 F) = 0$ and

$$E\frac{\partial u_x}{\partial x} = \nabla^2 F - (1+\nu)\frac{\partial^2 F}{\partial x^2}\ , \quad E\frac{\partial \tilde{u}_y}{\partial y} = \ldots \tag{I.8,9}$$

$$E\left(\frac{\partial \tilde{u}_y}{\partial x} + \frac{\partial \tilde{u}_x}{\partial y}\right) = -2(1+\nu)\frac{\partial^2 F}{\partial x\partial y}\ , \tag{I.10}$$

∇^2 being the two-dimensional Laplacian in x and y. The formulae for σ^E_{yy}, u^E_y and $\tilde{u}_y$ are obtained from σ^E_{xx}, u^E_x and $\tilde{u}_x$ by interchanging x and y.

In cylindrical coordinates, the corresponding formulae for the interior solution of plate extension and torsion are

$$\{\sigma^E_{\theta\theta},\ \sigma^E_{rr}\} = \{\frac{\partial^2}{\partial r^2},\quad \nabla^2 - \frac{\partial^2}{\partial r^2}\}[1 + \frac{\nu(h^2-3z^2)}{6(1+\nu)}\nabla^2]F \tag{I.11}$$

$$\sigma^E_{r\theta} = -\frac{\partial^2}{\partial r\partial\theta}\{\frac{1}{r}[1 + \frac{\nu(h^2-3z^2)}{6(1+\nu)}\nabla^2]F\}\ ,\qquad \sigma^E_{rz} = \sigma^E_{\theta z} = \sigma^E_{zz} = 0 \tag{I.12,13}$$

$$Eu^E_r = E\tilde{u}_r - \frac{\nu}{6}(h^2-3z^2)\frac{\partial}{\partial r}\nabla^2 F \tag{I.14}$$

$$Eu^E_\theta = E\tilde{u}_\theta - \frac{\nu}{6}(h^2-3z^2)\frac{1}{r}\frac{\partial}{\partial\theta}\nabla^2 F\ ,\qquad Eu^E_z = -\nu z\nabla^2 F \tag{1.15,16}$$

where $\nabla^2\nabla^2 F = 0$ with ∇^2 being the two-dimensional Laplacian in polar coordinates and where

$$E\frac{\partial\tilde{u}_r}{\partial r} = \nabla^2 F - (1+\nu)\frac{\partial^2 F}{\partial r^2} \tag{I.17}$$

$$E(\frac{1}{r}\frac{\partial\tilde{u}_\theta}{\partial\theta} + \frac{1}{r}\tilde{u}_r) = (1+\nu)\frac{\partial^2 F}{\partial r^2} - \nu\nabla^2 F \tag{I.18}$$

$$E(\frac{\partial\tilde{u}_\theta}{\partial r} - \frac{1}{r}u_\theta + \frac{1}{r}\frac{\partial\tilde{u}_r}{\partial\theta}) = -2(1+\nu)\frac{\partial^2}{\partial r\partial\theta}(\frac{1}{r}F) \tag{I.19}$$

APPENDIX II THE BENDING PART OF THE INTERIOR SOLUTION NEAR A POINT LOAD

Consider the bending part of the plate problem depicted in Figure 1. We apply the elastic reciprocal theorem

$$\iint_S \{\sigma^{(1)}_{ij}u^{(2)}_i - \sigma^{(2)}_{ij}u^{(1)}_i\}n_j ds = 0\ , \tag{II.1}$$

where S is the boundary of the region $r \leqslant b\ (<a)\ |z| \leqslant h$, and $b(>0)$ is independent of h. We take as the suffix (1) state the exact solution $\{u_j,\ \sigma_{ij}\}$ of the plate problem. For the suffix (2) state we take the axisymmetric state

$$u^{(2)}_r = -2rz,\quad u^{(2)}_z = r^2 - \frac{2\nu}{1-\nu}(h^2-z^2)\ ,\quad u^{(2)}_\theta = 0$$

$$\sigma^{(2)}_{rr} = -\frac{2Ez}{1-\nu}\ ,\quad \sigma^{(2)}_{rz} = \sigma^{(2)}_{zz} = \sigma^{(2)}_{r\theta} = \sigma^{(2)}_{\theta\theta} = 0 \tag{II.2}$$

Note that we have arranged that $u^{(2)}_z = 0$ at the points $r = 0,\ z = \pm h$. If we regard the point force P as being the limit of a normal pressure $P/\pi\delta^2$ applied over the area $z = h,\ r \leqslant \delta$ as $\delta\to 0$, then the contribution to the left side of (II.1) from the face $z = h$ is

$$2\pi \int_0^{\delta} \left(-\frac{P}{\pi\delta^2}\right) (r^2) r dr = -\frac{1}{2} P\delta^2$$

which tends to zero as $\delta \to 0$. Thus there is no contribution from the face $z = h$ in the limit of a point load at $r = 0$, $z = h$. There is certainly no contribution from the lower face $z = -h$ and so, taking into account the axi-symmetry, (II.1) reduces to

$$\int_{-h}^{h} [\sigma_{rr} u_r^{(2)} + \sigma_{rz} u_z^{(2)} - \sigma_{rr}^{(2)} u_r - \sigma_{rz}^{(2)} u_z]_{r=b} dz = 0 . \qquad (II.3)$$

Now in (II.3), $u_r^{(2)}$ etc., are given by (II.2) while the exact solution $\{u_j, \sigma_{ij}\}$ may be replaced (with exponentially small error as $h \to 0$) by the bending part of the interior solution $\{u_j^I, \sigma_{ij}^I\}$, derived from (5.1). (The extensional part of $\{u_j^I, \sigma_{ij}^I\}$ gives no contribution, in view of the symmetry of $\{u_j^{(2)}, \sigma_{ij}^{(2)}\}$.) On making these substitutions, we obtain after some simplification that

$$B = -\frac{2(8-3\nu)}{5(1-\nu)} h^2 A . \qquad (II.4)$$

Now it easily follows from the overall equilibrium of the region $r \leq b$, $|z| \leq h$ that $A = -P/8\pi D$ (where D is the flexural rigidity) as in the Kirchhoff theory. As a consequence, we have from (II.4)

$$B = \frac{2}{5} h^2 \left(\frac{8-3\nu}{1-\nu}\right) \frac{P}{8\pi D} . \qquad (II.5)$$

APPENDIX III THE EXTENSIONAL PART OF THE INTERIOR SOLUTION NEAR A POINT LOAD

Consider the extensional part of the plate problem depicted in Figure 1. From (3.5c) the corresponding interior solution is derivable from the scalar function

$$F = B' \ln r + C' r^2 \qquad (III.1)$$

where B', C' are constants. The corresponding stresses and displacements are derived from F by the formulae given in Appendix I.

The method of determining B' follows closely the method of determining B in Appendix II. We apply the elastic reciprocal theorem (II.1) to the same region, with the suffix (1) state being the exact solution $\{u_j, \sigma_{ij}\}$ of the plate problem. However, the suffix (2) state is taken this time to be

$$\sigma_{rr}^{(2)} = 1 , \quad \sigma_{rz}^{(2)} = \sigma_{zz}^{(2)} = \sigma_{r\theta}^{(2)} = 0, \qquad (III.2)$$

$$Eu_r^{(2)} = (1-\nu)r \ , \quad u_\theta^{(2)} = 0 \ , \quad Eu_z^{(2)} = -2\nu z \ . \tag{III.3}$$

This state is symmetrical about midplane $z = 0$, and so the bending part of the exact solution gives no contribution. The extensional part then gives the result

$$B' = -\frac{\nu P}{4\pi} \ . \tag{III.4}$$

APPENDIX IV SOLUTION FOR THE POINT LOAD PROBLEM BY REISSNER'S PLATE THEORY

In the framework of Reissner's Plate Theory, the relevant boundary value problem for a circular plate with a point force of magnitude P applied at the center of the plate may be formulated in terms of a (meridional slope) angle change variable ϕ which satisfies the second order differential equation

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{1}{r^2}\phi = \frac{P}{2\pi Dr} \qquad (0 < r < a) \tag{IV-1}$$

By symmetry, we expected to have

$$\phi(0) = 0 \tag{IV-2}$$

and for a simply supported edge

$$\left[\frac{d\phi}{dr} + \frac{\nu}{r}\phi\right]_{r=a} = 0 \tag{IV-3}$$

which is simply the condition of vanishing edge moment resultant.

The solution of the above two point boundary value problem for ϕ is

$$\phi(r) = -\frac{Pr}{4\pi D}\left[\ell n\left(\frac{a}{r}\right) + \frac{1}{1+\nu}\right]$$

From the stress strain relation $\phi + dw/dr = {}_QA_rQ$ and the overall equilibrium condition $Q_r = P/2\pi r$, we get

$$w = -\int_r^a \left\{\frac{A_Q P}{2\pi r} + \frac{Pr}{4\pi D}\left[\ell n\left(\frac{a}{r}\right) + \frac{1}{1+\nu}\right]\right\}dr$$

$$= \frac{P}{8\pi D}\left\{r^2 \ell n\left(\frac{a}{r}\right) - \frac{3+\nu}{2(1+\nu)}(a^2-r^2) - \frac{16}{5(1-\nu)}h^2 \ell n\left(\frac{a}{r}\right)\right\}$$

This expression is to be compared with (5.10). The only difference is the factor 16 in the last term; the corresponding factor in (5.10) is $2(8-3\nu)$.

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