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LECTURE NOTES ON THE LINEAR THEORY OF THIN ELASTIC SHELLS OF REVOLUTION

F.Y.M. Wan

Technical Report No. 84-9

August, 1984

Cover photo courtesy of the U.B.C. Museum of Anthropology:

Haida totem pole; main figure, possibly bear, holding wolf between legs, frog in mouth, wolf between ears.

LECTURE NOTES ON THE LINEAR THEORY OF THIN ELASTICS SHELLS OF REVOLUTION(1)

F.Y.M. Wan(2)
Institute of Applied Mathematics
The University of British Columbia
Vancouver, B.C. V6T 1Y4
Canada

August, 1984

(1) The research is supported in part by Canadian N.S.E.R.C. Operating Grant A9259 and by U.S.-NSF Grant No. MCS-8306592.

(2) Currently on leave as Professor of Applied Mathematics and Mathematics at the University of Washington, Seattle, WA 98195.

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Foreword

Over the years, this writer has taught courses on the theory of thin elastic shells at different institutions, mainly to students in applied mathematics. A set of lecture notes have been compiled from and for these courses. Portions of these notes are now published as technical reports of the Institute of Applied Mathematics (and Statistics, at one time) at the University of British Columbis. *These technical reports have limited distributions but are deposited in the UBC* Archive for future reference.

Because of time constraints, these shell courses usually focussed on the mathematical structure of the various special theories such as the linear theory, shallow shell theory, theory of shells of revolution, etc., and on the solution techniques applicable to the relevant boundary value problems. Some discussion of the foundations of shell theory, i.e., the adequacy of shell theory as an approximate solution of a three-dimensional elasticity problem, is usually included to make the subject meaningful for engineering applications. For this part of the course, it is necessary for the audience to have some background on the three-dimensional linear theory of elasticity. To accommodate students in applied mathematics without the requisite background, the material in elasticity theory essential to the discussion of foundations of shell theory is outlined in reference [1] at the end of this Foreword (not to be confused with references of the main body of this report given at the end of the report).

The foundation problem for shells is, at this writing, not completely understood and is, in any event, far too complicated for students exposed to the subject for the first time. The writer has found it more effective to discuss the corresponding problem for linear plate theory as an illustration of the nature of the foundation problems and the types of results attainable. For these reasons, the results of Friedrichs & Dressler and of E. Reissner are presented in reference [2]. It has always been the writer's intention to supplement [2] with a summary of his joint work with R. D. Gregory of the University of Manchester such as [3] and [4] to indicate an attractive and practical alternative solution of the foundation problem for plates. In principle, the same method of approach is also feasible for the corresponding shell problem though its implementation requires some fresh ideas. The compilation of the material on the foundation of plates, possibly including other results such as [5], will be the writer's next writing project.

To allow for a separate discussion of the structure and solution techniques of shell theory without any reference to three-dimensional elasticity theory, (and the foundation problem), the writer usually follows the approach of W. Gunther, E. Reissner and H. Schaefer, and develops shell theory as the mechanics of a two-dimensional deformable continuum. The constitutive relations will be adopted with little or no justifications whenever there is not sufficient time to discuss the foundations of the theory. The fundamentals of linear shell theory is presented in this way in [6], though the relationship between shell theory and three-dimensional elasticity theory is briefly discussed by way of a class of transversely rigid thin elastic shells. The theory developed there is for general orthogonal surface coordinates. A tensorial treatment to allow for oblique surface coordinates is avoided to gain time for shell theory proper. Most shell designs encountered by this writer can in fact be analyzed by working with orthogonal surface coordinates. In any event, it is not needed for the present of lecture notes.

The developments of two particular classes of shell problems have had significant impact on the theory and applications of shell structures. For their importance in engineering applications alone shallow shells and shells of revolution deserve special attention. In this set of lecture notes, we summarize the salient features of the linear elastostatics of shells of revolution. While an exhaustive discussion of the very rich subject of shells of revolution is not possible in the space of a technical report, our summary does present the main results of the subject which completes the solution process for an infinitesimal deformation theory under general loading. The report will be short on specific problems; there will be just enough of these to illustrate the main theoretical developments for our class of problems. A similar summary for both linear and nonlinear shallow shells is presented in [7]. As it is pointed out in [7], the very important subject of shell buckling will be treated in a separate report.

Through many years of association, the writer has undoubtedly absorbed many ideas from his Ph.D. thesis supervisor and research collaborator, Professor E. Reissner. It would be fair to say that a greater part of these lecture notes is an outgrowth of information assembled over the years from Professor Reissner's class lectures, his writings, and his conversations with this writer. In fact, whatever merits these lecture notes may have, the writer owes it all to his teacher and good friend, Eric.

References

1. F.Y.M Wan, "Lecture Notes on Problems in Elasticity: I. Fundamentals of Linear Theory of Elasticity," I.A.M.S. Tech. Rep. No. 82-17, Univ. of British Columbia, Vancouver, Canada, November, 1982.

2. __________, "Lecture Notes on Problems in Elasticity: II. Linear Plate Theory," I.A.M.S. Tech. Rept. No. 83-15, Univ. of British Columbia, Vancouver, Canada, September, 1983

3. R.D. Gregory and F.Y.M. Wan, "Decaying States of Plane Strain in a Semi-infinite Strip and Boundary Conditions for Plate Theory," J. Elasticity 14, 1984, 27-64.

4. __________, "On Plate Theories and Saint-Venant's Principle," I.A.M.S. Tech. Rep. No. 83-12, Univ. of British Columbia, Vancouver, Canada, August, 1983, and to appear in Intern'l. J. Solids and Structures, 1985.

5. J.G. Simmonds, "An Improved Estimate for the Error in the Classical, Linear Theory of Plate Bending," Quart, Appl. Math. 29, 1971, 439-447.

6. F.Y.M. Wan, "Lecture Notes on Theory of Thin Elastic Shells," I.A.M.S. Tech. Rep. No. 83-11, Univ. of British Columbia, Vancouver, Canada, July, 1983.

7. __________, "Lecture Notes on Shallow Shell Theory," I.A.M.* Tech. Rep. No. 84-8, Univ. of British Columbia, Vancouver, Canada, July, 1984.

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*The Institute of Applied Mathematics and Statistics of the University of British Columbia was formally re-named as the Institute of Applied Mathematics in March, 1984.

Part I - The Linear Elastostatics of Shells of Revolution

1 - Introduction

1. The Scope of This Volume

Even with the built-in simplifications associated with its inherent static-geometric duality, the general linear theory of thin elastic shells still poses rather difficult mathematical problems in partial differential equations. Qualitative understanding of the theory and tractable quantitative solutions for specific applications are possible only through additional mathematical simplifications associated with particular classes of problems. For shells of revolution, certain simplifications naturally suggest themselves. In cylindrical coordinates (r,θ,z) the coefficients of the linear partial differential equations governing the linear elastostatics of shells of revolution (whose middle surface is given parametrically by $z = z(\xi_1)$ and $r = r(\xi_1))$ are, as we shall soon see, independent of the polar angle θ. It follows that the loading and response of circumferentially complete shells of revolution (in the form of a dome or a frustum) have Fourier series representation in the form

$$f(\xi_1,\theta) = \sum_{n=0}^{\infty} [c_n(\xi_1)\cos(n\theta) + s_n(\xi_1)\sin(n\theta)]$$

Linearity then allows for a separate analysis of each Fourier coefficient, $c_n(\xi_1)$ or $s_n(\xi_1)$, $n = 0, 1, 2, \ldots$ Not so obvious however, are the different additional simplifications associated with the different harmonics. Also, not so easily seen are the simplifications and reduction associated with important classes of problems involving circumferentially incomplete shells of revolution including the special case of a shell of revolution slit along a meridian. It is our intention to summarize in this volume the main theoretical results of the rich body of knowledge accumulated from various simplifications and reductions for shells of revolution. These results are essential not only to a true understanding of the theory of shells of revolution but also to an effective solution for specific problems.

Historically, the first result in this direction was for the axisymmetric bending (and stretching) of spherical shells [1]. With all stress and strain distributions independent of the polar angle θ, the linear partial differential equations of force and moment equilibrium reduce to a system of first order ordinary differential equations in a meridional (independent) variable ξ_1. The requirement of overall axial force equilibrium gives a first integral of these ODEs. The existence of this first integral makes it possible to transform the only relevant differential equation of moment equilibrium into a second order ODE for a stress function and an angular displacement variable. Not having the benefit of the static-geometric duality at that time, it was considered a stroke of genius that Hans Reissner succeeded in obtaining the appropriate compatibility equation ~~and hence~~ for a second ODE ~~for~~ in the same two unknowns. The two simultaneous second order equations were further reduced to a single second order equation for a complex function. This second order equation in turn shows that the general solution for the axisymmetric bending of spherical shells can be expressed in terms of associated Legendre functions.

Shortly after the pioneering work for spherical shells, E. Meissner [2] used a similar approach to reduce the axisymmetric bending problem for general shells of revolution also to two simultaneous second order ODE's for a stress function and a strain function. His results are esthetically unsatisfactory. They do not preserve the static-geometric duality inherent in the shell equations (though not known at the time). Also, there is some difficulty in taking his results to the flat plate limit. Both of these shortcomings were removed later by a different (and more natural) choice of the stress function introduced by E. Reissner [3]. More recently, a similar reduction was accomplished for the more general Gunther-Reissner-Schaeffer type shell theory as well as for the axisymmetric torsion (and twisting) problem [4]. In both cases, allowance was made for shells in the form of a segment of a frustum (and therefore circumferentially incomplete) without destroying the elegant mathematical structure of the Reissner-Meissner-Reissner type formulation. As we shall see, this mathematical structure enables us to bring many exact, asymptotic and numerical solution techniques to bear on specific axisymmetric problems for shells of revolution which would not be tractable otherwise. References to other results of intermediate generality can be in [4].

Problems associated with the first harmonics in the Fourier decomposition in the polar angle θ are also amenable to simplifications and reduction similar to the axisymmetric case. However, simplifications and reduction for these so-called lateral (or wind) load problems were only recognized and implemented for the first time in 1959 by V. S. Chernina [5]. First integrals of the equilibrium and compatibility equations were derived for lateral load problems in [5] and the system of shell equations was also reduced in [6] to two simultaneous second order ODE's for a stress function and a strain function. Similar to Meissner's results for axisymmetric bending, Chernina's reduction also does not preserve the static-geometric duality inherent in shell theory and the resulting two ODE's take on an indeterminate form in the flat plate limit. These shortcomings were removed in [6,7]. To the degree of accuracy of shell theory, the two new second order simultaneous equations are remarkably similar to the Reissner-Meissner-Reissner equations for axisymmetric bending problems. The new reduction of [6,7] also allows for nonperiodic displacement fields which correspond to strain fields proportional to $\sin\theta$ and $\cos\theta$; thus the results obtained apply to laterally loaded ring shell sectors as well.

For more general loading leading to shell response with higher harmonics in the Fourier decomposition in the polar angle θ, it has been known for some time that the equations of the classical linear (and nonlinear) shallow shell theory can be reduced to two simultaneous fourth order PDE's for the transverse midsurface displacement and a stress function [8]. A similar reduction is also possible for spherical shells [9,10]. The reduction to two simultaneous equations has again led to exact and asymptotic solutions for specific problems which would not have been recognized otherwise. The success attained in the case of shallow shells and spherical shells naturally stimulated effort toward similar reduction for other classes of shells. This effort was further encouraged when the desired reduction was accomplished for circular cylindrical shells, at least within the error inherent in shell theory [11]. However, less encouraging results began to emerge shortly afterwards [12,13]. In the last part of these notes, we will describe the reductions for shallow shells, spherical shells and circular cylindrical shells. For the circular cylindrical shell case, we will accomplish the reduction using the procedure proposed in [14]. The same procedure will then be applied to conical shells [15] to show for the first time that a two simultaneous fourth order ordinary differential equations formulation is always possible for the <u>nth</u> harmonics $C_n(\xi_1)$ and $S_n(\xi_1)$ of the shell response ($n > 2$). It is seen from the development for a conical shell that the same reduction to two simultaneous fourth order differential equations can also be carried out for general shells of revolution and that, consistent with [12], a reduction to two simultaneous fourth order PDE's similar to situation of w and F of the shallow shell is not always possible.

2. Geometry of Surfaces of Revolution

The position vector for a point on a surface of revolution may be written as

$$\vec{r} = r(\xi_1)\vec{i}_r + z(\xi_1)\vec{i}_z$$

where $\vec{i}_r$ and $\vec{i}_z$ are the unit vectors in the radial and axial direction, respectively, for the cylindrical coordinate system (r,θ,z), with $\vec{i}_r = \cos\theta\vec{i}_x + \sin\theta\vec{i}_y$. The third unit vector $\vec{i}_\theta$ of the orthogonal triad $(\vec{i}_r,\vec{i}_\theta,\vec{i}_z)$ is given in terms of the cartesian unit vectors $\vec{i}_x$ and $\vec{i}_y$ by $\vec{i}_\theta = -\sin\theta\vec{i}_x + \cos\theta\vec{i}_y$ with $d(\vec{i}_r)/d\theta = \vec{i}_\theta$ and $d(\vec{i}_\theta)/d\theta = -\vec{i}_r$. From $\vec{r}$, we can compute the unit tangent vectors $\vec{t}_1$ and $\vec{t}_2$ along the coordinate curves on the surface, the arc length coefficients (i.e., the coefficients of the first fundamental form of the surface), the unit normal vector and the radii of curvature (or the coefficients of the second fundamental form).

We take $\xi_2 = \theta$ and indicate partial differentiation with respect to ξ_1 and θ by $(\ )'$ and $(\ )^\bullet$, respectively. We get from $\vec{r}$, $\vec{r}' = r'\vec{i}_r + z'\vec{i}_z$ and $\vec{r}^\bullet = r\vec{i}_\theta$ so that

$$\alpha_1 = \sqrt{(r')^2+(z')^2} \equiv \alpha, \qquad \alpha_2 = r$$

$$\vec{t}_1 = \cos\xi\vec{i}_r + \sin\xi\vec{i}_z, \qquad \vec{t}_2 = \vec{i}_\theta$$

where

$$\cos\xi = \frac{r'}{\alpha}, \qquad \sin\xi = \frac{z'}{\alpha}$$

The quantity ξ is evidently the angle made by the tangent of a surface meridian with the base plane (x,y-plane). The surface coordinates are orthogonal as $\vec{t}_1 \cdot \vec{t}_2 = 0$; it follows that

$$\vec{n} = \vec{t}_1 \times \vec{t}_2 = - \sin\xi \vec{i}_r + \cos\xi \vec{i}_z$$

and from the Weingarten formulas of differential geometry of surfaces (see Chapter 4, section (4) of Reference [16]),

$$\frac{1}{R_{11}} = - \frac{\xi'}{\alpha} = \frac{r''z'-r'z''}{\alpha^3} ,$$

$$\frac{1}{R_{22}} = - \frac{\sin\xi}{r} = - \frac{z'}{r\alpha} ,$$

$$\frac{1}{R_{12}} = \frac{1}{R_{21}} = 0$$

The surface coordinates (ξ,θ) are evidently lines of curvature coordinates. The quantities $\alpha_1 \equiv \alpha$, α_2 and R_{ij} are functions of ξ_1 only. That they do not depend on $\xi_2 \equiv \alpha$ is a feature of surfaces of revolution and will be useful for the analysis of shells of revolution.

In the subsequent development, the subscript 2 in all stress, strain and displacement measures for shells of revolution will be replaced by θ to reduce the amount of bookkeeping. To the extent that ξ_1 will always be a quantity which varies along a surface meridian (including the possibility of $\xi_1 = \xi$) the subscript 1 will be replaced by ξ whether or not we have $\xi_1 = \xi$. We note that it is not always possible to choose $\xi_1 = \xi$, given that ξ is a constant for a conical surface of revolution.

2 - Boundary Value Problems of Linear Shell Theory

1. Equilibrium and Compatibility Equations for Shells of Revolution

With α_j and R_{ij} as given in the previous section, we may now specialize the scalar differential equations of equilibrium for orthogonal surface coordinates (given in chapter 4, section (5) of [16]) to the special case of shells of revolution. These equilibrium equations are recorded here for references and/or further specializations in later sections.

$$(rN_{\xi\xi})' + (\alpha N_{\theta\xi})^{\bullet} - \alpha\cos\xi N_{\theta\theta} - r\xi' Q_\xi + r\alpha p_\xi = 0$$

$$(rN_{\xi\theta})' + (\alpha N_{\theta\theta})^{\bullet} + \alpha\cos\xi N_{\theta\xi} - \alpha\sin\xi Q_\theta + r\alpha p_\theta = 0$$

$$(rQ_\xi)' + (\alpha Q_\theta)^{\bullet} + r\xi' N_{\xi\xi} + \alpha\sin\xi N_{\theta\theta} + r\alpha p_n = 0$$

$$(rM_{\xi\xi})' + (\alpha M_{\theta\xi})^{\bullet} - \alpha\cos\xi M_{\theta\theta} - r\alpha Q_\xi - \alpha\sin\xi P_\theta + r\alpha q_\xi = 0$$

$$(rM_{\xi\theta})' + (\alpha M_{\theta\theta})^{\bullet} \ \alpha\cos\xi M_{\theta\xi} - r\alpha Q_\theta + r\xi' P_\xi + r\alpha q_\theta = 0$$

$$(rP_\xi)' + (\alpha P_\theta)^{\bullet} + r\alpha(N_{\xi\theta} - N_{\theta\xi}) - r\xi' M_{\xi\theta} + \alpha\sin\xi M_{\theta\xi} + r\alpha q_n = 0$$

where $(\ )' \equiv \partial(\ )/\partial\xi_1$ and $(\ )^{\bullet} \equiv \partial(\ )/\partial\theta$, respectively, and ξ_1, is generally different from the meridional slope angle ξ.

The compatibility equations for the strain measures of shells of revolution can be obtained immediately from these equilibrium equations with the application of the static-geometric duality of linear shell theory (see Chapter 6 of [16]). These compatibility equations are identical in form to the above equilibrium equations with all their stress measures replaced by the dual strain measures according to the following table:

$N_{\xi\xi}$	$N_{\theta\theta}$	$N_{\xi\theta}$	$N_{\theta\xi}$	Q_{ξ}	Q_{θ}	p_{ξ}	p_{θ}	p_{n}
$-\kappa_{\theta\theta}$	$-\kappa_{\xi\xi}$	$\kappa_{\theta\xi}$	$\kappa_{\xi\theta}$	λ_{θ}	$-\lambda_{\xi}$	0	0	0

$M_{\xi\xi}$	$M_{\theta\theta}$	$M_{\xi\theta}$	$M_{\theta\xi}$	P_{ξ}	P_{θ}	q_{ξ}	q_{θ}	q_{n}
$\varepsilon_{\theta\theta}$	$\varepsilon_{\xi\xi}$	$\varepsilon_{\theta\xi}$	$\varepsilon_{\xi\theta}$	γ_{θ}	$-\gamma_{\xi}$	0	0	0

The equilibrium and compatibility equations may be supplemented by a set of stress strain relations to form a complete system of governing equations for the elastostatics of shells. Unless indicated otherwise, these stress strain relations will be taken in the form

$$\varepsilon_{\xi\xi} = A(N_{\xi\xi} - \nu_s N_{\theta\theta}), \qquad \varepsilon_{\theta\theta} = A(N_{\theta\theta} - \nu_s N_{\xi\xi}),$$

$$\varepsilon_{\xi\theta} = \varepsilon_{\theta\xi} = A_s(N_{\xi\theta} + N_{\theta\xi}), \qquad \gamma_{\theta} = A_Q Q_{\theta}, \qquad \gamma_{\xi} = A_Q Q_{\xi}$$

$$M_{\theta\theta} = D(\kappa_{\theta\theta} + \nu_b \kappa_{\xi\xi}), \qquad M_{\xi\xi} = D(\kappa_{\xi\xi} + \nu_b \kappa_{\theta\theta}),$$

$$M_{\theta\xi} = M_{\xi\theta} = D_s(\kappa_{\theta\xi} + \kappa_{\xi\theta}), \qquad P_{\xi} = D_P \lambda_{\xi}, \qquad P_{\theta} = D_P \lambda_{\theta}$$

where $A_S = A(1+\nu_s)/2$ and $D_S = D(1-\nu_b)/2$. We note that the first half of these relations are the static-geometric duals of the second half if we observe the following dual relations among the elastic parameters for the shell:

A	ν_s	A_S	A_Q
$-D$	$-\nu_b$	$-D_S$	$-D_P$

For the conventional first approximation theory of isotropic (and linearly elastic) shells which are homogeneous across the shell thickness, we have

$$A = \frac{1}{Eh} , \qquad D = \frac{Eh^3}{12(1-\nu^2)} , \qquad \nu_s = \nu_b = \nu, \qquad D_P = A_Q = 0$$

where E, is Young's modulus, ν is Poisson's ratio and h is the shell thickness. In the subsequent development, we consider only shells for which these parameters do not vary in the circumferential direction.

To the extent that only quantities which appear in the stress strain relations are involved in the present formulation of the elastostatic problem of shells of revolution, it is customary to refer to this formulation as an _intrinsic_ theory of shells.

2. Displacement Components and Stress Functions

With $\xi_2 = \theta$ the strain-displacement relations for shells of revolution may be obtained from the corresponding relations for general orthogonal surface coordinates given in [16] (Chapter 4, section (5)). We record them here for future references and/or further specializations:

$$\varepsilon_{\xi\xi} = \frac{1}{\alpha}(u'_\xi - \xi' w), \qquad \varepsilon_{\theta\theta} = \frac{1}{r}(u^{\bullet}_\theta + \cos\xi u_\xi - \sin\xi w)$$

$$\varepsilon_{\xi\theta} = \frac{u'_\theta}{\alpha} - \omega, \qquad \varepsilon_{\theta\xi} = \frac{1}{r}(u^{\bullet}_\xi - \cos\xi u_\theta) + \omega$$

$$\gamma_\xi = \phi_\xi + \frac{1}{\alpha}(w' + \xi' u_\xi), \qquad \gamma_\theta = \phi_\theta + \frac{1}{r}(w^{\bullet} + \sin\xi u_\theta)$$

$$\kappa_{\xi\xi} = \frac{1}{\alpha}\phi'_\xi \qquad \kappa_{\theta\theta} = \frac{1}{r}(\phi^{\bullet}_\theta + \cos\xi\phi_\xi)$$

$$\kappa_{\xi\theta} = \frac{1}{\alpha}(\phi'_\theta + \xi'\omega) \qquad \kappa_{\theta\xi} = \frac{1}{r}(\phi^{\bullet}_\xi + \cos\xi\phi_\theta - \sin\xi\omega)$$

$$\lambda_\xi = \frac{1}{\alpha}(\omega' - \xi'\phi_\theta) \qquad \lambda_\theta = \frac{1}{r}(\omega^{\bullet} + \sin\xi\phi_\xi)$$

where r, α and ξ are functions of ξ_2 only.

From the above strain displacement relations, we can immediately obtain a stress function solution of the equilibrium equations by the static-geometric duality of linear shell theory. The stress function representations of the stress resultants and couples are identical in form to the above strain-displacement relations with all strain measures replaced by their dual stress measures according to the rules given in the last section and with the displacement components replaced by their dual stress functions accord to the following table

u_ξ	u_θ	w	ϕ_ξ	ϕ_θ	ω
U_ξ	U_θ	F	Φ_ξ	Φ_θ	Ω

The strain-displacement relations and the stress function representations along with a set of stress strain relations also form a complete system of governing differential equations for the elastostatics of shells. To the extent that displacement components and stress functions are not quantities involved in the stress strain relations, we shall refer to the present formulation of shell problems as an extrinsic formulation.

3. Boundary Conditions

A complete set of shell equations is to be supplemented by an appropriate set of boundary conditions to form a well-posed boundary value problem for the elastostatics of the shells. The appropriate set of boundary conditions is of course dictated by the particular physical problem. That the resulting boundary value problem is well-posed may be confirmed by variational considerations [16].

For problems with a prescribed displacement field along an ξ_1 = constant edge, we have from [16] the following six scalar displacement boundary conditions

$$\xi_1 = \xi_1^*: \qquad \Delta u_\xi = \Delta u_\theta = \Delta w = \Delta\phi_\xi = \Delta\phi_\theta = \Delta\omega = 0$$

where $\Delta f \equiv f(\xi_1^*,\theta) - f^*(\theta)$ and $(\ )^*$ is a prescribed quantity. For classical (i.e. the conventional first approximation) shell theory, the condition $P_\xi \equiv 0$ eliminates $\Delta\omega = 0$ as an Euler boundary condition of the variational principle for stresses, strains and displacements [16], while the condition $\gamma_\theta \equiv 0$ and the strain displacement relation

$$\gamma_\theta = \phi_\theta + \frac{1}{r}(w^{\bullet} + \sin\xi u_\theta)$$

do not allow ϕ_θ to be prescribed independently. Hence, we have for the classical theory the following four displacement boundary conditions along $\xi_1 = \xi_1^*$:

$$\xi_1 = \xi_1^*: \qquad \Delta u_\xi = \Delta u_\theta = \Delta w = \Delta\phi_\xi = 0$$

where $\Delta\phi_\xi = 0$ may be replaced by $\Delta w' = 0$ in view of the condition of vanishing transverse shearing strain $\gamma_\xi \equiv 0$. Appropriate displacement boundary conditions for "lower order" shell theories (such as membrane or inextensional bending theory) can also be obtained by specializing the corresponding conditions in [16] to shells of revolution with $\xi_2 = \theta$.

For problems with a prescribed stress resultant field and a stress couple field along an $\xi_1 = \xi_1^*$ edge, we have from [16] the following six stress boundary conditions:

$$\xi_1 = \xi_1^*: \quad \Delta N_{\xi\xi} = \Delta N_{\xi\theta} = \Delta Q_\xi = \Delta M_{\xi\xi} = \Delta M_{\xi\theta} = \Delta P_\xi = 0.$$

For classical shell theory, we must have $P_\xi^*(\theta) \equiv 0$ to be consistent with $P_\xi \equiv P_\theta \equiv 0$, while the requirement of no transverse shearing strain $\gamma_\xi \equiv \gamma_\theta \equiv 0$, *leads to the following four Kirchhoff-Bassett contracted stress* boundary conditions along an $\xi_1 = \xi_1^*$ edge [16]:

$$\xi_1 = \xi_1^*: \quad \Delta N_{\xi\xi} = \Delta N_{\xi\theta}^e = \Delta Q_\xi^e = \Delta M_{\xi\xi} = 0$$

where

$$N_{\xi\theta}^e = N_{\xi\theta} + \frac{M_{\xi\theta}}{R_{\theta\theta}}, \quad Q_\xi^e = Q_\xi + \frac{1}{r} \dot{M}_{\xi\theta}$$

(Note that $1/R_{\xi\theta} = 0$ so that $N_{\xi\xi}^e = N_{\xi\xi} + M_{\xi\theta}/R_{\xi\theta} = N_{\xi\xi}$.) Appropriate contracted boundary stress boundary conditons for lower order shell theories may be found in [16] and will not be discussed here.

It is of considerable interest to note that displacement boundary conditions may be alternatively formulated in terms of strain measures. This is particularly important for an intrinsic formulation as it would be rather cumbersome at best to integrate the strain-displacement relations before the general solution of the intrinsic theory can be specialized by the actual displacement boundary conditions. When all six displacement components are prescribed along an $\xi_1 = \xi_1^*$ edge, we may use the following six conditions on the strain measures for an intrinsic formulation of shells of revolution:

$$\xi_1 = \xi_1^*: \quad \Delta\kappa_{\theta\theta} = \Delta\kappa_{\theta\xi} = \Delta\lambda_\theta = \Delta\varepsilon_{\theta\theta} = \Delta\varepsilon_{\theta\xi} = \Delta\gamma_\theta = 0$$

The corresponding conditions for the classical shell theory are:

$$\xi_1 = \xi_1^*: \quad \Delta\kappa_{\theta\theta} = \Delta\kappa^e_{\theta\xi} = \Delta\lambda^e_\theta = \Delta\varepsilon_{\theta\theta} = 0$$

where $\kappa^e_{\theta\xi}$ and λ^e_θ are the static-geometric duals of $N^e_{\xi\theta}$ and Q^e_ξ, respectively. Both sets of strain conditions are obtained by specializing the corresponding conditions in [16] for shells of revolution with $\xi_2 = \theta$. It is clear from the strain displacement relations of section (3) that $\kappa^*_{\theta\theta}$, $\kappa^*_{\theta\xi}$, λ^*_θ, $\varepsilon^*_{\theta\theta}$, $\varepsilon^*_{\theta\xi}$ and γ^*_θ can be determined once we have u^*_ξ, u^*_θ, w^*, ϕ^*_ξ, ϕ^*_θ and ω^* as only differentiation with respect to θ is involved in the relevant strain displacement relations.

By the static-geometric duality, we may also formulate the stress boundary conditions in terms of stress functions if we wish to do so. The resulting boundary conditions for the stress functions are just the static-geometric duals of the displacement boundary conditions for displacement boundary value problems.

It is also clear that similar boundary conditions along a $\theta = \theta^*$ edge can also be written down. As these conditions are similar to those for ξ_1 = constant edge, they will not be recorded here.

Mixed boundary conditions with some of the conditions given in terms of stress resultant(s) and/or couple(s) and the remaining conditions in terms of displacements discussed in [16] for them to lead to a well-posed boundary value problem. The most convenient (admissable) form for a set of mixed conditions in a given formulation of the shell problem (intrinsic, extrinsic or others) will have to be investigated on an individual basis. In general, a set of mixed conditions given in terms of stress resultants, stress couples and displacement components may be left in its original form in an extrinsic formulation.

Part II - Axisymmetric Stress Distributions

3 - Intrinsic Formulation with Axisymmetry

1. Axisymmetric Bending and Stretching

For stress and strain measures which are independent of the polar angles θ, the equations of equilibrium and compatibility for shells of revolution given in Chapter 2 of these lecture notes naturally divide themselves into two uncoupled groups. The first group consists of the three equilibrium equations

$$(rN_{\xi\xi})' - \alpha\cos\xi\, N_{\theta\theta} - r\xi' Q_\xi + r\alpha p_\xi = 0$$

$$(rQ_\xi)' + r\xi' N_{\xi\xi} + \alpha\sin\xi\, N_{\theta\theta} + r\alpha p_n = 0$$

$$(rM_{\xi\xi})' - \alpha\cos\xi\, M_{\theta\theta} - r\alpha Q_\xi - \alpha\sin\xi\, P_\theta + r\alpha q_\xi = 0$$

and the three dual compatibility equations. The stress and strain measures involved in this group of equations are evidently associated with the stretching and bending action of the shell.

Six of the twelve stress strain relations for the shells of interest in these lecture notes given in Chapter 2 also contain only the same twelve stress and strain measures:

$$\varepsilon_{\theta\theta} = A(N_{\theta\theta} - \nu_s N_{\xi\xi}), \qquad \varepsilon_{\xi\xi} = A(N_{\xi\xi} - \nu_s N_{\theta\theta}), \qquad \gamma_\xi = A_Q Q_\xi$$

$$M_{\xi\xi} = D(\kappa_{\xi\xi} + \nu_b \kappa_{\theta\theta}), \qquad M_{\theta\theta} = D(\kappa_{\theta\theta} + \nu_b \kappa_{\xi\xi}), \qquad P_\theta = D_p \lambda_\theta$$

The three equilibrium equations, three dual compatibility equations and these six stress relations form a sixth order system of twelve (differential) equations for $N_{\xi\xi}$, $N_{\theta\theta}$, Q_ξ, $M_{\xi\xi}$, $M_{\theta\theta}$, P_θ and their dual strain measures.

Supplemented by suitable boundary conditions at the edge(s) of the shell, we have a well posed boundary value problem in ordinary differential equations.

Of the six stress boundary conditions along and $\xi_1 = \xi_1^*$ edge, three involve stress measures which appear in the axisymmetric bending (and stretching) problem*. They are:

$$\xi_1 = \xi_1^*: \qquad \Delta N_{\xi\xi} = \Delta Q_\xi = \Delta M_{\xi\xi} = 0$$

As we shall see in a later section the quantities $N^*_{\xi\xi}$ and Q^*_ξ cannot be completely arbitrary at one of the edges of the shell, as the shell structure must be in overall equilibrium. Note also that the three stress boundary conditions remain unchanged for classical shell theory, keeping in mind that we have $\Delta Q^e_\xi = \Delta(Q_\xi + r^{-1}\dot{M}_{\xi\theta}) = \Delta Q_\xi$ because of the stipulation of axisymmetric stress measures.

If displacement boundary conditions are prescribed along $\xi_1 = \xi_1^*$, they are most conveniently taken in the form of strain boundary conditions. Among the six appropriate strain boundary conditions given in Chapter 2, three involve the strain measures associated with the axisymmetric bending problem. They are

$$\xi_1 = \xi_1^*: \qquad \Delta\kappa_{\theta\theta} = \Delta\lambda_\theta = \Delta\varepsilon_{\theta\theta} = 0.$$

Keeping in mind $\Delta\lambda^e_\theta = \Delta(\lambda_\theta - r^{-1}\dot{\varepsilon}_{\theta\xi}) = \Delta\lambda_\theta$ because of axisymmetry these three conditions remain unchanged for the classical shell theory. By the static geometric duality, we expect that $K^*_{\theta\theta}$ and λ^*_θ cannot be prescribed arbitrarily at one of the shell edges because of the requirement of overall compability.

(*) It is customary in the shell literature to omit the reference to stretching for this class of shell problems.

2. Axisymmetric Torsion and Twisting

The second group of equilibrium and compatibility equations for axisymmetric stress and strain distributions consists of the three remaining equilibrium equations

$$(rN_{\xi\theta})' + \alpha\cos\xi N_{\theta\xi} - \alpha\sin\xi Q_\theta + r\alpha p_\theta = 0$$

$$(rM_{\xi\theta})' + \alpha\cos\xi M_{\theta\xi} - r\alpha Q_\theta + r\xi' P_\xi + r\alpha q_\theta = 0$$

$$(rP_\xi)' + r\alpha(N_{\xi\theta} - N_{\theta\xi}) - r\xi' M_{\xi\theta} + \alpha\sin\xi M_{\theta\xi} + r\alpha q_n = 0$$

and the three dual compatibility equations. They contain only the stress measures $N_{\xi\theta}$, $N_{\theta\xi}$, Q_θ, $M_{\xi\theta}$, $M_{\theta\xi}$ and P_θ and the dual strain measures. The six remaining stress strain relations not used in the axisymmetric bending problem are

$$\varepsilon_{\xi\theta} = \varepsilon_{\theta\xi} = A_S(N_{\xi\theta} + N_{\theta\xi}), \qquad \gamma_\theta = A_Q Q_\theta$$

$$M_{\theta\xi} = M_{\xi\theta} = D_S(\kappa_{\theta\xi} + \kappa_{\xi\theta}), \qquad P_\xi = D_P\lambda_\xi$$

with $A_S = A(1+\nu_S)/2$ and $D_S = D(1-\nu_b)/2$. They also involve only the same six stress measures and their dual strain measures. The sixth order system of three equilibrium equations, three compatibility equations and six stress strain relations for the twelve stress strain measures evidently governs the axisymmetric shearing and twisting actions of shells of revolution. Because of the types of physical problems involved, this general class of problems is commonly known as axisymmetric torsion (and twisting) of shells of revolution.

The sixth order system is supplemented by three boundary conditions at each edge of the shell. The three stress boundary conditions along an $\xi = \xi_1^*$ edge are:

$$\xi_1 = \xi_1^*: \qquad \Delta N_{\xi\theta} = \Delta M_{\xi\theta} = \Delta P_\xi = 0$$

For the classical theory, $\Delta P_\xi = 0$ should be trivially satisfied for consistency (in particular, we must have $P_\xi^* = 0$) while $\gamma_\theta \equiv 0$ leads to the Kirchhoff-Bassett contracted stress boundary condition

$$\xi_1 = \xi_1^*: \qquad \Delta N_{\xi\theta}^e = 0$$

with $N_{\xi\theta}^e = N_{\xi\theta} + M_{\xi\theta}/R_{\theta\theta} = N_{\xi\theta} - \sin\xi\, M_{\xi\theta}/r$. This is the only stress boundary for an ξ_1 = constant edge for this class of problems.

If displacement conditions are prescribed, they are most conveniently taken in terms of strain measures for our intrinsic formulation of the problem. The appropirate strain boundary conditions along an $\xi_1 = \xi_1^*$ edge are:

$$\xi_1 = \xi_1^*: \qquad \Delta\kappa_{\xi\theta} = \Delta\varepsilon_{\theta\xi} = \Delta\gamma_\theta = 0$$

For the class theory, $\Delta\gamma_\theta = 0$ should be trivially satisfied (in particular, we must have $\gamma_\theta^* = \phi_\theta^* + r^{-1}[(w^*)^{\cdot} + \sin\xi u_\theta^*] = 0$. (Note that we have not required that all displacement components to be axisymmetric as long as they lead to axisymmetric strain measures. We will return to this point in section (4) of this chapter.) At the same time, the condition $P_\xi = 0$ leads to the contracted strain condition

$$\xi_1 = \xi_1^*: \qquad \Delta\kappa_{\theta\xi}^e = 0$$

with $\kappa_{\theta\xi}^e = \kappa_{\theta\xi} - \varepsilon_{\theta\xi}/R_{\theta\theta} = \kappa_{\theta\xi} + \sin\xi(\varepsilon_{\theta\xi}/r)$.

Similar to the axisymmetric bending problem, overall equilibrium and compatibility, $N_{\xi\theta}^*$ and $M_{\xi\theta}^*$ for the stress case and $\kappa_{\theta\xi}^*$ and $\varepsilon_{\theta\xi}^*$ for the displacement case cannot be arbitrarily prescribed at one shell edge. We will return to this point in the next section.

3. First Integrals and Overall Equilibrium

For the axisymmetric bending problem, we may form the following combination of the two force equilibrium equations,

$$\sin\xi[(rN_{\xi\xi})' - \alpha\cos\xi N_{\theta\theta} - r\xi' Q_\xi + r\alpha p_\xi]$$
$$+ \cos\xi[(rQ_\xi)' + r\xi' N_{\xi\xi} + \alpha\sin\xi N_{\theta\theta} + r\alpha p_n] = 0$$

to get

$$[r(N_{\xi\xi}\sin\xi + Q_\xi\cos\xi)]' + r\alpha p_v = 0, \quad p_v \equiv p_\xi\sin\xi + p_n\cos\xi.$$

Upon integration, we get, for $\xi_i < \xi_1 < \xi_o$,

$$r(N_{\xi\xi}\sin\xi + Q_\xi\cos\xi) = rV(\xi_1), \quad rV \equiv \frac{F_z}{2\pi} - \int_{\xi_i}^{\xi_1} p_v(t)r(t)\alpha(t)dt$$

where F_z is a constant of integration. The existence of the above first integral of the equilibrium equation is expected from the requirement that the shell be in overall force equilibrium. The overall equilibrium of an arbitrary shell frustrum requires

$$\int_0^{2\pi}\int_{\xi_i}^{\xi_1} [(r\vec{N}_\xi)' + (\alpha\vec{N}_\theta)^{\bullet} + r(t)\alpha(t)\vec{p}(t,\theta)]dtd\theta = \vec{0},$$

which in component form is of the form

$$\vec{i}_x \int_0^{2\pi}\int_{\xi_i}^{\xi_1} [f_r(\xi_1)\cos\theta - \sin\theta\, f_\theta(\xi_1)]d\xi_1 d\theta$$

$$+ \vec{i}_y \int_0^{2\pi}\int_{\xi_i}^{\xi_1} [f_r(\xi_1)\sin\theta + f_\theta(\xi_1)\cos\theta]d\xi_1 d\theta$$

$$+ \vec{i}_z \int_0^{2\pi}\int_{\xi_i}^{\xi_1} [\{r(Q_\xi\cos\xi + N_{\xi\xi}\sin\xi)\}' + r\alpha p_v]d\xi_1 d\theta = \vec{0}.$$

(It will not be necessary to write out the expressions for $f_r(\xi_1)$, etc.) We get from this only one nontrivial condition of overall equilibrium in the direction of the axis of revolution, i.e. the axial direction:

$$2\pi[r(Q_\xi \cos_\xi + N_{\xi\xi}\sin\xi) + \int_{\xi_i}^{\xi_1} p_V(t)r(t)\alpha(t)dt] = F_Z$$

For the case of no surface load so that $p_V \equiv 0$, the constant of integration F_Z is the resultant axial force experienced by the shell which must be the same throughout. If stress conditions are prescribed at both circular edges of a shell frustum, $\xi_1 = \xi_i$ and $\xi_1 = \xi_0$, then we must have

$$r(\xi_i)[Q_\xi(\xi_i)\cos(\xi(\xi_i)) + N_{\xi\xi}(\xi_i)\sin(\xi(\xi_i))]$$

$$= r(\xi_0)[Q_\xi(\xi_0)\cos(\xi(\xi_0)) + N_{\xi\xi}(\xi_0)\sin(\xi(\xi_0))]$$

so that Q_ξ^* and N_ξ^* cannot be arbitrarily prescribed at both edges. The situation is modified in a trivial way if $p_V \neq 0$. In that case F_Z is the resultant axial force at the "lower" edge $\xi_1 = \xi_i$; at the "upper" edge $\xi_1 = \xi_0 (> \xi_i)$, the resultant axial force is now F_Z net the resultant force from p_V.

By the static geometric duality, we have also the following first integral of the compatibility equations for the axisymmetric bending problem

$$r(\kappa_{\theta\theta}\sin\xi - \lambda_\theta\cos\xi) = - C_B$$

where C_B is a constant of integration, the static geometric dual of (rV) (or $(F_Z/2\pi$ if $p_V = 0)$.

For axisymmetric torsion and twisting, we may form the combination

$$\sin\xi[(rM_{\xi\theta})' + \cos\xi\alpha M_{\theta\xi} - r\alpha Q_\theta + r\xi' P_\xi + r\alpha q_\theta]$$

$$- \cos\xi[(rP_\xi)' + r\alpha(N_{\xi\theta} - N_{\theta\xi}) - r\xi' M_{\xi\theta} + \sin\xi\alpha M_{\theta\xi} + r\alpha q_n]$$

$$- r[(rN_{\xi\theta})' + \cos\xi\alpha N_{\theta\xi} - \sin\xi\alpha Q_\theta + r\alpha p_\theta] = 0$$

of the three scalar equilibrium equations of the group to get

$$r^2\left(N_{\xi\theta} - \frac{\sin\xi}{r} M_{\xi\theta} + \frac{\cos\xi}{r} P_\xi\right) = T,$$

with

$$T = \frac{T_z}{2\pi} + \int_{\xi_i}^{\xi_1} (rp_\theta + \cos\xi q_n - \sin\xi q_\theta)r\alpha \, d\xi_1$$

where T_z is constant of integration. Physically, this first integral is a consequence of the requirement overall moment equilibrium. From

$$\int_0^{2\pi} \int_{\xi}^{\xi_1} \{(r\vec{M}_\xi)' + (\alpha\vec{M}_\theta)^{\bullet} + \vec{r}' \times (r\vec{N}_\xi) + \vec{r}^{\bullet} \times (\alpha\vec{N}_\theta) + r\alpha\vec{q}$$

$$+ \vec{r} \times [(r\vec{N}_\xi)' + (\alpha\vec{N}_\theta)^{\bullet} + r\alpha\vec{p}]\} \, d\xi_I \, d_\theta = \vec{0}.$$

we get as in the case of force equilibrium only one nontrivial condition associated with the resultant axial torque (turning about the axis of revolution):

$$-2\pi[r(M_{\xi\theta}\sin\xi - P_\xi\cos\xi - rN_{\xi\theta}) + \int_{\xi_i}^{\xi_1} (q_\theta\sin\xi - q_n\cos\xi - rp_\theta)r\alpha d\xi_1] = T_z$$

where the left hand side is the resultant moment turning about the positive $\vec{i}_z$ direction.

By the static geometric duality, we have also the following first integral of the compatibility equations for the axisymmetric shearing and twisting problem:

$$r(\varepsilon_{\theta\xi}\sin\xi + \gamma_{\theta}\cos\xi + r\kappa_{\theta\xi}) = C_T$$

where C_T is a constant of integration, the static geometric dual of T (or $T_z/2\pi$ if $q_n \equiv q_\theta \equiv p_\theta \equiv 0$).

The constants C_B and C_T in the two first integrals of the compatibility equations have their own physical significance which can be readily seen from the discussion in the next section on certain nonsymmetric displacement fields associated with axisymmetric strain fields.

4. Determination of Displacement Fields

Once we have the strain components from the solution of the boundary value problems for axisymmetric bending and stretching of section (1) and for axisymmetric torsion and twisting of section (2), we may then integrate the twelve strain displacement relations of section (2) of Chapter 2 to get the displacement components ϕ_ξ, ϕ_θ, ω, u_ξ, u_θ and w. The fact that the differential equations of compatibility have already been satisfied ensures that the twelve relations do not overdetermine the six displacement components. In the process of determining the displacement fields $\vec{\phi}$ and $\vec{u}$, it is important to realize that the displacement components themselves need not be axisymmetric although they must give rise to axisymmetric strain components.

To see what possible forms of unsymmetric displacement components may be, we note that

$$\alpha\vec{\kappa}_\xi = \vec{\phi}' = [-(\cos\xi\phi_\theta + \sin\xi\omega)'\vec{i}_r + (\cos\xi\omega - \sin\xi\phi_\theta)'\vec{i}_z + \phi_\xi'\vec{i}_\theta]$$

$$r\vec{\kappa}_\theta = \vec{\phi}^\bullet = [\{\phi_\xi^\bullet - (\cos\xi\phi_\theta + \sin\xi\omega)\}\vec{i}_\theta + (\omega^\bullet\cos\xi - \phi_\theta^\bullet\sin\xi)\vec{i}_z$$

$$- (\phi_\xi + \cos\phi_\theta^\bullet + \sin\xi\omega^\bullet)\vec{i}_r]$$

If we write with no loss in generality $f = \tilde{f}(\xi) + \tilde{f}(\xi_1,\theta)$, it follows from the above relations that

$$\hat{\phi}_\xi(\xi_1,\theta) = \hat{\phi}_\xi(\theta), \quad \cos\xi\hat{\omega} - \sin\xi\hat{\phi}_\theta = \hat{\phi}_z(\theta), \quad \cos\xi\hat{\phi}_\theta + \sin\xi\hat{\omega} = \hat{\phi}_r(\theta)$$

$$\hat{\phi}_\xi^\bullet(\theta) = \hat{\phi}_r(\theta) + \tilde{g}_\xi(\xi_1), \qquad \hat{\phi}_r^\bullet = -\hat{\phi}_\xi + \tilde{g}_r(\xi_1) \qquad \hat{\phi}_z^\bullet(\theta) = \tilde{g}_z(\xi_1)$$

or

$$\cos\xi\hat{\omega} - \sin\xi\hat{\phi}_\theta \equiv \hat{\phi}_z(\theta) = C_B\theta, \qquad \tilde{g}_\xi = -C_4, \qquad \tilde{g}_r = C_3$$

$$\hat{\phi}_\xi(\theta) = C_1\cos\theta + C_2\sin\theta + C_3, \qquad \hat{\phi}_r(\theta) = -C_1\sin\theta + C_2\cos\theta + C_4$$

where C_B, C_1, ..., C_4 are constants of integration still to be determined. The terms associated with C_1 and C_2 correspond to a rigid body rotation and may be deleted while the terms associated with C_3 and C_4 may be absorbed in $\tilde{\phi}_\xi(\xi_1)$ and $\tilde{\phi}_r(\xi_1) \equiv \cos\xi\tilde{\phi}_\theta(\xi_1) + \sin\xi\tilde{\omega}(\xi_1)$, respectively, leaving us with $\hat{\phi}_\xi \equiv \hat{\phi}_r \equiv 0$ and $\hat{\phi}_z = C_B\theta$, or

$$\hat{\phi}_\xi \equiv 0, \qquad \hat{\phi}_\theta = -C_B\theta\sin\xi, \qquad \hat{\omega} = C_B\theta\cos\xi,$$

and therewith

$$\phi_\xi = \tilde{\phi}_\xi(\xi_1), \qquad \phi_\theta = \tilde{\phi}_\theta(\xi_1) - C_B\theta\sin\xi, \qquad \omega = \tilde{\omega}(\xi_1) + C_B\theta\cos\xi$$

A similar analysis on the strain resultant vectors gives $\hat{u}_r \equiv 0$, $\hat{u}_\theta = C_B\theta r$ and $\hat{u}_z = C_T\theta$, or

$$\hat{u}_\xi = C_T\theta\sin\xi, \qquad \hat{u}_\theta = C_B\theta r, \qquad \hat{w} = C_T\theta\cos\xi,$$

where C_T is another constant of integration, and therewith

$$u_\xi = \tilde{u}_\xi(\xi_1) + C_T\theta\sin\xi, \qquad u_\theta = \tilde{u}_\theta(\xi_1) + C_B\theta r, \qquad w = \tilde{w}(\xi_1) + C_T\theta\cos\xi$$

Three of the corresponding components of strain resultant vectors and three of the corresponding components of strain couple vectors are just those given in section (2) of Chapter 2 with u_ξ, w, u_θ, ϕ_ξ, ϕ_θ and ω replaced by the corresponding $\tilde{u}_\xi$, $\tilde{w}$, etc. The remaining components of strain resultants and couples now take the form

$$\varepsilon_{\theta\theta} = \frac{1}{r}(\cos\xi\tilde{u}_\xi - \sin\xi\tilde{w}) + C_B, \qquad \varepsilon_{\theta\xi} = \frac{1}{r}(C_T\sin\xi - \cos\xi\tilde{u}_\theta) + \tilde{w}$$

$$\gamma_\theta = \tilde{\phi}_\theta + \frac{1}{r}(C_T\cos_\xi + \tilde{u}_\theta \sin\xi), \qquad \kappa_{\theta\theta} = \frac{1}{r}(-C_B\sin\xi + \tilde{\phi}_\xi\cos\xi)$$

$$\varepsilon_{\theta\xi} = -\frac{1}{r}(\tilde{\phi}_\theta\cos\xi + \tilde{\omega}\sin\xi) + C_B, \qquad \lambda_\theta = \frac{1}{r}(C_B\cos\xi + \tilde{\phi}_\xi\sin\xi)$$

With $\kappa_{\theta\theta}$ and λ_θ both expressed in terms of $\tilde{\phi}_\xi$, we have immediately the following linear relation between the two strain couple components

$$r(\kappa_{\theta\theta}\sin\xi - \lambda_\theta\cos\xi) = -C_B$$

Similarly, the expressions for $\varepsilon_{\theta\xi}$, γ_θ and $\kappa_{\theta\xi}$ may be combined to give a linear relation among the three quantities and C_T:

$$r[\varepsilon_{\theta\xi}\sin\xi + \gamma_\theta\cos\xi + r\kappa_{\theta\xi}] = C_T$$

These two relations are identical to the two first integrals of the compatibility equations obtained in the last section. From the nonperiodic displacement fields

$$\hat{\phi}_z = C_B\theta, \quad \hat{u}_\theta = C_B r\theta, \quad \hat{u}_z = C_T\theta$$

we see that $C_B 2\pi$ is a relative rotation about the axis of revolution between two sides of a radial slit, $\theta=0$ and $\theta=2\pi$, while $C_T 2\pi$ is a relative displacement in the direction of the axis of revolution between the two side of a radial slit. They characterize the only possible global incompatibilities or dislocations associated with axisymmetric strain measures.

4. Axisymmetric Stress Distributions for Shallow Shells of Revolution

1. Shallow Shell Approximations

The slope dz/dr of the meridional profile of a shell of revolution is related to the slope angle ξ (between the tangent at a point along the meridian and the radial line emanating from the same point parallel to the base plane, i.e., the (x,y) plane) by

$$\tan\xi = \frac{dz}{dr}$$

A shell of revolution is shallow if $(dz/dr)^2 \ll 1$ so that $1 + (dz/dr)^2 \approx 1$, $\tan\xi \approx \sin\xi \approx \xi \approx dz/dr$ and $\cos\xi \approx 1$.

It is evident that in-plane displacement components (in directions tangent to the shell's middle surface) are usually small compared to the out-of-plane displacement component (in direction normal to the shell's middle surface) for shell structures in general. A second feature characterizing a shallow shell stipulates that the in-plane displacement components are an order of magnitude smaller (by a factor z' for shells of revolution):

$$u_\xi,\ u_\theta = O(\xi w), \qquad \omega = O(\xi\phi_\xi, \xi\phi_\theta)$$

It follows from these magnitude relations that there is no distinction between the axial displacement component u_z and the normal component w as $u_z = \cos\xi w + \sin\xi u_\xi \simeq w[1+O(z'^2)] \simeq w$, etc. As we shall see, they also simplify the strain displacement relations in a qualitatively significant way.

In the remainder of this Chapter, we take $\xi_1 = r$ so that $\alpha^2 = (r')^2 + (z')^2 \approx 1$ for shallow shells. The strain-displacement relations are then simplified to read

$$\varepsilon_{\theta\theta} = \frac{1}{r}(u_\xi - z'w) + C_B, \qquad \varepsilon_{\xi\xi} = u_\xi' - z''w, \qquad \gamma_\xi = \phi_\xi + w'$$

$$\kappa_{\xi\xi} = \phi_\xi', \qquad \kappa_{\theta\theta} = \frac{1}{r}\,\phi_\xi, \qquad \lambda_\theta = \frac{1}{r}(z'\phi_\xi + C_B)$$

for axisymmetric bending, where now $(\;)' \equiv d(\;)/dr$, and

$$\varepsilon_{\theta\xi} = \frac{1}{r}(C_T z' - u_\theta) + \omega, \qquad \varepsilon_{\xi\theta} = u'_\theta - \omega, \qquad \gamma_\theta = \phi_\theta + \frac{1}{r}C_T$$

$$\kappa_{\theta\xi} = -\frac{1}{r}\phi_\theta \qquad \kappa_{\xi\theta} = \phi'_\theta \qquad \lambda_\xi = \omega' - z''\phi_\theta$$

for axisymmetric torsion and twisting.

To have a virtual work principle for a shallow shell theory, the equilibrium equations should be taken in the form

$$(rN_{\xi\xi})' - N_{\theta\theta} + rp_\xi = 0, \qquad (rQ_\xi)' + rz''N_{\xi\xi} + z'N_{\theta\theta} + rp_n = 0$$

$$(rM_{\xi\xi})' - M_{\theta\theta} - rQ_\xi - z'P_\theta + rq_\xi = 0$$

for axisymmetric bending, and

$$(rN_{\xi\theta})' + N_{\theta\xi} + rp_\theta = 0, \qquad (rM_{\xi\theta})' + M_{\theta\xi} + rz''P_\xi - rQ_\theta + rq_\theta = 0$$

$$(rP_\xi)' + r(N_{\xi\theta} - N_{\theta\xi}) + rq_n = 0$$

for axisymmetric torsion and twisting. It is not difficult to verify that there are six differential equations of compatibility which are the static geometric duals of these equilibrium equations.

Also, the stress boundary conditions along an $r = r^*$ edge consistent with the virtual work principle are

$$r = r^*: \quad \begin{cases} \Delta N_{\xi\xi} = \Delta Q_\xi = \Delta M_\xi = 0 & \text{(Axisymmetric Bending)} \\ \Delta N_{\xi\theta} = \Delta M_{\xi\theta} = \Delta P_\xi = 0 & \text{(Axisymmetric Torsion \& Twisting)} \end{cases}$$

The three conditions for the bending problem remains unchanged for the classical shell theory. The other three for the torsion problem (so that $C_T = 0$) reduce to a single condition $\Delta N_{\xi\theta} = 0$ (not $\Delta N^e_{\xi\theta} = 0$) for the classical theory as the contribution from u_θ to γ_θ is neglected in the shallow shell theory. The appropriate displacement boundary conditions in terms of strain measures may be inferred from these stress boundary conditions by way of the static geometric duality.

2. Reduction of the Axisymmetric Bending Problem

A first integral of the force equilibrium equations may be obtained either directly from the combination

$$z'[(rN_{\xi\xi})' - N_{\theta\theta} + rp_\xi] + [(rQ_\xi)' + rz''N_{\xi\xi} + z'N_{\theta\theta} + rp_n] = 0$$

or by specializing the first integral of the nonshallow shell case. By either approach, we have for a shell which spans the interval $r_i < r < r_o$

$$r(z'N_{\xi\xi} + Q_\xi) + \int_{r_i}^{r} p_v r dr = \frac{F_z}{2\pi} \qquad (r_i<r<r_o)$$

where $p_v \equiv z'p_\xi + p_n$. The first integral will be used instead of the differential equation of force equilibrium in the direction of the midsurface normal and will be considered as a relation which expresses Q_ξ in terms of $N_{\xi\xi}$. We write this relation in the form

$$Q_\xi = -\frac{z'}{r}\Phi_B + \frac{1}{r}(rV) \tag{4.1}$$

with

$$N_{\xi\xi} = \frac{1}{r}\Phi_B, \qquad rV = \frac{F_z}{2\pi} - \int_{r_i}^{r} p_v\, rdr \tag{4.2}$$

The differential equation of tangential force equilibrium can be used to express $N_{\theta\theta}$ also in terms of $N_{\xi\xi}$ or Φ_B:

$$N_{\theta\theta} = \Phi_B' + rp_\xi \tag{4.3}$$

Note that (4.1) - (4.3) are the static-geometric duals of the strain-displacement relations for shallow shells of the last section:

$$\lambda_\theta = -\frac{z'}{r}\phi_B + \frac{1}{r}C_B, \qquad \kappa_{\theta\theta} = -\frac{1}{r}\phi_B \qquad \kappa_{\xi\xi} = -\phi_B' \tag{4.4,5,6}$$

where we have set $\phi_B = -\phi_\xi$ so that ϕ_B is the dual of Φ_B. These expressions for strain couples may be used in the stress strain relations to give

$$M_{\xi\xi} = -D(\phi_B' + \frac{\nu_b}{r}\phi_B), \quad M_{\theta\theta} = -D(\nu_b\phi_B' + \frac{1}{r}\phi_B), \quad P_\theta = -\frac{D_P}{r}(z'\phi_B + C_B) \qquad (4.7\text{-}9)$$

The remaining (moment) equilibrium equation which has not been used up to this point may now be written in terms of ϕ_B and Φ_B. For shells with uniform thickness and material properties, this equation takes the form

$$\phi_B'' + \frac{1}{r}\phi_B' - \frac{1}{r^2}[1 + \frac{D_P}{D}(z')^2]\phi_B - \frac{z'}{rD}\Phi_B = \frac{1}{D}[q_\xi - \frac{1}{r}(rV) - D_P C_B \frac{z'}{r^2}] \qquad (4.10)$$

It is a straightforward calculation to obtain the dual compatibility equation in the form*

$$\Phi_B'' + \frac{1}{r}\Phi_B' - \frac{1}{r^2}[1 + \frac{A_Q}{A}(z')^2]\Phi_B - \frac{z'}{rA}\phi_B = -[rp_\xi' + (2+\nu_s)p_\xi] + \frac{1}{A}[\frac{1}{r}CB - A_Q(rV)\frac{z'}{r^2}] \,. \qquad (4.11)$$

The terms associated with D_P and A_Q on the left side of (4.10) and (4.11) should be deleted whenever D_P/D and A_Q/A are not large compared to unity as we have $(z')^2 \ll 1$.

Supplemented by suitable boundary conditions, the two coupled second order ordinary differential equations (ODE) determine ϕ_B and Φ_B. The stress resultants and couples for the axisymmetric bending problem are then obtained from ϕ_B and Φ_B by way of the auxiliary formulas (4.1)-(4.3) and (4.7)-(4.9). The strain measures are obtained from the dual auxiliary formulas.

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*p_ξ in (4.11) and (4.3) is sometimes replaced by the radial load component $p_H \equiv p_\xi - z'p_n$ as in the context of shallow shell theory, the radial stress resultant $N_{\xi\xi} - z'Q_\xi$ is indistinguishable from the tangential resultant $N_{\xi\xi}$ (but p_H is not equal to p_ξ).

For the three relevant displacement components, we note that $\phi_\xi = -\phi_B$ and

$$w' = \gamma_\xi - \phi_\xi = A_Q Q_\xi + \phi_B = \phi_B + \frac{1}{r} A_Q(rV - z'\Phi_B)$$

so that

$$w = w_0 + \int_{r_i}^{r} [\phi_B + \frac{1}{r} A_Q(rV - z'\Phi_B)]dr$$

Finally we have from the expression for $\varepsilon_{\theta\theta}$ that

$$u_\xi = r(\varepsilon_{\theta\theta} - C_B) + z'w = A(r\Phi_B' + r^2 p_\xi - \nu_s\Phi_B) - C_B r + z'w$$

It is of some interest to note that, consistent with the shallow shell approximation, $u_\xi - z'w$ is the radial component of the displacement vector and may be obtained directly from $u_r = r(\varepsilon_{\theta\theta} - C_B)$ without first having to compute w.

3. The Classical Theory for the Bending Problem

With $A_Q = D_P = 0$, the two simultaneous equations formulations of the axissymmetric bending problem simplifies somewhat but in a qualitatively significant way. The ODE's for Φ_B and ϕ_B now read:

$$\phi_B'' + \frac{1}{r}\phi_B' - \frac{1}{r^2}\phi_B - \frac{z'}{rD}\Phi_B = \frac{1}{Dr}[rq_\xi - (rV)]$$

$$(r_i < r < r_o)$$

$$\Phi_B'' + \frac{1}{r}\Phi_B' - \frac{1}{r^2}\Phi_B + \frac{z'}{rA}\phi_B = \frac{1}{Ar}C_B - [rp_H + (2+\nu_s)p_H]$$

The auxiliary equations for stress and strain measures become

$$N_{\xi\xi} = \frac{1}{r}\Phi_B, \qquad N_{\theta\theta} = \Phi_B' + rp_H, \qquad Q_\xi = \frac{1}{r}(rV - z'\Phi_B)$$

$$\kappa_{\theta\theta} = \frac{1}{r}\phi_B, \qquad \kappa_{\xi\xi} = -\phi_B', \qquad \lambda_\theta = -\frac{1}{r}(z'\phi_B + C_B)$$

The remaining stress and strain measures are obtained from the stress strain relations

$$\varepsilon_{\theta\theta} = A(\Phi_B' - \frac{\nu_s}{r}\Phi_B + rp_H), \qquad \varepsilon_{\xi\xi} = A(-\nu_s\Phi_B' + \frac{1}{r}\Phi_B - \nu_s rp_H)$$

$$M_{\xi\xi} = -D(\phi_B' + \frac{\nu_b}{r}\phi_B), \qquad M_{\theta\theta} = -D(\nu_b\phi_B' + \frac{1}{r}\phi_B)$$

The radial and axial displacement components are calculated from

$$u_r = r(\varepsilon_{\theta\theta} - C_B) = A(r\Phi_B' - \nu_s\Phi_B + r^2p_H) - C_B r$$

$$u_z \simeq w = w_o - \int_{r_i}^{r}\phi_\xi(t)dt = w_o + \int_{r_i}^{r}\phi_B(t)dt$$

In writing down the above equations for the clasical theory of shallow shells, we have adopted the conventional practice of replacing p_ξ by p_H and interpreting Φ_B/r as the radial resultant.

The seemingly trivial simplifications attained by specialization to the classical shell theory actually have some important consequences. If we multiply the second by $i\sqrt{A/D}$, and then add the resulting equation to the first, we effectively combine the two ODEs for ϕ_B and Φ_B into a single complex ODE of the same order for a complex function $\chi \equiv \phi_B + i\sqrt{A/D}\Phi_B$:

$$\chi'' + \frac{1}{r}\chi' - \frac{1}{r^2}\chi + \frac{iz'}{r\sqrt{DA}}\chi$$
$$= \frac{1}{rD}[rq_\xi - (rV)] - i\sqrt{\frac{A}{D}}[rp_H' + (2+\nu_s)p_H + \frac{C_B}{rA}]$$

There is considerable advantage working with a single second order ODE for χ instead of a fourth order system for ϕ_B and Φ_B. For example, the homogenous equation for χ has solutions in terms of Bessel functions if $z' = \xi_o(r/r_o)^n$ (where n may not be as integer); standard techniques such as WKB and Langer's method may be applied whenever $|z'/r\sqrt{DA}| \gg 1$, etc. In fact, for $n \neq -1$, we have with $x = r/r_o$,

$$\chi = C_1 J_\gamma(\sqrt{i}y) + C_2 J_{-\gamma}(\sqrt{i}y), \quad y = \gamma\beta x^{1/\gamma}$$

where $\beta^2 = \xi_o r_o/\sqrt{DA} = O(\xi_o r_o/h) \gg 1$ and $\gamma = 2/(n+1)$. If γ is an integer, then $J_{-\gamma}$ should be replaced by Y_γ or K_γ. $J_p(\sqrt{i}y)$ is more conveniently written in terms of Kelvin functions, $J_p(\sqrt{i}y) = \mathrm{ber}_p(y) - i\,\mathrm{bei}_p(y)$, with

$$\sqrt{2}\,\mathrm{ber}_{p+1}(y) = [\mathrm{ber}_p'(y) - \mathrm{bei}_p'(y)] - \frac{p}{y}[\mathrm{ber}_p(y) - \mathrm{bei}_p(y)]$$

$$\sqrt{2}\,\mathrm{bei}_{p+1}(y) = [\mathrm{ber}_p'(y) + \mathrm{bei}_p'(y)] - \frac{p}{y}[\mathrm{ber}_p(y) + \mathrm{bei}_p(y)]$$

and (for $y \gg 1$)

$$\{\mathrm{ber}_p(y),\ \mathrm{bei}_p(y)\} \sim \frac{e^{y/\sqrt{2}}}{\sqrt{2\pi y}}\{\cos\phi_p,\ \sin\phi_p\}, \qquad \phi_p = \frac{y}{\sqrt{2}} - \frac{\pi}{8} + \frac{p\pi}{2}$$

along with similar formulas for the Kelvin functions of the second kind ker_p and kei_p with $K_p(y/\sqrt{i})/i^p = \mathrm{ker}_p(y) - i\,\mathrm{kei}_p(y)$.

For n = -1, the equation for χ is equidimensional and we have for the complementary solution:

$$\chi = C_1 x^{\gamma_1} + C_2 x^{\gamma_2}$$

where γ_1 and γ_2 are the two roots of $\gamma^2 = 1 - \xi_o r_o i/\sqrt{DA}$. We will illustrate the solution process for specific problems in the next few sections.

4. A Shallow Spherical Cap Under Its Own Weight

Consider a shallow spherical cap with $z' = \xi_0(r/r_0) \equiv \xi_0 x$ which is closed at the apex so that $0 \leqslant r \leqslant r_0$. The cap is placed in its face-up position so that the gravity vector is downward ($-g\vec{i}_z$). We are interested here in the elastostatics of the cap under its own weight with the shell "clamped" to a rigid support at its only edge $r = r_0$, i.e., the edge is constrained from displacement and rotation. We seek the solution for this shell problem within the framework of the classical shell theory so that the clamped edge conditions take the form

$$r = r_0: \qquad w = \phi_\xi = u_r = u_\theta = 0$$

The last condition is automatically satisfied in the theory of axisymmetric bending after we set $C_B = 0$ for shells complete in the circumferential direction.

With $\vec{p} = -\rho h g \vec{i}_z$ for our face-up spherical cap, where ρ is the mass density of the cap material, h is the shell thickness and g is the gravitational acceleration, we have $p_V = -\rho h g$, $p_H \equiv 0$* and $q_\xi \equiv 0$ so that

$$rV = \frac{1}{2} \rho g h r^2$$

where $F_z = 0$ as there is no resultant axial force applied at the apex $r = 0$. The ODE for χ now takes the form

$$\chi'' + \frac{1}{r}\chi' - \frac{1}{r^2}\chi + i\frac{\beta^2}{r_0^2}\chi = -\frac{1}{2D}\rho g h r,$$

---

*Readers should verify that there would be an extra term of order $\xi_0^{\,2}/\beta^2$ on the right hand side of the ODE for χ if p_ξ had been used instead of p_H.

with $\beta^2 = \xi_0 r_0/\sqrt{DA}$. An exact particular solution of the inhomogeneous equation is

$$\chi_\rho = i\,\frac{\rho g r_0^2}{2\xi_0}\sqrt{\frac{A}{D}}\,x$$

With the exact complementary solution given in terms of Kelvin functions, we have as the exact solution for χ.

$$\chi = C_1[\mathrm{ber}_1(\beta x) - i\,\mathrm{bei}_1(\beta x)] + C_2[\mathrm{ker}_1(\beta x) - i\,\mathrm{kei}_1(\beta x)] + \frac{i r_0^2 \rho h g}{2\xi_0}\sqrt{\frac{A}{D}}\,x$$

where C_1 and C_2 are two complex constants of integration. For the stresses and displacements of the shell to be finite at the apex, we must have

$$C_2 = 0$$

leaving us with

$$\chi = (C_r + iC_i)[\mathrm{ber}_1\,(\beta x) - i\,\mathrm{bei}_1(\beta x)] + \frac{r_0^2 \rho h g}{2\xi_0}\sqrt{\frac{A}{D}}\,x,$$

where we have set $C_1 = C_r + iC_i$, or

$$\phi_B = C_r \mathrm{ber}_1(\beta x) + C_i \mathrm{bei}_1(\beta x)$$

$$\sqrt{\frac{A}{D}}\,\Phi_B = C_i \mathrm{ber}_1(\beta x) - C_r \mathrm{bei}_1(\beta x) + \sqrt{\frac{A}{D}}\,\frac{r_0^2 \rho h g}{2\xi_0}\,x$$

The two real constants C_i and C_r are determined by the clamped edge conditions

$$r = r_0: \qquad \phi_B = r\varepsilon_{\theta\theta} = 0$$

where we have made use of the relation $u_r = r\varepsilon_{\theta\theta}$ to get the condition of no radial displacement at the edge in terms of the stress function Φ_B. These two conditions give the following two simultaneous equations for C_r and C_i:

$$C_r \text{ber}_1(\beta) + C_i \text{bei}_1(\beta) = 0$$

$$C_i[\beta \text{ber}_1^{\bullet}(\beta) - \nu_s \text{ber}_1(\beta)] - C_r[\beta \text{bei}_1^{\bullet}(\beta) - \nu_s \text{bei}_1(\beta)] = -\sqrt{\frac{A}{D}}\ \frac{r_0^2 \rho h g}{2\xi_0}\ (1-\nu_s)$$

The solution of this system is

$$\{C_r,\ C_i\} = \sqrt{\frac{A}{D}}\frac{r_0^2 \rho h g(1-\nu_s)}{2\xi_0 \beta\Delta} \qquad \{\text{bei}_1(\beta),\ -\ \text{ber}_1(\beta)\}$$

$$\Delta = \ \text{ber}_1(\beta)\text{ber}_1^{\bullet}(\beta) + \text{bei}_1^{\bullet}(\beta)\text{bei}_1^{\bullet}(\beta)] - \frac{\nu_s}{\beta}\,[\text{ber}_1^2(\beta) + \text{bei}_1^2(\beta)]$$

where a dot indicates differentiation with respect to the argument of the function. We may integrate ϕ_B to get w if we wish, with $w = 0$ at $r = r_0$.

With $\beta >> 1$ for the cap to behave as a shell, we see from the asymptotic behavior of the Kelvin functions that the complementary solution terms $\text{ber}_1(\beta x)$ and $\text{bei}_1(\beta x)$ are significant only near the edge $r = r_0$ and that their contribution is $O(\beta^{-1})$ in Φ_B even at (and near) the edge. Thus, the stress resultants $N_{\theta\theta}$ and $N_{\xi\xi}$ are both of the order $\rho h g r_0/2\xi_.$ throughout the shell with the associated direct stress magnitude given by

$$\sigma_D \equiv O(\frac{N_{\theta\theta}}{h},\ \frac{N_{\xi\xi}}{h}) = O(\frac{\rho g r_0}{\xi_0})$$

On the other hand, the stress couples $M_{\xi\xi}$ and $M_{\theta\theta}$ are both of order $\sqrt{DA}\ r_0\rho hg/\xi_0$ near the edge $r = r_0$ and are insignificantly small away from the edge. The associated <u>bending stress</u> magnitude is given by

$$\sigma_B \equiv O(\frac{M_{\xi\xi}}{h^2},\ \frac{M_{\theta\theta}}{h^2}) = O(\frac{\rho g r_0}{\xi_0})$$

near the edge and again negligibly small away from $r = r_0$. It follows that the bending stresses are significant only near the shell edge while the direct stresses are significant throughout the shell.

It is of interest to note that the particular solution of the ODE for χ may also be obtained by setting D = 0 in (4.10). Such a solution is called the membrane solution as it corresponds to the limiting case of a shell having no bending stiffness. It is fortuitous that for the present problem, the membrane solution turns out to be an exact particular solution of the shell problem.

5. Regular Perturbation Solution

For meridional profiles which do not admit an exact solution for the ODE for χ, we may take advantage of the fact that $\beta^2 \equiv r_o\xi_o/\sqrt{DA}$ (where $\xi_o = z'(r_o)$) is large compared to unity and seek an asymptotic solution for the boundary value problem. To obtain an asymptotic solution, we set $x = r/r_o$ and write the ODE for χ (with $q_\xi \equiv 0$ for simplicity) as

$$\ddot{\chi} + \frac{1}{x}\dot{\chi} - \frac{1}{x^2}\chi + i\beta^2\tau(x)\chi$$

$$= -\beta^2\{\frac{e_v}{x} f_v(x) + i[\frac{e_H}{\beta^2} f_H(x) + \frac{C_B}{\xi_o x}]\}$$

where a dot indicates differentiation with respect to x, and where

$$\beta^2\tau(x) = \frac{r_o^2 z'}{r\sqrt{DA}}, \qquad p_V \equiv \bar{p}_V g_V(x), \qquad p_H \equiv \bar{p}_H g_H(x)$$

$$e_v = \frac{r_o^2\bar{p}_V}{\xi_o}\sqrt{\frac{A}{D}}, \qquad e_H = r_o^2 p_H\sqrt{\frac{A}{D}}, \qquad f_H(x) = (xg_H)^{\bullet} + (2+\nu_s)g_H$$

$$f_v(x) = -\frac{1}{x}\int^x g_v(t)t\,dt$$

with $\bar{p}_H$ and $\bar{p}_V$ chosen so that $|g_H(x)| \leq 1$ and $|g_V(x)| \leq 1$.

With $\beta^2 >> 1$, we may seek a parametric series solution in powers of β^{-2}:

$$\chi \sim \sum_{m=0}^{\infty} \chi_m(x)\beta^{-2m} \equiv \chi_p(x)$$

If e_V, e_H and C_B/ξ_o are all $O(1)$ in magnitude, then we have from the ODE for χ

$$\chi_o = \frac{1}{\tau(x)} \{i\frac{e_V}{x} f_V(x) - \frac{C_B}{\xi_o x}\}$$

$$\chi_1 = \frac{i}{\tau(x)} L[\chi_o] - \frac{e_H}{\tau(x)} f_H(x)$$

$$\chi_{n+1} = \frac{i}{\tau(x)} L[\chi_n] \qquad (n = 1,2,...)$$

where the linear differential operator L is defined by

$$L[\ ] \equiv [\ ]^{\cdot\cdot} + \frac{1}{x}[\ ]^{\cdot} - \frac{1}{x^2}[\ ]$$

It follows that

$$\phi_B \sim \frac{C_B}{\xi_o x \tau(x)} - \frac{1}{\beta^2 \tau}\{e_H f_H + e_V L[\frac{f_V}{x\tau}]\} + O(\beta^{-4}\}$$

$$\Phi_B \sim \frac{r_o^2 \bar{p}_V}{\xi_o}[\frac{f_V}{x\tau}] - \frac{1}{\beta^2\tau} L[\frac{C_B}{\xi_o x\tau}] + O(\beta^{-4})$$

Note that successive terms in the perturbation series are determined algebraically and that the solution process gives no constant of integration. Given that the structure of the ODE for χ is of the singular perturbation type, the parametric series solution obtained above constitutes only a particular solution of the ODE (or only the outer solution in the method of matched asymptotic expansions). The general solution for χ will be obtained by seeking a WKB type solution for the homogeneous ODE for χ in the next section.

Before leaving the topic of a regular perturbation solution, it should be noted that the leading term for ϕ_B and Φ_B obtained above correspond to the limiting case of a membrane and inextensional bending solution of the shell problem. As in the case of the shallow spherical cap under its own dead weight (in the face-up position), the leading term solution for Φ_B can be obtained by setting $D = 0$ (corresponding to a shell with no bending stiffness and hence a membrane) in the ODE obtained from moment equilibrium (with $q_\xi \equiv 0$). On the other hand, the leading term solution for Φ_B can be obtained by setting $A = 0$ (corresponding to a shell with no stretching compliance and hence inextensible) in the ODE associated with strain compatibility. In either case, we have $\beta^2 = \sqrt{DA}/\xi_0 r_0 = 0$ so that the parametric series solution for X reduces only to its leading term. The two limiting solutions corresponding to membrane and inextensional bending approximation play an important role in simple approximate solutions for many shell problems. For example, the membrane solution in the case of a spherical cap under its own dead weight constitutes an accurate approximation of the exact solution except in a narrow region adjacent to the shell edge. As we shall see later, similar situations arise in many other problems including problems involving finite deformation.

6. WKB Solution, Langer's Method and Turning Point Problems

For a shell frustum with $r_i > 0$, we may seek the complementary solution of the ODE for X by the WKB method. It is customary to first transform the ODE into Liouville normal form for such a solution. For the shallow shell problem, a change of the dependent variable $\bar{X} = \sqrt{x}X$ transform the dimensionless homogeneous ODE for X in section (5) into

$$\bar{X}^{\cdot\cdot} - \frac{3}{4x^2}\bar{X} + i\beta^2\tau(x)\bar{X} = 0.$$

We seek an asymptotic (complementary) solution of the above ODE in the form

$$\bar{X} \sim e^{\beta\zeta(x)}\sum_{n=0}^{\infty}\bar{X}_n(x)\beta^{-n} \equiv \bar{X}_c$$

Upon substituting the series solution into the ODE and collecting terms of the same powers of β, we get

$$\beta^2[(\zeta')^2 + i\tau]\bar{X}_0 + \beta\{[(\zeta')^2 + i\tau]\bar{X}_1 + 2\zeta'\bar{X}_0^{\cdot}\}$$

$$+ \{[(\zeta')^2 + i\tau]\bar{X}_2 + 2\zeta'\bar{X}_1^{\cdot} - \bar{X}_0^{\cdot\cdot} - \frac{3}{4x^2}\bar{X}_0\} + O(\beta^{-1}) = 0 .$$

For this equation to be satisfied identically in β, we must have

$$(\zeta')^2 + i\tau = 0 \qquad \text{or} \qquad \xi' = \pm\sqrt{-i\tau}$$

$$2\zeta'\bar{X}_0^{\cdot} = 0 \qquad\qquad \bar{X}_0(x) = C_0$$

$$2\zeta'\bar{X}_1^{\cdot} - \frac{3}{4x^2}\bar{X}_0 = 0 \qquad\qquad \bar{X}_1^{\cdot} = \frac{3C_0}{8x^2\zeta'}$$

etc. For each of the two roots of ζ', we have a different solution for $\bar{X}$.

We take a linear combination of these complementary solutions for $\chi = \bar{\chi}/\sqrt{x}$ in the form:

$$\chi \sim \chi_c \equiv \frac{1}{\sqrt{x}} \{e^{\beta\eta(x)}[C_1\cos\beta\eta + C_2\sin\beta\eta] + e^{-\beta\eta(x)}[C_3\cos\beta\eta - C_4\sin\beta\eta]\}$$

$$+ \frac{i}{\sqrt{x}} \{e^{\beta\eta(x)}[C_2\cos\beta\eta - C_1\sin\beta\eta] + e^{-\beta\eta(x)}[C_4\cos\beta\eta - C_3\sin\beta\eta]\}$$

with

$$\eta(x) = \int^x \sqrt{\tau(x)/2}\, dt$$

where C_1, C_2, C_3 and C_4 are four real constants of integration. They are to be determined by the boundary conditions at $r = r_i$ and $r = r_o$ applied to $\chi \sim \chi_p + \chi_c$

For a shell closed at the apex, we have $r_i = 0$ and the WKB solution obtained above would not be appropriate near $x = 0$. The singularity at $x = 0$ of the ODE for χ makes a nonnegligible contribution to the solution (at least in a neighborhood of the apex). Its effect must be included in the leading term solution by Langer's method. To do so, we change both the dependent and independent variable and transform the homogeneous ODE for χ into

$$\frac{d^2\tilde{\chi}}{dy^2} + \frac{1}{y}\frac{d\tilde{\chi}}{dy} - [\frac{1}{y^2} - G(y)]\tilde{\chi} + i\beta^2\,\tilde{\chi} = 0,$$

where $$\tilde{\chi}(y) = \psi(x)\chi(x)$$

$$G(y) = \frac{1}{\tau(x)}\,[\psi(\psi^{-1})^{\cdot\cdot} + \frac{\psi}{x}(\psi^{-1})^{\cdot} + \frac{1}{y^2}\{1 - \frac{1}{\psi^4(x)}\}]$$

is $O(1/y)$ as $y \to 0$ if $\tau(x)$ is bounded away from zero, by choosing

$$\dot{y} = \sqrt{\tau}(x) \qquad\qquad \psi(x) = \sqrt{x\dot{y}(x)/y(x)}$$

with $\psi(0) = 1$. The term multiplied by G(y) is small compared to $i\beta^2\tilde{\chi}$ away from $y = 0$ and is small compared to y^{-2} near $y = 0$ and hence may be neglected in a leading term asymptotic solution. It follows that the leading term asymptotic solution uniformly valid for $0 \leq x \leq 1$ is

$$\tilde{\chi} \sim \tilde{\chi}_c \equiv C_1 J_1(\sqrt{i}\beta y) + C_2 Y_1(\sqrt{i}\beta y)$$

provided that $\tau(x)$ does not vanish throughout $0 \leq x \leq 1$. The two complex constants of integration are to be determined by the appropriate boundary conditions for the problem applied to $\chi \sim \chi_p + \chi_c$.

If $\tau(x)$ vanishes in the interval $x_i < x < 1$ we have a turning point problem in ODE and a different type of asymptotic solution is appropriate. For example, we have $z'(r) = \xi_0(x-x_a)$ $(x_i \leq x \leq 1$ and $x_i < x_a < 1)$ for a toroidal cap so that $\tau(x) = x-x_a$ vanishes at $x = x_a$. To obtain an appropriate asymptotic solution for $\beta^2 >> 1$, we again change both the independent and dependent variable in the ODE for χ. Since $x > 0$ for the present problem, we take $\hat{\chi} = \psi\chi$ and $y = y(x)$ but now with $\dot{y}^2 y = \tau$ and $\psi^2 = x\dot{y}$ so that the (homogeneous) ODE for χ becomes

$$\frac{d^2\hat{\chi}}{dy^2} + [i\beta^2 y + \hat{G}(y)]\hat{\chi} = 0$$

where

$$\hat{G}(y) = \frac{\tau}{y}\{\psi(x)[\psi^{-1}]^{\cdot\cdot} + \frac{\psi}{x}[\psi^{-1}]^{\cdot} - \frac{1}{x^2}\}$$

It can be shown that $\hat{G}(y)$ is a bounded for $x_i \leq x \leq 1$ so that the term involving $\hat{G}$ may be neglected for a leading term asymptotic solution. In that case, we have

$$\hat{\chi} \sim \chi_c \equiv C_1 A_i(i^{3/2}y) + C_2 B_i(i^{3/2}y)$$

where C_1 and C_2 are two complex constants of integration and A_i and B_i are the Airy function of the first and second kind, respectively. The two complex constants (or four real constants) are determined by appropriate boundary conditions for the problem applied to the general (asymptotic) solution $X \sim X_c + X_p$.

7. Reduction of Axisymmetric Torsion Problem

As in the nonshallow shell case, we expect a first integral for the equilibrium equations. We may get this by specializing the nonshallow shell result or obtaining directly from the shallow shell equilibrium equations. In either case, we get

$$rP_\xi + r^2 N_{\xi\theta} = T(r) \equiv T_z - \int_{r_i}^{r} (r^2 p_\theta + r q_n)\,dr$$

where T_z is a constant of integration, the resultant axial torque at the inner edge $r = r_i$. By static geometric duality, we have the following first integral of the compatibility equations:

$$r\gamma_\theta + r^2 \kappa_{\theta\xi} = C_T$$

With the two first integrals, the reduction of equations for axisymmetric torsion is particularly simple for the classical theory. For that case, we have

$$P_\xi \equiv 0, \qquad \gamma_\theta \equiv 0$$

so that the first integrals give

$$N_{\xi\theta} = \frac{T(r)}{r^2}, \qquad \kappa_{\theta\xi} = \frac{C_T}{r^2} .$$

The differential equation of force equilibrium and its dual then give

$$N_{\theta\xi} = -rp_\theta - (rN_{\xi\theta})' = -\,rp_\theta - \left(\frac{T}{r}\right)'$$

$$\kappa_{\xi\theta} = -(r\kappa_{\theta\xi})' = -\,\left(\frac{C_T}{r}\right)'$$

The stress strain relations then give $M_{\xi\theta} = M_{\theta\xi}$ and $\varepsilon_{\xi\theta} = \varepsilon_{\theta\xi}$:

$$M_{\xi\theta} = M_{\theta\xi} = \frac{1}{2} D(1-\nu_b)(\kappa_{\xi\theta} + \kappa_{\theta\xi}), \qquad \varepsilon_{\xi\theta} = \varepsilon_{\theta\xi} = \frac{1}{2} A(1+\nu_s)(N_{\xi\theta} + N_{\theta\xi})$$

The remaining stress measure Q_θ and its dual λ_ξ are given by the moment equilibrium equation in the $\vec{t}_\xi$ direction and its dual compatibility equation:

$$rQ_\theta = (rM_{\xi\theta})' + M_{\theta\xi} + rq_\theta = \frac{1}{r}[(r^2M_{\xi\theta})' + r^2q_\theta]$$

$$r\lambda_\xi = (r\varepsilon_{\theta\xi})' + \varepsilon_{\xi\theta} = \frac{1}{r}(r^2\varepsilon_{\theta\xi})'$$

It remains to determine the displacement components. From the strain displacement relations, we have immediately

$$\phi_\theta = - r\kappa_{\theta\xi} = - \frac{C_T}{r}, \qquad (\frac{u_\theta}{r})' = \frac{1}{r}(\varepsilon_{\xi\theta} + \varepsilon_{\theta\xi}) - C_T \frac{z'}{r^2}$$

$$2\omega = \frac{1}{r}[(ru_\theta)' - C_T z']$$

It should be noted that the stress and strain measures are all determined by algebraic relations (not differential equations). The only stress boundary condition at an edge in the classical theory must be satisfied by the only constant integration from the first integral which gives the axial torque. It is evident that we cannot have two arbitrarily prescribed conditions on $N_{\xi\theta}$ at two different edge as the shell must be in overall equilibrium. Also, the constant C_T is associated with nonperiodic circumferential displacement and should set equal to zero if the shell is complete in the circumferential direction.

For the general theory with $D_P \neq 0$ and $A_Q \neq 0$, we let

$$rN_{\xi\theta} = \Phi_T, \qquad r\kappa_{\theta\xi} = \phi_T$$

and, as before, we have immediately

$$N_{\theta\xi} = -\Phi_T' - rp_\theta, \qquad \kappa_{\xi\theta} = -\phi_T'$$

From the first integrals, we have also

$$P_\xi = -\Phi_T + \frac{T}{r}, \qquad \gamma_\theta = -\phi_T + \frac{C_T}{r}$$

The stress strain relations then give

$$M_{\xi\theta} = M_{\theta\xi} = \frac{1}{2} D(1-\nu_b)(\kappa_{\xi\theta} + \kappa_{\theta\xi}) = -\frac{1}{2} D(1-\nu_b)(\phi_T' - \frac{1}{r}\phi_T)$$

$$\varepsilon_{\xi\theta} = \varepsilon_{\theta\xi} = \frac{1}{2} A(1+\nu_s)(N_{\xi\theta} + N_{\theta\xi}) = -\frac{1}{2} A(1+\nu_s)(\Phi_T' - \frac{1}{r}\Phi_T + rp_\theta)$$

$$Q_\theta = \frac{1}{A_Q}(-\phi_T + \frac{C_T}{r}), \qquad \lambda_\xi = \frac{1}{D_P}(-\Phi_T + \frac{T}{r})$$

The moment equilibrium equation

$$(rM_{\xi\theta})' + M_{\theta\xi} + rz''P_\xi - rQ_\theta + rq_\theta = 0$$

and the dual compatibility equation

$$(r\varepsilon_{\theta\xi})' + \varepsilon_{\xi\theta} - rz''\gamma_\theta - r\lambda_\xi = 0$$

then give two simultaneous second order ODEs for Φ_T and ϕ_T. With the constant of integration in T(r), the solution of the two ODEs can satisfy the three stress boundary conditions $\Delta P_\xi = \Delta N_{\xi\theta} = \Delta M_{\xi\theta} = 0$ at one edge but only two more stress conditions at a second edge as the shell must be in overall equilibrium. The sixth condition for a shell with two edges can be prescribed in terms of the circumferential displacement $\Delta u_\theta = 0$ with no restriction.

5. Reduction for Axisymmetric Problems of Nonshallow Shells

1. Reduction of the Axisymmetric Bending Problem

The reduction is similar to the shallow shell case. We begin with the first integral expressing the axial stress resultant

$$N_{\xi\xi}\sin\xi + Q_\xi\cos\xi = \frac{1}{r}(rV) \equiv \frac{1}{r}\left\{\frac{F_z}{2\pi} - \int_{\xi_i}^{\xi_1} Pv(t)r(t)\alpha(t)dt\right\}$$

and designate the radial resultant by Φ_B/r so that

$$N_{\xi\xi}\cos\xi + Q_\xi\sin\xi \equiv \frac{1}{r}\Phi_B$$

The corresponding dual relations are

$$-\kappa_{\theta\theta}\sin\xi + \lambda_\theta\cos\xi = \frac{C_B}{r}$$

$$-\kappa_{\theta\theta}\cos\xi + \lambda_\theta\sin\xi = \frac{1}{r}\,\phi_B$$

The four relations may be inverted to give

$$rN_{\xi\xi} = (rV)\sin\xi + \Phi_B\cos\xi, \qquad rQ_\xi = (rV)\cos\xi - \Phi_B\sin\xi$$

$$-r\kappa_{\theta\theta} = C_B\sin\xi + \phi_B\cos\xi, \qquad r\lambda_\theta = C_B\cos\xi - \phi_B\sin\xi$$

The differential equation of force equilibrium in the radial direction then gives

$$N_{\theta\theta} = \frac{1}{\alpha}\Phi_B' + rpH, \qquad (\ )' \equiv \frac{d(\ )}{d\xi_1}$$

and the dual compatibility equation gives

$$-\kappa_{\xi\xi} = \frac{1}{\alpha}\phi_B'$$

Next, we use the stress strain relations to express the remaining stress and strain measures in terms of ϕ_B and Φ_B:

$$M_{\xi\xi} = D(\kappa_{\xi\xi} + \nu_b\kappa_{\theta\theta}) = -D\{\frac{1}{\alpha}\phi_B' + \frac{\nu_b}{r}(\cos\xi\phi_B + \sin\xi C_B)\}$$

$$M_{\theta\theta} = D(\kappa_{\theta\theta} + \nu_b\kappa_{\theta\theta}) = -D\{\frac{\nu_b}{\alpha}\phi_B' + \frac{1}{r}(\cos\xi\phi_B + \sin\xi C_B)\}$$

$$P_\theta = D_p\lambda_\theta = \frac{D_p}{r}(\phi_B\sin\xi - C_B\cos\xi)$$

$$\varepsilon_{\theta\theta} = A(N_{\theta\theta} - \nu_s N_{\theta\theta}) = A\{\frac{1}{\alpha}\Phi_B' + rp_H - \frac{\nu_s}{r}[\cos\xi\Phi_B + \sin\xi(rV)]\}$$

$$\varepsilon_{\xi\xi} = A(N_{\xi\xi} - \nu_s N_{\theta\theta}) = A\{\frac{\nu_s}{\alpha}\Phi_B' - \nu_s rp_H + \frac{1}{r}[\cos\xi\Phi_B + \sin\xi(r_B V)]\}$$

$$\gamma_\xi = A_Q Q_\xi = \frac{A_Q}{r}[(rV)\cos\xi - \Phi_B\sin\xi]$$

The moment equilibrium equation which has not been used up to this point may now be written in terms of Φ_B and ϕ_B

$$\phi_B'' + \frac{(Dr/\alpha)'}{(Dr/\alpha)}\phi_B' - [(\frac{r'}{r})^2 + \frac{D_p}{D}(\frac{z'}{r})^2 - \frac{(\nu_b Dr'/\alpha)'}{(Dr/\alpha)}]\phi_B - \frac{z'}{(Dr/\alpha)}\Phi_B$$

$$= C_B\{\frac{r'z'}{r^2}(1 - \frac{D_p}{D}) - \frac{(\nu_b Dz'/\alpha)'}{(Dr/\alpha)}\} - \frac{r'(rV) - r\alpha q_\xi}{(Dr/\alpha)}$$

The dual compatibility equation may also be written in terms of ϕ_B and Φ_B:

$$\Phi_B'' + \frac{(Ar/\alpha)'}{(Ar/\alpha)} \Phi_B' - [(\frac{r'}{r})^2 + \frac{A_Q}{A}(\frac{z'}{r})^2 + \frac{(\nu_s Ar'/\alpha)'}{(Ar/\alpha)}]\Phi_B + \frac{z'}{(Ar/\alpha)} \phi_B$$

$$= (rV) \frac{r'z'}{r^2} (1 - \frac{A_Q}{A}) + \frac{[\nu_s Az'(rV)/\alpha]'}{(Ar/\alpha)} + \frac{C_B r'}{(Ar/\alpha)} - [\nu_s r'\alpha p_H + \frac{(Ar^2 p_H)'}{(Ar/\alpha)}]$$

For shells with uniform thickness and material properties, these two equations simplify slightly to read

$$\phi_B'' + \frac{(r/\alpha)'}{(r/\alpha)} \phi_B' - [(\frac{r'}{r})^2 + \frac{D_p}{D}(\frac{z'}{r})^2 - \nu_b \frac{(r'/\alpha)'}{(r/\alpha)}]\phi_B - \frac{z'}{(Dr/\alpha)} \Phi_B$$

$$= C_B\{\frac{r'z'}{r^2}(1 - \frac{D_p}{D}) - \nu_b \frac{(z'/\alpha)'}{(r/\alpha)}\} - \frac{r'(rV) - r\alpha q_\xi}{(Dr/\alpha)}$$

$$\Phi_B'' + \frac{(r/\alpha)'}{(r/\alpha)} \Phi_B' - [(\frac{r'}{r})^2 + \frac{A_Q}{A}(\frac{z'}{r})^2 + \nu_s \frac{(r'/\alpha)'}{(r/\alpha)}]\Phi_B + \frac{z'}{(Ar/\alpha)} \phi_B$$

$$= (rV) \frac{r'z'}{r^2}(1 - \frac{A_Q}{A}) + \nu_s \frac{[z'(rV)/\alpha]'}{(r/\alpha)} + \frac{C_B r'}{Ar/\alpha} - [\nu s r'\alpha p_H + \frac{(r^2 p_H)'}{r/\alpha}]$$

The fourth order system of ODEs and one additional constant of integration in the expression for rV allow us to satisfy at most five stress boundary conditions for any two constant ξ_1 edges as the shell must be in overall equilibrium.

One of the six appropriate boundary conditions for the shell frustum must be prescribed in terms of the axial displacement component. With

$$u_z' = (\sin\xi u_\xi + \cos\xi w)' = (z'\varepsilon_{\xi\xi} + r'\gamma_\xi) - r'\phi_\xi$$

$$= z'\varepsilon_{\xi\xi} + r'\gamma_\xi + r'\phi_B$$

where $\varepsilon_{\xi\xi}$ and γ_ξ are known in terms of Φ_B and ϕ_B, we have

$$u_z(\xi_1) = u_0 + \int_{\xi_i}^{\xi_1} (z'\varepsilon_\xi + r'\gamma_\xi + r'\phi_B)d\xi_1$$

The appropriate form of the stress and displacement boundary conditions along a given edge $\xi_1 = \xi_1^*$ can be found in section (4) of Chapter 2 of these notes.

Finally, the constant C_B is associated with a circumferentially nonperiodic displacement field (see Section (4) of Chapter (3)). It should be set to zero for shells complete in the circumferential direction.

2. Circular Cylindrical Shells

For a circular cylindrical shell with a midsurface radius r_o, the two simultaneous ODEs for Φ_B and ϕ_B simplifies considerably. For a shell of uniform thickness and material properties, they take the form

$$\phi_B'' - \frac{D_p}{D}\phi_B - \frac{r_o}{D}\Phi_B = \frac{r_o{}^2 q_\xi}{D}$$

$$\Phi_B'' - \frac{A_Q}{A}\Phi_B + \frac{r_o}{A}\phi_B = \nu_s(r_o V)' - r_o^2 p_H'$$

where we $\xi_1 = z/r_o$ so that $\alpha = r_o$, $\sin\xi = 1$ and $\cos\xi = 0$. The corresponding auxiliary equations for stress and strain measures become

$$r_o N_{\xi\xi} = (r_o V), \qquad r_o Q_\xi = -\Phi_B, \qquad N_{\theta\theta} = \frac{1}{r_o}\Phi_B' + r_o p_H$$

$$M_{\xi\xi} = -\frac{D}{r_o}\{\phi_B' + \nu_b C_B\}, \qquad M_{\theta\theta} = -\frac{D}{r_o}\{\nu_b\phi_B' + C_B\}, \qquad P_\theta = -\frac{D_p}{r_o}\phi_B$$

The two ODEs for ϕ_B and Φ_B are of constant coefficients; the complementary solutions are in terms of exponential functions. To motivate subsequent developments, we observe that the two ODEs may be combined into a single complex ODE of the same order

$$\chi'' - \left(\frac{D_p}{D} - \lambda\frac{r_o}{A}\right)\chi = \frac{r_o{}^2 q_\xi}{D} + \lambda[\nu_s(r_o V)' - r_o^2 p_H']$$

where λ is a root of the quadratic equation

$$\frac{r_o}{A}\lambda^2 + \left(\frac{A_Q}{A} - \frac{D_p}{D}\right)\lambda + \frac{r_o}{D} = 0$$

and $\chi = \phi_B + \lambda\Phi_B$. With $r_o^2/DA = O(r_o^2/h^2) >> 1$, λ is usually a complex number. In general, it is easier to work with a single second order complex ODE than with a fourth order system, especially in the case of equations with variable coefficients.

For the classical theory with $A_Q = D_P = 0$, we have $\lambda = \pm \sqrt{A/D}\, i$ and

$$\chi'' + i\bar{\beta}^2\chi = \frac{r_o^2 q_\xi}{D} - i\sqrt{\frac{A}{D}}\,[\nu_s r_o^2 p_V + r_o^2 p_H']$$

with $\chi = \phi_B + i\sqrt{A/D}\,\Phi_B$ and $\bar{\beta}^2 = r_o/\sqrt{DA}$. The complementary solution for χ may be taken in the form

$$\chi_c = [e^{\bar{\beta}\xi_1/\sqrt{2}}\{C_1 \cos\frac{\bar{\beta}\xi_1}{\sqrt{2}} + C_2 \sin\frac{\bar{\beta}\xi_1}{\sqrt{2}}\} + e^{\bar{\beta}\xi_1/\sqrt{2}}\{C_3 \cos\frac{\bar{\beta}\xi_1}{\sqrt{2}} - C_4 \sin\frac{\bar{\beta}\xi_1}{\sqrt{2}}\}]$$

$$+ [e^{\bar{\beta}\xi_1/\sqrt{2}}\{C_2 \cos\frac{\bar{\beta}\xi_1}{\sqrt{2}} - C_1 \sin\frac{\bar{\beta}\xi_1}{\sqrt{2}}\} + e^{\bar{\beta}\xi_1/\sqrt{2}}\{C_4 \cos\frac{\bar{\beta}\xi_1}{\sqrt{2}} + C_3 \sin\frac{\bar{\beta}\xi_1}{\sqrt{2}}\}] .$$

χ_C is significant only near an edge of the shell if $\bar{\beta}\ell/r_o = \ell/[DAr_o^2]^{1/4} \equiv \beta >> 1$. We now consider three specific problems within the framework of the classical theory:

I. Axial End Loads: The shell is subject to no surface loads so that $q_\xi = p_V = p_H = 0$. At the ends $z = 0$ and $z = \ell$, the edges experience no transverse shear resultants and no bending moments so that $M_{\xi\xi} = Q_\xi = 0$. The two ends are subject to equal and opposite axial stress resultants of magnitude N_o. For this problem we have

$$(rV) = \frac{F_z}{2\pi}, \qquad N_{\xi\xi} = \frac{(rV)}{r_o} = \frac{F_z}{2\pi r_o}$$

The ODE for χ is homogeneous; so are the four boundary conditions for Φ_B and ϕ_B (as we have $C_B = 0$ for a complete shell of revolution):

$$z = 0, \ell: \qquad \Phi_B = -\frac{D}{r_0}\phi_B' = 0$$

It follows that $\Phi_B \equiv \phi_B \equiv 0$ and therewith

$$M_{\xi\xi} = M_{\theta\theta} = Q_\xi = N_{\theta\theta} = 0 \qquad (0 < \xi_1 \leqslant \ell/r_0)$$

The boundary conditions on $N_{\xi\xi}$ at the two ends are satisfied by taking $F_z = 2\pi r_0 N_0$ so that

$$N_{\xi\xi}(\xi_1) = N_0 \qquad (0 \leqslant \xi_1 \leqslant \ell/r_0)$$

Having determined all stress and strain measures for the classical theory, we may then calculate the displacement components from the strain-displacement relations if we wish. For a circular cylindrical shell, the strain displacement relations for the axisymmetric bending problem take the form

$$u_r = r_0\varepsilon_{\theta\theta}, \qquad \phi_\xi = -\phi_B$$

$$u_z = u_0 + \int_{\xi_i}^{\xi_1} (r_0\varepsilon_{\xi\xi})d\xi_1 \qquad (\equiv u_\xi)$$

where u_0 is a constant of integration. We see from these relations that the BVP for ϕ_B and Φ_B continues to admit only a trivial solution if the moment free condition $M_{\xi\xi} = 0$ is replaced by the condition of no rotation - $\phi_B = 0$ at one or both edges. On the other hand, if the condition of no transverse shear resultant, $Q_\xi = 0$, is replaced by the condition of no radial displacement at one or both edges, Φ_B and ϕ_B will no longer vanish identically as

$$u_r = r_o \varepsilon_{\theta\theta} = r_o A[\frac{1}{r_o} \Phi_B' - \nu_s \frac{(rV)}{r_o}] = A(\Phi_B' - \nu_s \frac{Fz}{2\pi}) = 0$$

is an inhomogeneous boundary condition.

II. Uniform Radial Surface Load Distribution: For this case, we have $p_V = q_\xi = 0$ and $p_H = p_0$ throughout the shell. If both shell edges are free of traction, we have also $N_{\xi\xi} = (rV)/r_0 = F_z/2\pi r_0 = 0$ at $z = 0$ and $z = \ell$ so that $F_z = 0$. The boundary value problem for ϕ_B and Φ_B is again homogeneous; it admits only the trivial solution $\phi_B \equiv \Phi_B \equiv 0$ for $(0 \leqslant \xi_1 \leqslant \ell/r_0)$. Correspondingly, we have

$$N_{\xi\xi} = Q_\xi = M_{\xi\xi} = M_{\theta\theta} = 0, \qquad N_{\theta\theta} = r_0 p_0$$

throughout the shell. Again ϕ_B and Φ_B would not vanish identically if the condition $Q_\xi = 0$ at one or both edge is replaced by $u_r = 0$. For $\beta^2 = \bar{\beta}^2 \ell^2/r_0{}^2 = \ell^2/(DAr_0^2)^{1/2} >> 1$, the contribution from ϕ_B and Φ_B in this case is in the nature of a boundary layer phenomenon.

III. A Semi-infinite Shell with Edge Moment and Transverse Edge Load: With $p_r \equiv p_H \equiv q_\xi \equiv 0$ and $N_{\xi\xi} = 0$ at the edges, we have $(rV) = F_z/2\pi = 0$. For the stress and displacement components to remain bounded at infinity, we take $C_1 = C_2 = 0$ in the complementary solution χ_C so that

$$\phi_B = e^{\bar{\beta}\xi_1/\sqrt{2}} \{C_3 \cos \frac{\bar{\beta}\xi_1}{\sqrt{2}} - C_4 \sin \frac{\bar{\beta}\xi_1}{\sqrt{2}}\}$$

$$\Phi_B = \sqrt{\frac{D}{A}}\, e^{\bar{\beta}\xi_1/\sqrt{2}} \{C_4 \cos \frac{\bar{\beta}\xi_1}{\sqrt{2}} + C_3 \sin \frac{\bar{\beta}\xi_1}{\sqrt{2}}\}$$

It remains to choose C_4 and C_3 to satisfy the edge conditions

$$\xi_1 = 0: \qquad M_{\xi\xi} = -\frac{D}{r_o}\phi_B' = M_o \qquad Q_\xi = -\frac{\Phi B}{r_o} = Q_o$$

These two conditions requires that C_3 and C_4 be the solution of

$$-\frac{D}{r_o}\left[\left(-\frac{\bar\beta}{\sqrt{2}}\right)C_3 - \frac{\bar\beta}{\sqrt{2}}C_4\right] = M_0, \qquad \sqrt{\frac{D}{A}}\,C_4 = -r_oQ_o$$

or

$$C_4 = -\sqrt{\frac{A}{D}}\,r_oQ_o, \qquad C_3 = \frac{r_oM_o\sqrt{2}}{D\bar\beta} + \sqrt{\frac{A}{D}}\,r_oQ_o$$

The expressions for ϕ_B and Φ_B then become

$$\phi_B = \frac{r_o}{D\bar\beta}\,e^{-\bar\beta\xi_1/\sqrt{2}}\left[\left(\sqrt{2}\,M_0 + \frac{1}{\bar\beta}r_oQ_o\right)\cos\frac{\bar\beta\xi_1}{\sqrt{2}} + \frac{1}{\bar\beta}r_oQ_o\sin\frac{\bar\beta\xi_1}{\sqrt{2}}\right]$$

$$\Phi_B = \bar\beta\,e^{-\bar\beta\xi_1/\sqrt{2}}\left[-\frac{1}{\bar\beta}r_oQ_o\cos\frac{\beta\xi_1}{\sqrt{2}}\left(\sqrt{2}\,M_0 + \frac{1}{\bar\beta}r_oQ_o\right)\sin\frac{\bar\beta\xi_1}{\sqrt{2}}\right]$$

Near the edge $\xi_1 = 0$ of the shell, we have for the direct stress σ_D and bending stress σ_B the following order of magnitude estimates:

$$\sigma_D = 0\left(\frac{N_{\theta\theta}}{h}\right) = 0\left(\frac{\Phi_B'}{r_oh}\right) = 0\left(\frac{\bar\beta r_oQ_o}{r_oh}, \frac{\bar\beta^2M_o}{r_oh}\right)$$

$$\sigma_B = 0\left(\frac{M_{\theta\theta}}{h^2}\right) = 0\left(\frac{D\phi_B'}{r_oh^2}\right) = 0\left(\frac{M_o}{h^2}, \frac{r_oQ_o}{\bar\beta h^2}\right)$$

It follows that

$$\frac{\sigma_D}{\sigma_B} = 0\left(\frac{\bar\beta^2h}{r_o}\right) = 0\left(\frac{r_oh}{\sqrt{DA}\,r_o}\right) = 0(1)$$

so that the direct and bending stresses near the edge are of comparable magnitude.

3. A Second Order Complex Equation for Spherical, Conical and Torodal Shells

For a spherical shell with midsurface radius a, we take $r = a\sin\xi$, and $z = a\cos\xi$, so that $\alpha = a$ and $\xi = \pi - \xi_1$. For shells with uniform thickness and material properties, the two ODEs for ϕ_B and Φ_B become

$$\phi_B'' + \cot\xi_1 \phi_B' - \{\csc^2\xi_1 - [1-\nu_b - \frac{D_P}{D}]\}\phi_B + \frac{a}{D}\Phi_B = f_e(\xi_1)$$

$$\Phi_B'' + \cot\xi_1 \Phi_B' - \{\csc^2\xi_1 - [1+\nu_s - \frac{A_Q}{A}]\}\Phi_B - \frac{a}{A}\phi_B = f_c(\xi_1)$$

where

$$f_e(\xi_1) = - C_B \cot\xi_1(1-\nu_b - \frac{D_P}{D}) - \frac{a}{D}\{\cot\xi_1(rV) - aq_\xi\}$$

$$f_c(\xi_1) = - (rV)\cot\xi_1(1+\nu_s - \frac{A_Q}{A}) - \nu_s(rV)' + \frac{a}{A} C_B\cot\xi_1$$
$$-[(2+\nu_s)a^2\cos\xi_1 p_H + a^2\sin\xi_1 p_H']$$

The two second order ODEs may be combined into a single second order ODE for a complex function $\chi \equiv \phi_B + \lambda\Phi_B$ of the form

$$\chi'' + \cot\xi_1\chi' + [\mu(\mu+1) - \csc^2\xi_1]\chi = f_e + \lambda f_c \equiv f(\xi i;\lambda)$$

where

$$\lambda = \frac{A}{2a}\{(\frac{A_Q}{A} - \frac{D_P}{D} - \nu_b - \nu_s) + i\frac{2a}{\sqrt{DA}}\sqrt{1 - \frac{DA}{4a^2}(\frac{A_Q}{A} - \frac{D_P}{D} - \nu_b - \nu_s)^2}\}$$

$$\mu(\mu+1) = (1 + \frac{\nu_s - \nu_b}{2} - \frac{D_P}{2D} - \frac{A_Q}{2A}) - i\frac{a}{\sqrt{DA}}\sqrt{1 - \frac{DA}{4a^2}(\frac{A_Q}{A} - \frac{D_P}{D} - \nu_b - \nu_s)^2}$$

For the classical theory with $D_P = A_Q = 0$ and $\nu_s = \nu_b = \nu$, we have

$$\lambda = -\frac{\nu A}{a} + i\sqrt{\frac{D}{A}}\sqrt{1 - \frac{\nu^2 DA}{a^2}}$$

$$\mu(\mu+1) = 1 - i\frac{a}{\sqrt{DA}}\sqrt{1 - \frac{\nu^2 DA}{a^2}}$$

With the second order ODE for χ, it is now not difficult to see that the complementary solution for ϕ_B and Φ_B is a linear combination of associated Legendre functions with

$$\chi_c = \phi_B + \lambda\Phi_B = C_1 P_\mu^1(\cos\xi_1) + C_2 Q_\mu^1(\cos\xi_1)$$

where C_1 and C_2 are two complex constants of integration. The parameters μ and λ take on different values depending on the particular theory employed for the analysis. Applications of this result to specific problems, including the problem of a complete spherical shell with equal and opposite point load at the two poles, can be found in [16] and elsewhere.

For a conical shell with a constant slope angle ξ, we take $r = \ell\xi_1\cos\xi$ and $z = \ell\xi_1\sin\xi - z_i$ for $\xi_i \leqslant \xi_1 \leqslant \xi_0$ with $z_i = \ell\xi_i\sin\xi$ so that $\alpha = \ell$. For shells of uniform thickness and material properties, the two ODEs for ϕ_B and Φ_B in the classical theory ($A_Q = D_P = 0$, $\nu_b = \nu_s = \nu$) take the form:

$$\phi_B'' + \frac{1}{\xi_1}\phi_B' - \frac{1}{\xi_1^2}\phi_B - \frac{\ell\tan\xi}{D\xi_1}\Phi_B = f_e(\xi_1)$$

$$\Phi_B'' + \frac{1}{\xi_1}\Phi_B' - \frac{1}{\xi_1^2}\Phi_B - \frac{\ell\tan\xi}{A\xi_1}\phi_B = f_c(\xi_1)$$

where

$$f_e(\xi_1) = \frac{\tan\xi}{\xi_1^2} C_B - \frac{\ell}{D\xi_1}[(rV) - \ell\xi_1 q_\xi]$$

$$f_c(\xi_1) = \frac{\tan\xi}{\xi_1^2}(rV) + \nu_s\tan\xi\,\frac{(rV)'}{\xi_1} + \frac{\ell}{A\xi_1} C_B$$

$$- [(2+\nu_s)p_H + \xi_1 p_H']\ell^2\cos\xi$$

The two ODEs can be combined into a single ODE of the same order for $\chi \equiv \phi_B + i\sqrt{A/D}\,\Phi_B$ in the form

$$\chi'' + \frac{1}{\xi_1}\chi' - \frac{1}{\xi_1^2}\chi + i\,\frac{\ell\tan\xi}{\xi_1\sqrt{DA}}\chi = f_e(\xi_1) + i\sqrt{\frac{A}{D}}\,f_c(\xi_1)$$

The complementary solution of the above ODE is easily seen to be in terms of Bessel functions:

$$\chi_c = A_1 J_2(\beta\sqrt{i}\,\xi_1) + A_2 Y_2(\beta\sqrt{i}\,\xi_1)$$

$$= [C_1 \mathrm{ber}_2(\beta\sqrt{\xi_1}) + C_2 \mathrm{bei}_2(\beta\sqrt{\xi_1}) + C_3 \mathrm{ker}_2(\beta\sqrt{\xi_1}) + C_4 \mathrm{kei}_2(\beta\sqrt{\xi_1})]$$

$$+ i\,[C_2 \mathrm{ber}_2(\beta\sqrt{\xi_1}) - C_1 \mathrm{bei}_2(\beta\sqrt{\xi_1}) + C_4 \mathrm{ker}_2(\beta\sqrt{\xi_1}) - C_3 \mathrm{kei}_2(\beta\sqrt{\xi_1})]$$

where $\beta^2 = 4\ell\tan\xi/\sqrt{DA} >> 1$ for shell behavior.

Consider next a toroidal shell with

$$r = a + b\sin\xi_1 \equiv a + bs_1, \qquad z = -\,b\cos\xi_1 \equiv -\,bc_1 \qquad (0 \leqslant \xi_1 \leqslant 2\pi)$$

so that

$$r' = bc_1, \qquad z' = bs_1, \qquad \alpha = b$$

With $\lambda = b/a < 1$, the two ODEs for ϕ_B and Φ_B become

$$\phi_B'' + \frac{\lambda c_1}{1+\lambda s_1}\phi_B' - \left[\frac{\lambda^2 c_1^2}{(1+\lambda s_1)^2} + \frac{D_P}{D}\frac{\lambda^2 s_1^2}{(1+\lambda s_1)^2} + \nu_b \frac{\lambda s_1}{(1+\lambda s_1)^2}\right]\phi_B$$

$$- \frac{b\lambda s_1}{A(1+\lambda s_1)}\Phi_B = f_e(\xi_1)$$

$$\Phi_B'' + \frac{\lambda c_1}{1+\lambda s_1}\Phi_B' - \left[\frac{\lambda^2 c_1^2}{(1+\lambda s_1)^2} + \frac{A_Q}{A}\frac{\lambda^2 s_1^2}{(1+\lambda s_1)^2} + \nu_s \frac{\lambda s_1}{(1+\lambda s_1)}\right]\Phi_B$$

$$- \frac{b\lambda s_1}{A(1+\lambda s_1)}\phi_B = f_e(\xi_1)$$

where

$$f_e(\xi_1) = C_B\{\frac{\lambda^2 s_1 c_1}{(1+\lambda s_1)^2}(1 - \frac{D_P}{D}) - \nu_b \frac{\lambda c_1}{1+\lambda s_1}\} - \frac{b}{D}\{\frac{\lambda c_1 (rV)}{1+\lambda s_1} - bq_\xi\}$$

$$f_e(\xi_1) = (rV)\ \{\frac{\lambda^2 s_1 c_1}{(1+\lambda s_1)^2}(1 - \frac{A_Q}{A}) + \frac{\nu_s \lambda c_1}{1+\lambda s_1}\} - \nu_s \frac{\lambda s_1}{1+\lambda s}(rV)' + \frac{b c_B \lambda c_1}{A(1+\lambda s_1)}$$

$$- [(2+\nu_s)b^2 c_1 p_H + ab(1+\lambda s_1)p_H']$$

For the classical theory with $A_Q = D_P = 0$ and $\nu_s = \nu_b = \nu$, the two equations for ϕ_B and Φ_B may be combined to get a single ODE for the complex function $\chi = \phi_B + \mu\Phi_B$:

$$\chi'' + \frac{\lambda c_1}{1+\lambda s_1}\chi' - \frac{\lambda^2 c_1{}^2}{(1+\lambda s_1)^2}\chi + i\,\frac{b}{\sqrt{DA}}\sqrt{1 - \frac{\nu^2 DA}{b^2}}\ \frac{\lambda s_1}{1+\lambda s_1}\chi$$

$$= f_e(\xi_1) + \mu f_c(\xi_1)$$

where $s_1 = \sin\xi_1$, $c_1 = \cos\xi_1$ and

$$\mu = \frac{\nu A}{b} + i\sqrt{\frac{A}{D}[1 - \frac{\nu^2 DA}{b^2}]}$$

An exact solution of the homogeneous equation for χ in terms of known function does not seem possible. Furthermore, the second order ODE has a simple turning point at $\xi_1 = 0$ and $\xi_1 = \pi$ where $\sin\xi_1 = 0$. Applications Langer's method for an asymptotic solution for χ is appropriate whenever we have $\beta^2 = \lambda b/\sqrt{DA} = b^2/\sqrt{DAa^2} >> 1$. Such an asymptotic solution and the use of this solution for a number of practical problems can be found in [17].

4. Finite Difference Solutions

The reduction of the shell equations for axisymmetric bending (and stretching) problems to the canonical form of two simultaneous second order ODEs for ϕ_B and Φ_B is evidently a very important step in our effort to understand the shell behavior for this class of problems. For example, the canonical form has enabled us to deduce the exact solution of certain problems in terms of known functions. It has also enabled us to bring out the singular perturbation structure of (axisymmetric bending) shell problems which is not at all obvious from the original form of the shell equations. This special structure of the canonical ODEs in turn allows us to bring a wide range of asymptotic techniques such as the WKB method, Langer's method, etc., to bear on our problems for accurate approximate (asymptotic) solutions*. In addition to all these significant features, the canonical form is an attractive arrangement for efficient numerical solutions of the relevant boundary value problems.

For our discussion of an efficient numerical solution process here, it suffices to consider shells with uniform thickness and material properties and to take the two canonical ODEs in their Liouville normal form,

$$\phi^{\cdot\cdot} - [\sigma(x) + \nu_b\eta(x)] - \beta^2\tau(x)\Phi = f_1(x)$$

$$\Phi^{\cdot\cdot} - [\sigma(x) - \nu_s\eta(x)] + \beta^2\tau(x)\phi = f_2(x)$$

where $\sigma(x)$, $\eta(x)$, $\tau(x)$, $f_1(x)$ and $f_2(x)$ are known functions of the dimensionless independent variable x, $(\;)^{\cdot}$ indicates differentiation with respect to x and β^2 is the Reissner number for the shell. ^{2} is large compared to unity if the shell is not plate-like. The two ODEs are supplemented by two appropriate

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* The asymptotic solution process for shallow shells of Chapter 4 (Section (6)), can be extended to the non-shallow shell case. See also [17,18].

boundary conditions at each of the shell edges, $x=x_i$ and $x=x_o$. We take these in the form

$$x = x_i: \qquad a_{\phi i}\dot{\phi} + b_{\phi i}\phi = c_{\phi i}, \qquad a_{\Phi i}\dot{\Phi} + b_{\Phi i}\Phi = c_{\Phi i}$$

$$x = x_o: \qquad a_{\phi o}\dot{\phi} + b_{\phi o}\phi = c_{\phi o}, \qquad a_{\Phi o}\dot{\Phi} + b_{\Phi o}\Phi = c_{\Phi o}$$

where $a_{\phi i}$, etc. are known constants. The form of these four boundary conditions includes the appropriate stress and/or displacement boundary conditions as special cases. A more general set of boundary conditions would be needed for an elastically supported edge or for a curved tube with a closed cross section.

For the purpose of a finite difference solution, we introduce a mesh of equal spacings $\{x_1, x_2, \ldots, x_N\}$ for the interval $[x_i, x_0]$ with $x_1 = x_i$, $x_N = x_0$, $x_k = x_i + k\Delta$ and $\Delta = (x_o - x_i)/(N-1)$. For each of the interior mesh points $\{x_2, \ldots, x_{N-1}\}$ we have a finite difference analogue of the ODEs constructed by replacing the second derivative of a function $\psi(x)$ at a point $x = x_j$ by the finite difference expression $[\psi_{j+1} - 2\psi_j + \psi_{j-1}]/\Delta^2$ where $\psi_j \equiv \psi(x_j)$. We write the two finite difference equations as a single equation for the vector unknown $Y_j = [\phi_j, \Phi_j]^T$:

$$Y_{n-1} + B_n Y_n + Y_{n+1} = D_n, \qquad (n = 2,3, \ldots, N-1) \qquad (5.1)$$

where

$$B_k = \begin{bmatrix} -2 - \Delta^2(\sigma_k + \nu_b \eta_k) & -\beta^2\Delta^2\tau_k \\ \beta^2\Delta^2\tau_k & -2 - \Delta^2(\sigma_k - \nu_s \eta_k) \end{bmatrix}, \qquad D_n = \begin{bmatrix} \Delta^2 f_n \\ \Delta^2 g_n \end{bmatrix}$$

At each of the two end points, we have also a finite difference analogue of the two boundary conditions. There are a number of different ways for constructing the analogues. For our purpose, it suffices to take the simplest approach (which may not be optimal) by using a forward difference formula for the derivatives at $x = x_i$ and a backward difference formula for the derivatives at $x = x_o$. In that case, the two boundary conditions at $x = x_i$ may be written a single equation for Y_1 of the form

$$B_1 Y_1 + C_1 Y_2 = D_1 \qquad (5.2)$$

while the two boundary conditions at $x = x_o$ may be written as

$$A_N Y_{N-1} + B_N Y_N = D_N \qquad (5.3)$$

where the actual form of the coefficients B_1, C_1, D_1, B_N A_N and D_N depends on the difference formulas used. In fact, more elaborate formulas for the first derivatives may bring in additional unknowns, Y_3, Y_4, ... for (5.2) and Y_{N-2}, Y_{N-3}, ... for (5.3).

Equations (5.1) - (5.3) constitute a system of 2N linear equations for the 2N unknowns $\{\phi_1, \ldots \phi_N\}$ and $\{\Phi_1, \ldots, \Phi_N\}$. The block tridiagonal form of the system enables us to solve it very efficiently by forward elimination and back substitution in a way that takes advantage of the special kind of sparse structure of the coefficient matrix. Equation (5.2) may be first solved for Y_1 to get

$$Y_1 = \bar{C}_1 Y_2 + \bar{D}_1,$$

with $\bar{C}_1 = \bar{B}_1^1 C_1$ and $\bar{D}_1 = \bar{B}_1^1 D_1$. The result may then be used to eliminate Y_1 from equation (5.1) with n=2 to get

$$Y_2 = \bar{C}_2 Y_3 + \bar{D}_2,$$

where $\bar{C}_2 = -(\bar{C}_1 + B_2)^{-1} C_2$ and $\bar{D}_2 = (\bar{C}_1 + B_2)^{-1}(D_2 - \bar{D}_1)$.

By repeating the same process, we get

$$Y_n = \bar{C}_n Y_{n+1} + \bar{D}_n \qquad (n=1,2,\ \ldots,\ N-1) \qquad (5.4)$$

where $\bar{C}_n = -(\bar{C}_{n-1} + B_n)^{-1} C_n$ and $\bar{D}_n = (\bar{C}_{n-1} + B_n)^{-1}(D_n - \bar{D}_{n-1})$.

Finally, we have from (5.3)

$$Y_N = \bar{D}_N \equiv (A_N \bar{C}_{N-1} + B_N)^{-1}(D_N - A_N \bar{D}_{N-1})$$

which gives the solution for $\phi(x_0)$ and $\Phi(x_0)$. We back-substitute through (5.4) for n = N-1, N-2, ..., to get successively Y_{N-1}, Y_{N-2}, ..., Y_1. The solution process involves only the inversion of N (2x2) matrices and the multiplication of two (2x2) matrices 2N times. The operation counts is O(N) which is to be compared with the $O(N^3)$ count if the coefficient matrix of the linear system is not sparse (which would be the case if we had sought a numerical solution of the original system of shell equations. In addition, the storage requirement is also an order of magnitude less than that for the shell equations before the reduction to canonical form.

The above finite difference solution process is only the simplest (but crude) method for a numerical solution. In addition to the possibility of a more refined difference analogue based on finite difference formulas of higher order accuracy or some spline approximation, we may improve the efficiency of the numerical scheme by using a variable mesh with more mesh points (of smaller spacing) inside the boundary layer region of pronounced effect of edge bending. Here, the availability of some exact solutions makes it possible for us to estimate the boundary layer width and hence the appropriate mesh size for our problem, etc. The exact solutions also offer a bench mark for the accuracy of the numerical solutions. Finally, it should be mentioned that the reduction of operation counts by two orders of magnitude is even more important for problems involving finite deformations. Numerical solution scheme for nonlinear BVP in ODE generally require an iterative solution of a repeated modified linear system. As we shall see later, axisymmetric bending problems involving finite deformations may also reduce to a canonical form with nearly the same structure.

Part III - Laterally Loaded Shells of Revolution

6 - Sinusoidal Stress Distributions in Shells of Revolution

1. Intrinsic Formulation of Shell Problems

In many applications, the surface load components p_ξ, p_θ and p_n for shells of revolution vary in the circumferential direction in the form of $\sin\theta$ and/or $\cos\theta$. One example is a dome-shaped shell of revolution in its "face-side" position so that the gravity vector is perpendicular to the shell's axis of revolution. With no loss in generality, we may let $\vec{i}_x$ be in the direction of the gravity vector. In that case, the surface load vector $\vec{p}$ is given by $\vec{p} = \rho g h \vec{i}_x$ where ρ is the mass density of the shell material, g is the gravitational acceleration and h is the shell thickness. It follows that

$$p_\xi = \vec{p}\cdot\vec{t}_\xi = \rho g h \cos\xi\cos\theta, \qquad p_\theta = \vec{p}\cdot\vec{t}_\theta = -\rho g h \sin\theta$$

$$p_n = \vec{p}\cdot\vec{n} = -\rho g h \sin\xi\cos\theta ,$$

where ξ is the meridional slope angle, i.e., the angle between the meridional tangent and the horizon, $\cos\xi = r'/\alpha$ and $\sin\xi = z'/\alpha$ with $\alpha^2 = (r')^2 + (z')^2$. In a linear theory of shell, the stress and strain distributions of a shell of revolution are proportional to $\sin\theta$ and/or $\cos\theta$ whenever the surface load components are proportional to $\sin\theta$ and/or $\cos\theta$. With

$$\{p_\xi, p_n\} = \{\bar{p}_\xi(\xi_1), \bar{p}_n(\xi_1)\}\cos\theta, \qquad p_\theta = \bar{p}_\theta(\xi_1)\sin\theta, \tag{6.1}$$

it can be verified that the stress resultants and stress couples are of the form

$$\begin{aligned}\{N_{\xi\xi}, N_{\theta\theta}, Q_\xi, M_{\xi\xi}, M_{\theta\theta}, p_\theta\} &= \{\bar{N}_{\xi\xi}(\xi_1), \dots, \bar{p}_\theta(\xi_1)\}\cos\theta \\ \{N_{\xi\theta}, N_{\theta\xi}, Q_\theta, M_{\xi\theta}, M_{\theta\xi}, p_\xi\} &= \{\bar{N}_{\xi\theta}(\xi_1), \dots, \bar{p}_\xi(\xi_1)\}\sin\theta\end{aligned} \tag{6.2}$$

Correspondingly, we have

$$\{\kappa_{\xi\xi},\ \kappa_{\theta\theta},\ \lambda_{\theta},\ \varepsilon_{\xi\xi},\ \varepsilon_{\theta\theta},\ \gamma_{\xi}\} = \{\bar{\kappa}_{\xi\xi}(\xi_1),\ \ldots,\ \bar{\gamma}_{\xi}(\xi_1)\}\cos\theta$$

$$\{\kappa_{\varepsilon\xi},\ \kappa_{\theta\xi},\ \lambda_{\xi},\ \varepsilon_{\xi\theta},\ \varepsilon_{\theta\xi},\ \gamma_{\theta}\} = \{\bar{\kappa}_{\xi\theta}(\xi_1),\ \ldots,\ \bar{\gamma}_{\theta}(\xi_1)\}\sin\theta$$

for the strain measures. Other surface loads which are of the form (6.1) include the conventional approximate expressions for wind loads. Shells of revolution without surface loads but subject to edge loads may also develop stress and strain distributions with a $\sin\theta/\cos\theta$ type of circumferential variations. Examples of such edge loads include the case of a side force and/or a tilting moment applied to a rigid plug with the edge of the shell welded to the plug. Shells of revolution having stress and strain measures which vary like $\sin\theta/\cos\theta$ in the circumferential direction are called <u>laterally loaded</u> shells of revolution.

For laterally loaded shells of revolution with stress, strain and load measures of the form (6.1) - (6.3), the equilibrium equations are effectively six linear ordinary differential equations for the ξ_1 dependent portions of these shell quantities. For <u>classical shell theory</u> with $P_\xi \equiv P_\theta \equiv \gamma_\xi \equiv \gamma_\theta \equiv 0$, and $q_\xi \equiv q_\theta \equiv q_n \equiv 0$, we have

$$(r\bar{N}_{\xi\xi})' + \alpha\bar{N}_{\theta\xi} - \alpha\cos\xi\bar{N}_{\theta\theta} - r\xi'\bar{Q}_{\xi} + r\alpha\bar{p}_{\xi} = 0$$

$$(r\bar{N}_{\xi\theta})' - \alpha\bar{N}_{\theta\theta} + \alpha\cos\xi\bar{N}_{\theta\xi} - z'\bar{Q}_{\theta} + r\alpha\bar{p}_{\theta} = 0$$

$$(r\bar{Q}_{\xi})' + \alpha\bar{Q}_{\theta} + r\xi'\bar{N}_{\xi\xi} + z'\bar{N}_{\theta\theta} + r\alpha\bar{p}_{n} = 0$$

$$(r\bar{M}_{\xi\xi})' + \alpha\bar{M}_{\theta\xi} - \alpha\cos\xi\bar{M}_{\theta\theta} - r\alpha\bar{Q}_{\xi} = 0$$

$$(r\bar{M}_{\xi\theta})' - \alpha\bar{M}_{\theta\theta} + \alpha\cos\xi\bar{M}_{\theta\xi} - r\alpha\bar{Q}_{\theta} = 0$$

$$r\alpha(\bar{N}_{\xi\theta} - \bar{N}_{\theta\xi}) - z'\bar{M}_{\xi\theta} + \xi'r\bar{M}_{\theta\xi} = 0$$

These six equilibrium equations along with the six dual compatibility equations and the stress strain relations

$$\bar{M}_{\xi\xi} = D(\bar{\kappa}_{\xi\xi} + \nu_b \bar{\kappa}_{\theta\theta}), \qquad \bar{M}_{\theta\theta} = D(\bar{\kappa}_{\theta\theta} + \nu_b \bar{\kappa}_{\xi\xi})$$

$$\bar{M}_{\xi\theta} = \bar{M}_{\theta\xi} = \frac{1}{2} D(1-\nu_b)(\bar{\kappa}_{\xi\theta} + \bar{\kappa}_{\theta\xi})$$

$$\bar{\varepsilon}_{\theta\theta} = A(\bar{N}_{\theta\theta} + \nu_s \bar{N}_{\xi\xi}), \qquad \bar{\varepsilon}_{\xi\xi} = A(\bar{N}_{\xi\xi} + \nu_s \bar{N}_{\theta\theta})$$

$$\bar{\varepsilon}_{\xi\theta} = \bar{\varepsilon}_{\theta\xi} = \frac{1}{2} A(1-\nu_s)(\bar{N}_{\xi\theta} + \bar{N}_{\theta\xi})$$

form a system of twenty equations for the twenty unknowns, $\bar{N}_{\xi\xi}$, $\bar{N}_{\theta\theta}$, $\bar{N}_{\theta\xi}$, $\bar{N}_{\xi\theta}$, $\bar{Q}_{\xi}$, $\bar{Q}_{\theta}$, $\bar{M}_{\xi\xi}$, $\bar{M}_{\theta\theta}$, $\bar{M}_{\xi\theta}$, $\bar{M}_{\theta\xi}$ and their dual strain measures. Unlike the axisymmetric stress distribution case, they do not uncouple into two separate groups. As such, the problem of laterally loaded shells of revolution is substantially more complicated than the axisymmetric load problem.

2. First Integrals

Just as the axisymmetric problem, the equilibrium equations for laterally loaded shells of revolution have two first integrals. First, we form the combination

$$0 = \cos\xi[(r\bar{N}_{\xi\xi})' + \ldots + \alpha r\bar{p}_{\xi}] - [(r\bar{N}_{\xi\theta})' + \ldots + r\alpha\bar{p}_{\theta}] - \sin\xi[(r\bar{Q}_{\xi})' \ldots + r\alpha\bar{p}_{n}]$$

of the three force equilibrium equations to get

$$[r(\cos\xi\bar{N}_{\xi\xi} - \sin\xi\bar{Q}_{\xi} - \bar{N}_{\xi\theta})]' + [r(\bar{p}_{\xi}\cos_{\xi} - \bar{p}_{\theta} - \bar{p}_{n}\sin\xi)]' = 0$$

or

$$r[\cos\xi\bar{N}_{\xi\xi} - \sin\xi\bar{Q}_{\xi} - \bar{N}_{\xi\theta}] = \frac{1}{\pi} P_x(\xi_1) \tag{6.6a}$$

with

$$P_x(\xi_1) = P_o - \pi\int_{\xi_i}^{\xi_1} (\bar{p}_{\xi}\cos\xi - \bar{p}_{\theta} - \bar{p}_{n}\sin\xi)\alpha r d\xi_1$$

where P_o is a constant of integration. The existence of this first integral is expected from the requirement that the shell be in overall force equilibrium:

$$\int_0^{2\pi}\int_{\xi_i}^{\xi_1} [(r\vec{N}_{\xi})' + (\alpha\vec{N}_{\theta})^{\bullet} + r\alpha\vec{p}]d\xi_1 d\theta = \vec{0}$$

As αN_θ is periodic in θ with period 2π, it follows that

$$\left[\int_0^{2\pi} r\vec{N}_{\xi}d\theta\right]_{\xi_i}^{\xi_1} + \int_0^{2\pi}\int_{\xi_i}^{\xi_1} r\alpha\vec{p}d\xi_1 d\theta = \vec{0}$$

The left side has only one nonvanishing component in the $\vec{i}_x$ direction; thus overall force equilibrium for a laterally loaded shell of revolution leads to only one nontrivial condition (6.6) of vanishing resultant side force in that direction.

Similarly, overall moment equilibrium for a laterally loaded shell of revolution gives only one nontrivial condition on the stress resultants and couples. For classical shell theory with $P_\xi \equiv P_\theta \equiv 0$ (and $\gamma_\xi \equiv \gamma_\theta \equiv 0$), this condition takes the form

$$r[\bar{M}_{\xi\xi} - \cos\xi\bar{M}_{\xi\theta} - r\cos\xi\bar{Q}_\xi - r\sin\xi\bar{N}_{\xi\xi}] = -\frac{z}{\pi} P_x + \frac{1}{\pi} T_y \qquad (6.7a)$$

with

$$\frac{1}{\pi} T_y = \int^{\xi_1} [(r\cos\xi + z\sin\xi)\bar{p}_n + (r\sin\xi - z\cos\xi)\bar{p}_\xi + z\bar{p}_\theta]\alpha r d\xi_1 \qquad (6.7b)$$

Evidently, T_y is a resultant tilting moment turning about the y-axis.

By the static geometric duality, we now know without a separate derivation the existence of two first integrals of the differential equations of compatibility. For classical shell theory with $\gamma_\xi \equiv \gamma_\theta \equiv 0$, they take the form

$$-r[\cos\xi\bar{\kappa}_{\theta\theta} + \sin\xi\bar{\lambda}_\theta + \bar{\kappa}_{\theta\xi}] = \frac{1}{\pi} \Omega_x \qquad (6.8)$$

$$r[\bar{\varepsilon}_{\theta\theta} + \cos\xi\bar{\varepsilon}_{\theta\xi} - r\cos\xi\bar{\lambda}_\theta + r\sin\xi\bar{\kappa}_{\theta\theta}] = -\frac{z}{\pi} \Omega_x + \frac{1}{\pi} U_y \qquad (6.9)$$

In these, the quantities Ω_x and U_y are constants of integration. As we shall see in the next section, these quantities have the interpretation of relative edge rotation and displacement, i.e., dislocations of the Volterra type for shells.

With the four first integrals (6.6), (6.7), (6.8) and (6.9), the system of shell equations for classical shell theory is effectively reduced from an eighth order to a fourth order system. We will in section (4) make use of these first integrals to reduce this system to two coupled second order equations of a form which is very similar to that of the Reissner-Meissner-Reissner system for axisymmetric bending (and stretching) of shells of revlution.

3. Non-periodic Displacement Fields

With a view toward an interpretation of the constants of integration Ω_x and U_y in (6.8) and (6.9), we recall from plate bending and generalized plane stress problems that strain measures of a laterally loaded flat plate in bending and extension with a $\sin\theta$ and/or $\cos\theta$ dependence on the polar angle θ may correspond to a nonperiodic displacement with a $\theta\cos\theta$ and/or $\theta\sin\theta$ dependence. This suggests that we consider nonperiodic displacement fields of the form

$$\{u_\varepsilon,\ w,\ \phi_\varepsilon\} = \{\bar{u}_\varepsilon,\ \bar{w},\ \bar{\phi}_\varepsilon\}\cos\theta + \{\hat{u}_\xi,\ \hat{w},\ \hat{\phi}_\xi\}\theta\sin\theta$$

$$\{u_\theta,\ \phi_\theta,\ \omega\} = \{\bar{u}_\theta,\ \bar{\phi}_\theta,\ \bar{\omega}\}\sin\theta + \{\hat{u}_\theta,\ \hat{\phi}_\theta,\ \hat{\omega}\}\theta\cos\theta$$

where terms of the form $\bar{f}$ and $\hat{f}$ are functions of ξ_1 only. Upon substituting the above expressions into the linear strain-displacement relations for a shell of revolution and requiring the resulting strain measures be periodic in θ, we get

$$\hat{u}_\varepsilon = \cos\xi u_o + (z\cos\xi - r\sin\xi)\phi_o,$$

$$\hat{w} = -\sin\xi u_o - (r\cos\xi - z\sin\xi)\phi_o,$$

$$\hat{u}_\theta = u_o + z\phi_o$$

$$\hat{\phi}_\xi = \phi_o, \qquad \hat{\phi}_\theta = \phi_o\cos\xi, \qquad \hat{\phi}_n = \phi_o\sin\xi.$$

where ϕ_o and u_o are arbitrary constants.

The corresponding strain measures are

$$\bar{\varepsilon}_{\xi\xi} = \frac{1}{\alpha}\,\bar{u}'_{\xi} + \frac{1}{R_{\xi}}\,\bar{w}, \qquad \bar{\varepsilon}_{\theta\theta} = \frac{1}{r}[\bar{u}_{\theta} + \cos\xi\bar{u}_{\xi} - \sin\xi\bar{w} + z\phi_{o} + u_{o}]$$

$$\bar{\varepsilon}_{\xi\theta} = \frac{1}{\alpha}\,\bar{u}'_{\theta} - \bar{w}, \qquad \bar{\varepsilon}_{\theta\xi} = \frac{1}{r}[\bar{u}_{\xi} + \cos\xi\bar{u}_{\theta} - r\bar{\omega} - \cos\xi u_{o} - (z\cos\xi - r\sin\xi)\phi_{o}]$$

$$\bar{\gamma}_{\xi} = \bar{\phi}_{\xi} + \frac{1}{\alpha}\,\bar{w}' - \frac{1}{R_{\xi}}\,\bar{u}_{\xi}, \qquad \bar{\gamma}_{\theta} = \bar{\phi}_{\theta} - \frac{1}{r}\,\bar{w} + \frac{\sin\xi}{r}\,(\bar{u}_{\theta} - u_{o}) - (\cos\xi + \frac{z}{r}\sin\xi\phi_{o})$$

$$\bar{\kappa}_{\xi\xi} = \frac{1}{\alpha}\,\bar{\phi}'_{\xi}, \qquad \bar{\kappa}_{\theta\theta} = \frac{1}{r}\,[\bar{\phi}_{\theta} + \cos\xi(\bar{\phi}_{\xi} + \phi_{o})]$$

$$\bar{\kappa}_{\xi\theta} = \frac{1}{\alpha}\,\bar{\phi}'_{\theta} - \frac{1}{R_{\xi}}\,\bar{\omega}, \qquad \bar{\kappa}_{\theta\xi} = -\frac{1}{r}\,(\bar{\phi}_{\xi} + \cos\xi\bar{\phi}_{\theta} + \sin\xi\bar{\omega} - \phi_{o})$$

$$\bar{\lambda}_{\xi} = \frac{1}{\alpha}\,\bar{\omega}' + \frac{1}{R_{\xi}}\,\bar{\phi}_{\theta}, \qquad \bar{\lambda}_{\theta} = \frac{1}{r}\,[\bar{\omega} - \sin\xi(\bar{\phi}_{\xi} + \phi_{o})]$$

From these follow the relations

$$\cos\xi\bar{\kappa}_{\theta\theta} + \bar{\kappa}_{\theta\xi} + \sin\xi\bar{\lambda}_{\theta} = \frac{2}{r}\,\phi_{o}$$

$$r(\sin\xi\bar{\kappa}_{\theta\theta} - \cos\xi\bar{\lambda}_{\theta}) + \bar{\varepsilon}_{\theta\theta} + \cos\xi\bar{\varepsilon}'_{\theta\xi} = \frac{2}{r}\,(u_{o} + z\phi_{o})$$

A comparison of (6.8) and (6.9) with the above relations leads to the identifications

$$\phi_{o} = -\frac{1}{2\pi}\,\Omega_{x}, \qquad u_{o} = \frac{1}{2\pi}\,U_{y} \qquad (6.11)$$

which allow us to interpret Ω_x and U_y as Volterra type dislocations. Nonperiodic displacement fields are not permissable for shells of revolution which are complete in the circumferential direction; hence, the constants Ω_x and U_y must be set to zero for such a shell. In contrast, nonperiodic displacement fields are needed in the solution of a slit shell. A physically meaningful problem of this type is the asymmetric twisting of a ring shell segment formulated in [6].

Once the strain and curvature measures of the shell are known, the reduced strain displacement relations (6.10) may be solved as a system of first order ODE's for $\bar{u}_\xi$, $\bar{u}_\theta$, $\bar{w}$, $\bar{\phi}_\xi$, $\bar{\phi}_\theta$ and $\bar{\omega}$. The system is not overdetermined as the strain and curvature change measures satisfy the compatibility conditions (6.8) and (6.9). The solution for the ξ_1-dependent portion of the displacement components for the classical theory (with $\gamma_\xi \equiv \gamma_\theta \equiv 0$) may be given in the form

$$\bar{\phi}_\xi = \int^{\xi_1} \bar{\kappa}_{\xi\xi}\alpha d\xi_1, \qquad \bar{\phi}_\theta = r\bar{\kappa}_{\theta\theta} - \cos\xi(\bar{\phi}_\xi - \frac{1}{2\pi}\Omega_x)$$

$$\bar{\omega} = r\bar{\lambda}_\theta - \sin\xi(\bar{\phi}_\xi - \frac{1}{2\pi}\Omega_x)$$

$$\bar{u}_\theta = r\bar{\varepsilon}_{\theta\theta} - z\bar{\phi}_\xi - \frac{1}{2\pi}(U_y - z\Omega_x) + \int^{\xi_1}(z\bar{\kappa}_{\xi\xi} - \cos\xi\bar{\varepsilon}_{\xi\xi})\alpha d\xi_1 \qquad (6.12)$$

$$\bar{u}_\xi = r^2\bar{\lambda}_\theta - r\bar{\varepsilon}_{\theta\xi} - r\sin\xi\bar{\phi}_\xi - \cos\xi(\bar{u}_\theta - \frac{1}{2\pi}U_y) + (r\sin\xi - z\cos\xi)\frac{1}{\pi}\Omega_x$$

$$\bar{w} = r^2\bar{\kappa}_{\theta\theta} - (z\sin\xi + r\cos\xi)(\bar{\phi}_\xi - \frac{1}{\pi}\Omega_x).$$

$$+ \sin\xi[r\bar{\varepsilon}_{\theta\theta} - \frac{1}{\pi}U_y + \int^{\xi_1}(z\bar{\kappa}_{\xi\xi} - \cos\xi\bar{\varepsilon}_{\xi\xi})\alpha d\xi_1$$

4. Reduction to Two Simultaneous Equations

We re-write (6.6a) and (6.8) as

$$\cos\xi \bar{N}_{\xi\xi} - \sin\xi \bar{Q}_{\xi} = \frac{1}{\pi r} P_x(\xi_1) + \frac{1}{r} \bar{N}_{\xi\theta}$$

and

$$\cos\xi \bar{\kappa}_{\theta\theta} + \sin\xi \bar{\lambda}_{\theta} = - \frac{1}{\pi r} \Omega_x - \frac{1}{r} \bar{\kappa}_{\theta\xi} \; ,$$

respectively. The reduction procedure for the axisymmetric bending problem suggests that we take $\cos\xi \bar{N}_{\xi\xi} - \sin\xi \bar{Q}_{\xi}$ and $\cos \bar{\kappa}_{\theta\theta} + \sin\xi \bar{\lambda}_{\theta}$ as the two primary dependent variables. We adopt the equivalent choices by setting

$$\bar{N}_{\xi\theta} = \frac{1}{r} \Phi, \qquad \bar{\kappa}_{\theta\xi} = \frac{1}{r} \phi \tag{6.13}$$

With (6.13), we now consider the sixth equilbrium equation, the stress strain relation for $\bar{M}_{\xi\theta} = \bar{M}_{\theta\xi}$ and their static geometric duals as four linear algebraic equations for $\bar{M}_{\xi\theta} = \bar{M}_{\theta\xi}$, $\bar{\varepsilon}_{\xi\theta} = \bar{\varepsilon}_{\theta\xi}$, $\bar{N}_{\theta\xi}$ and $\bar{\kappa}_{\xi\theta}$ in terms of ϕ and Φ. We solve these equations to get

$$\bar{\varepsilon}_{\xi\theta} = \bar{\varepsilon}_{\theta\xi} = \frac{A(1+\nu_s)}{r(1+\varepsilon_0^2)} [\Phi - \frac{1}{2} \rho D(1-\nu_b)\phi], \qquad \bar{N}_{\theta\xi} = \frac{1-\varepsilon_0^2}{1+\varepsilon_0^2} \frac{1}{r} \Phi - \frac{D(1-\nu_b)}{1-\varepsilon_0^2} \frac{\rho}{r} \phi$$

$$\bar{M}_{\theta\xi} = \bar{M}_{\xi\theta} = \frac{D(1-\nu_b)}{r(1+\varepsilon_0^2)} [\phi + \frac{1}{2} \rho A(1+\nu_s)\Phi], \qquad \bar{\kappa}_{\xi\theta} = \frac{1-\varepsilon_0^2}{1+\varepsilon_0^2} \frac{1}{r} \phi + \frac{A(1+\nu_s)}{1-\varepsilon_0^2} \frac{\rho}{r} \Phi$$

where $\rho = (1/R_\theta) - (1/R_\xi)$ and $\varepsilon_0^2 = \frac{1}{4} DA(1-\nu^2)\rho^2$.

With the help of (6.14) and the remaining four stress strain relations, the four first integrals, the second equilibrium equation (with $\bar{Q}_\theta$ eliminated by way of the fifth equilibrium equation) and the dual compatibility equations may be treated as six linear algebraic equations for the six unknowns $\bar{Q}_\xi$, $\bar{M}_{\xi\xi}$, $\bar{M}_{\theta\theta}$ and their duals in terms of Φ, Φ', ϕ and ϕ' and load terms. We emphasize here that no higher (than first) derivatives of Φ and ϕ appear in the solution of this system for the three stress and three strain measures. Also, $\bar{N}_{\xi\xi}$, $\bar{N}_{\theta\theta}$, $\bar{\kappa}_{\xi\xi}$ and $\bar{\kappa}_{\theta\theta}$, obtained from the inverted stress strain relations

$$\bar{N}_{\xi\xi} = \frac{\bar{\varepsilon}_{\xi\xi}+\nu_s\bar{\varepsilon}_{\theta\theta}}{A(1-\nu_s^2)} , \qquad \bar{N}_{\theta\theta} = \frac{\bar{\varepsilon}_{\theta\theta}+\nu_s\bar{\varepsilon}_{\xi\xi}}{A(1-\nu_s^2)} , \text{ etc.,}$$

and $\bar{Q}_\theta$ and $\bar{\lambda}_\xi$, obtained from the fifth equilibrium and compatibility, are also known in terms of ϕ, ϕ', Φ, Φ' and load terms alone.

The only two remaining equations in the whole system of shell equations for laterally loaded shells of revolution are the fourth equilibrium equation and the dual compatibility equation. Upon expressing all stress and strain measures in these two first order differential equations in terms of ϕ, ϕ', Φ, Φ' and load terms by way of the results obtained above, we get two simultaneous second order ODE's for ϕ and Φ in the form

$$\phi'' - \frac{A(a+\nu_s)}{R_\theta(1+\varepsilon_1^2)}\Phi'' = f_1, \qquad \Phi'' + \frac{D(1-\nu_b)}{R_\theta(1+\varepsilon_1^2)}\phi'' = f_2, \tag{6.15}$$

where $1/R_\theta = - z'/r\alpha = - \sin\xi/r$, $\varepsilon_1^2 = O(DA/R^2)$ with $R = \min\{R_\theta,R_\xi\}$ and where only Φ', Φ, ϕ', ϕ and load terms appear in f_1 and f_2.

The <u>exact</u> equations (6.15) for Φ and ϕ are rather complicated. We limit ourselves here to the case $DA = O(h^2)$ and $\nu_s = \nu_b$, and to shell properties and loadings varying significantly only over a distance of the order $\ell >> h$. To make use of these restrictions to simplify the ODE's for ϕ and Φ, we first solve (6.15) for ϕ'' and Φ'' to get

$$[1 + \frac{\varepsilon_4^2}{(1+\varepsilon_1^2)^2}]\Phi'' = f_2 - \frac{D(1-\nu_b)}{R_\theta(1+\varepsilon_1^2)} f_1,$$

$$[1 + \frac{\varepsilon_4^2}{(1+\varepsilon_1^2)^2}]\phi'' = f_1 - \frac{D(1-\nu_b)}{R_\theta(1+\varepsilon_1^2)} f_2,$$

where $\varepsilon_4^2 = O(DA/R_\theta^2)$. We then omit all h/R and h^2/ℓ^2 terms from these equations to get the following two differential equations for ϕ and Φ:

$$\phi'' + \frac{(Dr/\alpha)'}{Dr/\alpha}\phi' - [4(\frac{r'}{r})^2 - \frac{\{(1-\nu_b)Dr'/\alpha\}'}{Dr/\alpha} + 2(\frac{z'}{r})^2]\phi - \frac{z'}{Dr/\alpha}$$

$$= \frac{1}{Dr/\alpha}\ [\frac{P_x r}{\pi}\ (\frac{z}{r})' + \frac{T_y}{\pi}\frac{r'}{r}] - \frac{1}{Dr/\alpha}\ [\frac{\nu_b D}{r\alpha}\ \{\frac{\Omega_x}{\pi}\ (rr' + zz') - \frac{U_y}{\pi}\ z'\}]' + \frac{\Omega_x}{\pi}\ \frac{(1+\nu_b)\alpha^2}{r^2}$$

$$(6.16)$$

$$\Phi'' + \frac{(Ar/\alpha)'}{Ar/\alpha}\Phi' - [4\ (\frac{r'}{r})^2 - \frac{\{(1+\nu_s)Ar'/\alpha\}'}{Ar/\alpha} + 2(\frac{z'}{r})^2]\Phi + \frac{z'}{Ar/\alpha}\phi =$$

$$= -\frac{1}{Ar/\alpha}\ [\frac{\Omega_x r}{\pi}\ (\frac{z}{r})' + \frac{U_y}{\pi}\frac{r'}{r}] + \frac{1}{Ar/\alpha}\ [\frac{\nu_s A}{r\alpha}\ \{\frac{P_x}{\pi}(rr' + zz') - \frac{T_y}{\pi}\ z'\}]' + \frac{P_x}{\pi}\ \frac{(1-\nu_s)\alpha^2}{r^2}$$

$$- \frac{(Ar^2\bar{p}_\theta)'}{Ar/\alpha} + (1 - \nu_s)\alpha r'\bar{p}_\theta, \qquad (6.17)$$

When $z \equiv 0$, these equations reduce to the exact equations for a flat plate. When r = constant, they differ from the exact equations for a homogeneous circular cylindrical shell of constant thickness only by terms of order h^2/R_θ^2 compared to other terms in the same equations. When $z \neq 0$ and r is not a constant, it is possible to effect a further reduction to a single second order equation for a complex stress function for spherical, conical and toroidal shells (with constant R_ξ). The method of reduction is the same as that for the corresponding axisymmetric problems. For simplicity, we have omitted the details leading up to (6.16) and (6.17) (which are remarkably like the Reissner-Meissner-Reissner equations for axisymmetric bending of shells of revolution) as well as the auxiliary equations for the stress and strain measures in terms of ϕ and Φ. These details can be found in [6,7] where applications of the results are also given. A simpler but more complete reduction of the corresponding shallow shell problem and some applications of the results will be discussed in the next chapter.

7 - Laterally Loaded Shallow Shells of Revolution

1. Governing Differential Equations

With the same characterization of shallow shells of revolution given previously for axisymmetric stress distribution problems, the strain displacement relations for laterally loaded shells of revolution in the classical theory simplify to

$$\bar{\kappa}_{\xi\xi} = \bar{\phi}'_{\xi}, \qquad \bar{\kappa}_{\xi\theta} = \bar{\phi}'_{\theta}, \qquad \bar{\lambda}_{\xi} = \bar{\omega}' - z''\bar{\phi}_{\theta}, \qquad \bar{\lambda}_{\theta} = \frac{1}{r}\left(\bar{\omega} + z'\bar{\phi}_{\xi} - z'\frac{\Omega_x}{2\pi}\right)$$

$$\bar{\kappa}_{\theta\xi} = -\frac{1}{r}\left(\bar{\phi}_{\xi} + \bar{\phi}_{\theta} + \frac{\Omega_x}{2\pi}\right), \qquad \bar{\kappa}_{\theta\theta} = \frac{1}{r}\left(\bar{\phi}_{\theta} + \bar{\phi}_{\xi} - \frac{\Omega_x}{2\pi}\right) \tag{7.1}$$

$$\bar{\varepsilon}_{\xi\xi} = \bar{u}'_{\xi} - z''\bar{w}, \qquad \bar{\varepsilon}_{\xi\theta} = \bar{u}'_{\theta} - \bar{\omega}, \qquad \bar{\gamma}_{\xi} = \bar{\phi}_{\xi} + \bar{w}'$$

$$\bar{\varepsilon}_{\theta\xi} = -\frac{1}{r}\left[\bar{u}_{\xi} + \bar{u}_{\theta} - r\bar{\omega} - \frac{U_y}{2\pi} + (z - rz')\frac{\Omega_x}{2\pi}\right] \tag{7.2}$$

$$\bar{\varepsilon}_{\theta\theta} = \frac{1}{r}\left[\bar{u}_{\xi} + \bar{u}_{\theta} - z'\bar{w} + \frac{U_y}{2\pi} - z\frac{\Omega_x}{2\pi}\right], \qquad \bar{\gamma}_{\theta} = \bar{\phi}_{\theta} - \frac{1}{r}\bar{w} + \frac{\Omega_x}{2\pi}$$

where we have taken $\xi_1 = r$ so that $(\ )' \equiv d(\ )/dr$ in this chapter. Consistent with the principle of virtual work, the corresponding differential equations of force and moment equilibrium for the r-dependent portion of the stress resultants and couples are

$$(r\bar{N}_{\xi\xi})' + \bar{N}_{\theta\xi} - \bar{N}_{\theta\theta} + r\bar{p}_{\xi} = 0, \qquad (r\bar{N}_{\xi\theta})' - \bar{N}_{\theta\theta} + \bar{N}_{\theta\xi} + r\bar{p}_{\theta} = 0,$$

$$(r\bar{Q}_{\xi})' + \bar{Q}_{\theta} + rz''\bar{N}_{\xi\xi} + z'\bar{N}_{\theta\theta} + r\bar{p}_n = 0 \tag{7.3}$$

$$(r\bar{M}_{\xi\xi})' + \bar{M}_{\theta\xi} - \bar{M}_{\theta\theta} + r\bar{Q}_{\xi} = 0, \qquad (r\bar{M}_{\xi\theta})' - \bar{M}_{\theta\theta} + \bar{M}_{\theta\xi} + r\bar{Q}_{\theta} = 0,$$

$$\bar{N}_{\xi\theta} - \bar{N}_{\theta\xi} = 0 \tag{7.4}$$

for the case of $P_j \equiv 0$ and no surface moment load so that $\bar{q}_\xi \equiv \bar{q}_\theta \equiv \bar{q}_n \equiv 0$. With $\bar{\gamma}_\xi \equiv \bar{\gamma}_\theta = 0$ for the classical theory, the r-dependent portions of the strain measures satisfy six compatibility equations which are the static-geometric duals of the equilibrium equations. The stress and strain measures are related by the stress strain relations

$$\bar{M}_{\xi\xi} = D(\bar{\kappa}_{\xi\xi} + \nu_b \bar{\kappa}_{\theta\theta}), \qquad \bar{M}_{\theta\theta} = D(\bar{\kappa}_{\theta\theta} + \nu_b \bar{\kappa}_{\xi\xi})$$

$$\bar{M}_{\xi\theta} = \bar{M}_{\theta\xi} = \frac{1}{2} D(1-\nu_b)\ (\bar{\kappa}_{\xi\theta} + \bar{\kappa}_{\theta\xi}) \tag{7.5}$$

$$\bar{\varepsilon}_{\theta\theta} = A(\bar{N}_{\theta\theta} - \nu_s \bar{N}_{\xi\xi}), \qquad \bar{\varepsilon}_{\xi\xi} = A(\bar{N}_{\xi\xi} + \nu_s \bar{N}_{\theta\theta})$$

$$\bar{\varepsilon}_{\xi\theta} = \bar{\varepsilon}_{\theta\xi} = \frac{1}{2} A(1+\nu_s)\ (\bar{N}_{\xi\theta} + \bar{N}_{\theta\xi})$$

along with $\gamma_\xi \equiv \gamma_\theta \equiv 0$ and $P_\xi \equiv P_\theta \equiv 0$ for the classical shell theory.

The four appropriate stress boundary conditions at an $r = r^*$ edge now take the form

$$r = r^*: \qquad \Delta\bar{N}_{\xi\xi} = \Delta\bar{N}_{\xi\theta} = \Delta\bar{Q}^e_\xi = \Delta\bar{M}_{\xi\xi} = 0 \tag{7.7}$$

where $\bar{Q}^e_\xi = \bar{Q}_\xi + \bar{M}_{\xi\theta}/r$. Because of the overall equilibrium requirement, boundary conditions at both edges of a shell frustum may not all be prescribed in terms of stress measures. The four appropriate displacement conditions at an $r = r^*$ edge are

$$r = r^*: \qquad \Delta\bar{u}_\xi = \Delta\bar{u}_\theta = \Delta\bar{w} = \Delta\bar{\phi}_\xi = 0 \tag{7.8}$$

They imply four strain boundary conditions which are the static-geometric duals of the four stress boundary conditions. For overall compatibility, all boundary conditions at the two edges of shell frustum may not all be prescribed in terms of the strain measures. Hence, a combination of stress and strain (or displacement) measures are to be prescribed at each edge of the shell frustum. The four first integral will in general determine the admissibility of a particular set of mixed edge conditions.

2. Reduction to Two Simultaneous Differential Equations

As in the case of nonshallow shells, overall equilibrium consideration leads to two first integrals of the differential equations of equilibrium:

$$\bar{N}_{\xi\xi} - \bar{N}_{\xi\theta} = \frac{1}{\pi r} P_x(r), \tag{7.9}$$

$$\bar{M}_{\xi\xi} - \bar{M}_{\xi\theta} - r(\bar{Q}_\xi + z'\bar{N}_{\xi\xi}) = \frac{1}{\pi r} [T_y - zP_x] \tag{7.10}$$

with

$$P_x = P_o - \pi \int_{r_i}^{r} (\bar{p}_\xi - \underline{z'\bar{p}_n} - \bar{p}_\theta)rdr \cong P_o - \pi \int_{r_i}^{r} (\bar{p}_\xi - \bar{p}_\theta)rdr$$

$$T_y = T_o - \pi \int_{r_i}^{r} [(z-rz')\bar{p}_\xi - z\bar{p}_\theta - (\underline{zz'} + r)\bar{p}_n]rdr \tag{7.11}$$

where P_o and T_o are two constants of integration. The underlined terms should be omitted for lateral load problems to be consistent with the definition of a shallow shell, since with $p_z \equiv 0$, we have $\bar{p}_n = -z'\bar{p}_r$ and $\bar{p}_\xi = \bar{p}_r$ for shallow shells. By the static-geometric duality, we have also two first integrals of the compatibility equations of the form.

$$-\bar{\kappa}_{\theta\theta} - \bar{\kappa}_{\theta\xi} = \frac{1}{\pi r} \Omega_x, \tag{7.12}$$

$$\varepsilon_{\theta\theta} + \varepsilon_{\theta\xi} - r(\lambda_\theta - z'\kappa_{\theta\theta}) = \frac{1}{\pi r} [U_y - z\Omega_x] \tag{7.13}$$

where Ω_x and U_y are two constants of integration whose geometrical interpretation can be found in chapter (6) section (3). The relations (7.12) and (7.13) can be verified by direct substitution of the expressions for the strain measures in terms of the displacement components. We will make use of these first integrals to reduce the system of equations for laterally loaded shallow shells of revolution to a canonical form similar to that for the axisymmetric bending problem.

Analogous to the axisymmetric bending case, we set

$$\bar{N}_{\xi\theta} = \frac{\Phi}{r}, \qquad \bar{\kappa}_{\theta\xi} = \frac{\phi}{r} \tag{7.14}$$

so that we have from the first integrals (7.9) and (7.12)

$$\bar{N}_{\xi\xi} = \frac{1}{r}\left(\Phi + \frac{P_x}{\pi}\right), \qquad -\bar{\kappa}_{\theta\theta} = \frac{1}{r}\left(\phi + \frac{\Omega_x}{\pi}\right) \tag{7.15}$$

Next, we get from the sixth equilibrium equation and the dual compatibility equation

$$\bar{N}_{\theta\xi} = \frac{\Phi}{r}, \qquad \bar{\kappa}_{\xi\theta} = \frac{\phi}{r} \tag{7.16}$$

and from the second equilibrium equation and its dual compatibility equation

$$\bar{N}_{\theta\theta} = \Phi' + \frac{1}{r}\Phi + rp_\theta, \qquad -\bar{\kappa}_{\xi\xi} = \phi' + \frac{1}{r}\phi \tag{7.17}$$

The stress strain relations then give

$$\bar{\varepsilon}_{\xi\theta} = \bar{\varepsilon}_{\theta\xi} = A(1+\nu_s)\frac{\Phi}{r}, \qquad M_{\xi\theta} = M_{\theta\xi} = D(1-\nu_b)\frac{\phi}{r} \tag{7.18}$$

$$\bar{M}_{\xi\xi} = -D[\phi' + (1+\nu_b)\frac{\phi}{r} + \nu_b\frac{\Omega_x}{\pi r}],$$

(7.19)

$$\bar{M}_{\theta\theta} = -D[\nu_b\phi' + (1+\nu_b)\frac{\phi}{r} + \frac{\Omega_x}{\pi r}]$$

$$\bar{\varepsilon}_{\theta\theta} = A[\Phi' + (1-\nu_s)\frac{\Phi}{r} - \nu_s\frac{P_x}{\pi r} + r\bar{p}_\theta]$$

$$\bar{\varepsilon}_{\xi\xi} = A\,[-\nu_s\Phi' + (1-\nu_s)\frac{\Phi}{r} + \frac{P_x}{\pi r} - \nu_s r\bar{p}\theta] \tag{7.20}$$

The first integrals (7.10) and (7.13) then give

$$r\bar{Q}_\xi = - D[\phi' + \frac{2}{r}\phi + \nu_b \frac{\Omega_x}{\pi r}] - z'\Phi + (z-z'r)\frac{P_x}{\pi r} - \frac{T_y}{\pi r}$$

$$r\bar{\lambda}_\theta = - A[\Phi' + \frac{2}{r}\Phi - \nu_s \frac{P_x}{\pi r} + r\bar{p}_\theta] - z'\phi + (z-z'r)\frac{\Omega_x}{\pi r} - \frac{U_y}{\pi r}$$

while $\bar{Q}_\theta$ and $\bar{\lambda}_\xi$ are given in terms of ϕ and Φ by the fifth equilibrium equation and the dual compatibility equation:

$$r\bar{Q}_\theta = D[\phi' + \frac{2}{r}\phi + \frac{\Omega_x}{\pi r}],$$

$$r\bar{\lambda}_\xi = A\,[\Phi' + \frac{2}{r}\Phi + \frac{P_x}{\pi r} - \nu_s r\bar{p}_\theta] \tag{7.22}$$

We now substitute the expressions for $\bar{M}_{\xi\xi}$, $\bar{M}_{\theta\theta}$, $\bar{M}_{\theta\xi}$ and $\bar{Q}_\xi$ into the fourth equilibrium equation to get

$$\phi'' + \frac{1}{r}\phi' - \frac{4}{r^2}\phi - \frac{z'}{rD}\Phi = (1+\nu_b)\frac{\Omega_x}{\pi r^2} + \frac{(rz'-z)}{D}\frac{P_x}{\pi r^2} + \frac{1}{D}\frac{T_y}{\pi r^2} \tag{7.23}$$

From the dual compatibility equation, we get a second equation for ϕ and Φ:

$$\Phi'' + \frac{1}{r}\Phi' - \frac{4}{r^2}\Phi + \frac{z'}{rA}\phi =$$

$$= (1-\nu_s)\frac{P_x}{\pi r^2} - \frac{(rz'-z)}{A}\frac{\Omega_x}{\pi r^2} - \frac{1}{A}\frac{U_y}{\pi r^2} - \nu_s\bar{p}_\xi - (r\bar{p}_\theta)' \tag{7.24}$$

Supplemented by appropriate boundary conditions, the two second order differential equations (7.23) and (7.24) determine $\Phi(r)$ and $\phi(r)$. The stress and strain measures of the shell can then be computed by way of equations (7.14) - (7.22). The displacement components can also be obtained by integrating (7.1) and (7.2).

3. Shallow Spherical Cap in a Face-Side Position

With the gravity vector in the direction of the x-axis and the axis of revolution of the shell in the direction of the z-axis, we have for a shallow spherical shell of uniform mass density, ρ, and thickness, h, $\vec{p} = \rho g h \vec{i}_x$ so that

$$p_\xi = \rho g h \cos\theta, \qquad p_\theta = -\rho g h \sin\theta, \qquad p_n = -\rho g h z' \cos\theta \tag{7.25}$$

where $z' = \xi_0(r/r_0)$ and ξ_0 and r_0 are known constants. For a spherical cap extending over the region $0 \leqslant r \leqslant r_0$, $0 \leqslant \theta \leqslant 2\pi$, the expressions

$$P_x(r) = -\pi\rho g h r^2, \qquad T_y(r) = -\frac{1}{4r_0}\pi\rho g h \xi_0 r^4$$

follow from the load components (7.25), and at the same time the condition of uni-valued displacement fields requires $\Omega_x = U_y = 0$.

The two coupled ODE's for ϕ and Φ for our shallow spherical cap take the form

$$\phi'' + \frac{1}{r}\phi' - \frac{4}{r^2}\phi - \frac{\xi_0}{Dr_0}\Phi = -\frac{3\rho g h \xi_0}{4r_0 D} r^2$$

$$\Phi'' + \frac{1}{r}\Phi' - \frac{4}{r^2}\Phi + \frac{\xi_0}{Ar_0}\phi = 0$$

An exact particular solution for the above system is

$$\Phi_p = \frac{3}{4}\rho g h r^2, \qquad \phi_p = 0$$

while the exact complementary solutions may be given in terms of Bessel functions. For the stress and displacement measures to be bounded at the appex, the general solution for ϕ and Φ may be taken in the form

$$\phi = c_1 \mathrm{ber}_2(\beta x) + c_2 \mathrm{bei}_2(\beta x)$$

$$\Phi = c_2 \mathrm{ber}_2(\beta x) - c_1 \mathrm{bei}_2(\beta x) + \frac{3}{4} \rho g h r^2$$

where $\beta^2 = \xi_0 r_0/\sqrt{DA}$, $x = r/r_0$ and ber_2 and bei_2 are the Kelvin functions of the first kind. For a cap clamped to a rigid support at its edge $r = r_0$, all displacement components and the rotation ϕ_ξ must vanish. We take these boundary conditions in the form $\bar{\kappa}_{\theta\theta} = 0$ and $\bar{\varepsilon}_{\theta\theta} = 0$ at $r = r_0$ or, from (7.15) and (7.20).

$$r = r_0: \qquad \phi = 0, \qquad \Phi' + \frac{1-\nu_s}{r} \Phi - \frac{\nu_s}{\pi r} P_x + r p_\theta = 0$$

Note that the remaining contracted strain boundary conditions are automatically satisfied once the conditions on $\bar{\kappa}_{\theta\theta}$ and $\bar{\varepsilon}_{\theta\theta}$ are satisfied. The two conditions on ϕ and a linear combination of Φ and Φ' at $r = r_0$ completely determine c_1 and c_2 and hence the solution of our problem.

4. The Side Force and Tilting Moment Problem for a Shell Frustum

Consider a shallow shell frustum extending over the region $r_i \leq r \leq r_0$ and $0 \leq \theta \leq 2\pi$, which is free of surface loads in the shell interior. The outer edge $r = r_0$ of the frustum is clamped to a rigid wall which is not allowed to move. The inner edge $r = r_i$ is clamped to a movable rigid plug. The plug is subject to a side force P_0 in the x-direction and/or a tilting moment T_0 turning about the y-axis to produce a lateral plug displacement δ_x and a plug rotation β_y. Of interest are the relations between δ_x and β_y on the one hand and P_0 and T_0 on the other hand, as well as the stress distributions in the shell. The formulation of section (2) of this chapter with $\bar{p}_\xi \equiv \bar{p}_\theta \equiv \bar{p}_n \equiv 0$ and $\Omega_x = U_y = 0$ is applicable to this problem.

The displacement boundary conditions at the edges for our problem are

$$r = r_0: \qquad \phi_\xi = u_\theta = u_\xi = w = 0 \tag{7.26}$$

$$\begin{aligned} r = r_i: \qquad & \phi_\xi = \beta_y \cos\theta, \quad u_\theta = -(\delta_x + z\beta_y)\sin\theta \\ & u_r = (\delta_x + z\beta_y)\cos\theta, \quad u_z = -r\beta_y\cos\theta \end{aligned} \tag{7.27}$$

where u_r and u_z are the radial and axial displacement component, respectively. These conditions may of course be expressed in terms of r-dependent quantities $\bar{\Phi}_\xi$, $\bar{u}_\theta$, etc. As displacement components are integrals of strain and curvature change measures (see (6.12)), the four displacement boundary conditions at each edge of the shell frustum would generally give rise to four conditions involving integrals of the unknown ϕ and Φ. For laterally loaded shallow shells of revolution, the integrated strain-displacement relations (6.12) simplify to read

$$\bar{\phi}_\xi = -\int_r^{r_o} \bar{\kappa}_{\xi\xi} dr, \quad \bar{\phi}_\theta = r\bar{\kappa}_{\theta\theta} - \bar{\phi}_\xi, \quad \bar{\omega} = r\bar{\lambda}_\theta - z'\bar{\phi}_\xi$$

$$\bar{u}_\theta = r\bar{\varepsilon}_{\theta\theta} - z\bar{\phi}_\xi + \int_r^{r_o} (\bar{\varepsilon}_{\xi\xi} - z\bar{\kappa}_{\xi\xi}) dr$$

$$\bar{u}_\xi = r^2\bar{\lambda}_\theta - r\bar{\varepsilon}_{\theta\xi} - rz'\bar{\phi}_\xi - \bar{u}_\theta, \quad \bar{w} = r^2\bar{\kappa}_{\theta\theta} - r\bar{\phi}_\xi$$

where we have incorporated the univalued displacement condition (by setting $\Omega_x = U_y = 0$) and have chosen the constants of integration in anticipation of the application of the displacment boundary conditions (7.26) at $r = r_o$.

It turns out that we can avoid the unattractive requirement of carrying out the integration involved in (7.28) in order to determine the actual stress distributions in the shell if P_o and T_o are prescribed. The eight displacement boundary conditions (7.26) and (7.27) are evidently equivalent to the two integrated conditions

$$-\int_{r_i}^{r_o} \bar{\kappa}_{\xi\xi} dr = \beta_y, \qquad -\int_{r_i}^{r_o} (\bar{\varepsilon}_{\xi\xi} - z\bar{\kappa}_{\xi\xi}) dr = \delta_x \tag{7.29}$$

and four local conditions $\bar{\lambda}_\theta - \bar{\varepsilon}_{\theta\xi}/r = 0$ and $\bar{\varepsilon}_{\theta\theta} = 0$ at $r = r_i$ and $r = r_o$. Note that the conditions $\bar{\kappa}_{\theta\theta} = 0$ at $r = r_i$ and $r = r_o$ follow from the above four local conditions and the first integral of the compatibility equations (7.13). In view of (7.12), we use the equivalent local conditions

$$r = r_i, r_o: \qquad \bar{\kappa}_{\theta\xi} = 0, \qquad \bar{\varepsilon}_{\theta\theta} = 0 \tag{7.30}$$

instead as they are more simply expressed as conditions on ϕ and Φ:

$$r = r_i, r_o: \qquad \phi = 0, \qquad \Phi' + \frac{1}{r}(1-\nu_s)\Phi - \frac{\nu_s}{\pi r} P_o = 0 \tag{7.30'}$$

The boundary value problem defined by the differential equations (7.23) and

(7.24) (with $U_y = \Omega_x = 0$, $P_x = P_o$ and $T_y = T_o$) and the four boundary conditions (7.30') determine ϕ and Φ in terms of P_o and T_o. The linearity of the problem allows us to write

$$\phi = P_o\phi_P + T_o\phi_T, \qquad \Phi = P_o\Phi_P + T_o\Phi_T \tag{7.31}$$

where ϕ_P, etc., are independent of P_o and T_o. Thus, if the applied force and moment are known, Φ and ϕ and therefore the stress distributions of the shell are completely determined without the explicit solution of the displacement components.

The two integrated conditions (7.29) relate P_o and T_o to the plug displacement δ_x and plug rotation β_y. Upon expressing $\bar{\kappa}_{\xi\xi}$ and $\bar{\varepsilon}_{\xi\xi}$ in terms of ϕ and Φ and carrying out the integration with respect to r, we get two linear flexibility relations

$$\delta_x = P_o B_{\delta P} + T_o B_{\delta T}, \qquad \beta_y = P_o B_{\beta P} + T_o B_{\beta T} \tag{7.32}$$

where

$$\{B_{\beta P}, B_{\beta T}\} = \int_{r_i}^{r_o} \{(r\phi_P)', (r\phi_T)'\} \frac{1}{r}\, dr$$

$$B_{\delta P} = -\int_{r_i}^{r_o} \{A[-\nu_s \Phi_P' + (1-\nu_s)\frac{1}{r}\Phi_P + \frac{1}{\pi r}] + \frac{z}{r}(r\phi_P)'\}dr$$

$$B_{\delta T} = -\int_{r_i}^{r_o} \{A[-\nu_s \Phi_T' + (1-\nu_s)\frac{1}{r}\Phi_T] + \frac{z}{r}(r\phi_T)'\}dr$$

are independent of P_o and T_o. For a given shell, the flexibility coefficients $B_{\delta P}$, $B_{\delta T}$, $B_{\beta P}$ and $B_{\beta T}$ may be determined once and for all. Given P_o and T_o, the flexibility relations (7.32) determine δ_x and β_y. Conversely, given δ_x and β_y, they serve to determine the force P_o and moment T_o needed to produce them. For sufficiently thin shells so that their Reissner's number β is large compared to unity, it can be shown that the actual integration may be avoided in the evaluation of the flexibility coefficients (see [7]).

Part IV - Shells of Revolution Under Arbitrary Loading

8. Shallow Shells of Revolution

1. Shallow Shell Equations

Under the order of magnitude relations of the shallow shell theory adopted in chapter 4, the strain displacement relations for shells of revolution given in section (2) of chapter 2 simplify considerably. With $\xi_1 = r$, we have

$$\gamma_\xi = \phi_\xi + w' \quad , \qquad \gamma_\theta = \phi_\theta + \frac{1}{r} w^{\cdot}$$

$$\kappa_{\xi\xi} = \phi_\xi' \quad , \quad \kappa_{\xi\theta} = \phi_\theta' \quad , \quad \kappa_{\theta\xi} = \frac{1}{r}(\phi_\xi^{\cdot} - \phi_\theta) \quad , \quad \kappa_{\theta\theta} = \frac{1}{r}(\phi_\theta^{\cdot} + \phi_\xi)$$

$$\lambda_\xi = \omega' - z''\phi_\theta \quad , \qquad \lambda_\theta = \frac{1}{r}(\omega^{\cdot} + z'\phi_\xi)$$

$$\varepsilon_{\xi\xi} = u_\xi' - z''w \quad , \qquad \varepsilon_{\theta\theta} = \frac{1}{r}(u_\theta^{\cdot} + u_\xi - z'w)$$

$$\varepsilon_{\xi\theta} = u_\theta' - \omega \quad , \qquad \varepsilon_{\theta\xi} = \frac{1}{r}(u_\xi^{\cdot} - u_\theta) + \omega \quad .$$

The corresponding equations of equilibrium for the stress measures are

$$(rN_{\xi\xi})' + N_{\theta\xi}^{\cdot} - N_{\theta\theta} + rp_\xi = 0 \quad , \quad (rN_{\xi\theta})' + N_{\theta\theta}^{\cdot} + N_{\theta\xi} + rp_\theta = 0 \quad ,$$

$$(rQ_\xi)' + Q_\theta^{\cdot} + rz''N_{\xi\xi} + z'N_{\theta\theta} + rp_n = 0 \quad ,$$

$$(rM_{\xi\xi})' + M^{\cdot}_{\theta\xi} - M_{\theta\theta} - z'P_\theta - rQ_\xi = 0 \quad , \quad (rM_{\xi\theta})' + M^{\cdot}_{\theta\theta} + M_{\theta\xi} + rz''P_\xi - rQ_\theta = 0 \quad ,$$

$$(rP_\xi)' + P^{\cdot}_\theta + r(N_{\xi\theta} - N_{\theta\xi}) = 0$$

where, for simplicity, we have assumed the absence of surface moment load components so that $q_\xi = q_\theta = q_n = 0$. These equilibrium equations are consistent with our strain displacement relations for shallow shells of revolution so that the virtual work axiom is satisfied.

The equilibrium equations and strain displacement relations are supplemented by the twelve stress strain relations given in chapter 2. With suitably prescribed boundary conditions, this set of shallow shell equations defines a well-posed boundary value problem in linear PDE. Along an r = constant edge, the stress boundary conditions are the same as those for nonshallow shells. For the classical shell theory, the appropriate contracted stress boundary conditions consistent with the principle of virtual work are

$$r = r^* : \qquad \Delta N_{\xi\xi} = \Delta N_{\xi\theta} = \Delta Q^e_\xi = \Delta M_{\xi\xi} = 0$$

with $Q^e_\xi = Q_\xi + r^{-1} M^{\cdot}_{\xi\theta}$. Evidently the boundary condition on the effective shear resultant $N^e_{\xi\theta} = N_{\xi\theta} + M_{\xi\theta}/R_{\theta\theta}$ is replaced by $N_{\xi\theta}$ itself. The displacement boundary conditions along an r = constant edge are the same as those for nonshallow shells of revolution. For the classical theory, the corresponding strain boundary conditions are

$$r = r^* : \qquad \Delta\kappa_{\theta\theta} = \Delta\kappa_{\theta\xi} = \Delta\lambda^e_\theta = \Delta\varepsilon_{\theta\theta} = 0$$

with $\lambda_\theta^e = \lambda_\theta - r^{-1}\dot{\varepsilon}_{\theta\xi}$. The condition on $\kappa_{\theta\xi}^e = \kappa_{\theta\xi} - \varepsilon_{\theta\xi}/R_{\theta\theta}$ for nonshallow shells is replaced by a condition on $\kappa_{\theta\xi}$ itself.

The twelve strain measures defined in terms of the six displacement components satisfy six compatibility equations which are the static geometric duals of the equilibrium equations. The equilibrium equations may be satisfied identically by suitable stress function representations of the stress measures. Except for particular solutions associated with the surface load terms, the stress function representations are the static geometric duals of the strain displacement relations. These additional relations will also be useful in the reduction of the boundary value problem to its canonical form.

2. Reduction of Equations for the Classical Theory

With $\gamma_\xi = \gamma_\theta = 0$, we have

$$\phi_\xi = -w' \quad , \qquad \phi_\theta = -\frac{1}{r}w^{\bullet} \tag{8.1}$$

$$\kappa_{\xi\xi} = -w'' \quad , \quad \kappa_{\xi\theta} = -\left(\frac{w^{\bullet}}{r}\right)' = \kappa_{\theta\xi} \quad , \quad \kappa_{\theta\theta} = -\left(\frac{1}{r}w' + \frac{1}{r^2}w^{\bullet\bullet}\right) \tag{8.2}$$

These relations are identical to the corresponding relations for the transverse bending of flat plates in polar coordinates. With the three moment equilibrium equations also identical to the flat plate case when $P_\xi \equiv P_\theta = 0$, we get from these equations

$$N_{\xi\theta} = N_{\theta\xi} \quad , \quad Q_\xi = -D(\nabla^2 w)' \quad , \quad Q_\theta = -r^{-1}D(\nabla^2 w)^{\bullet} \tag{8.3}$$

where $\nabla^2(\ ) = (\ )'' + r^{-1}(\ )' + r^{-2}(\ )^{\bullet\bullet}$. The expressions for Q_ξ and Q_θ may be used in the third equilibrium equations. However, unlike the case of a flat plate, this equation now involves the stress resultant $N_{\xi\xi}$ and $N_{\theta\theta}$ and thus cannot be an equation for w alone.

As for a flat sheet, the stress resultants may be expressed in terms of the Airy stress function F through the static-geometric dual of the development leading to the expression for $\kappa_{\xi\xi}$, etc., the only difference being the appearance of inhomogeneous terms associated with the surface load intensities. If p_ξ and p_θ can be expressed in terms of a load potential* L with

*The load potential was denoted by $-\Omega$ for the flat plate case so that $L = -\Omega$ for flat plates.

$$p_\xi = L' \quad , \quad p_\theta = r^{-1}L^{\cdot} \quad ,$$

then the first two equilibrium equations are satisfied identically by setting

$$N_{\xi\xi} = (\frac{1}{r}F' + \frac{1}{r^2}F^{\cdot\cdot}) - L \quad , \quad N_{\theta\theta} = F'' - L$$
$$N_{\xi\theta} = N_{\theta\xi} = -(\frac{F^{\cdot}}{r})' \quad . \tag{8.4}$$

Now the remaining force equilibrium equation becomes a single fourth order PDE for w and F :

$$+D\nabla^2\nabla^2 w = p_n + L[F;z] - (z'' + \frac{1}{r}z')L \tag{8.5}$$

with

$$L[f,g] = (\frac{1}{r}g' + \frac{1}{r^2}g^{\cdot\cdot})f'' + (\frac{1}{r}f' + \frac{1}{r^2}f^{\cdot\cdot})g'' \quad , \tag{8.6}$$

keeping in mind that z is independent of θ for shells of revolution.

The dual development of the compatibility equations gives a second PDE for w and F . The fourth and fifth compatibility equations give

$$\lambda_\theta = A[(\nabla^2 F) - (1 - \nu_s)L]' \quad , \quad \lambda_\xi = -\frac{A}{r}[\nabla^2 F - (1 - \nu_s)L]^{\cdot} \quad . \tag{8.7}$$

The third compatibility equation may then be written as

$$-A\nabla^2\nabla^2 F = L[w;z] - A(1 - \nu_s)\nabla^2 L \quad . \tag{8.8}$$

If $z \equiv 0$, the two equations (8.5) and (8.8) uncouple and become

$$D\nabla^2\nabla^2 w = p_n \quad , \qquad +A\nabla^2\nabla^2 F = A(1 - \nu_s)\nabla^2 L$$

which are the governing PDE for plate bending and for generalized plane stress, respectively.

The two PDEs (8.5) and (8.8) may be combined into a single fourth order complex PDE for a complex potential $X = w + i\sqrt{A/D}\, F$:

$$\nabla^2\nabla^2 X + \frac{i}{\sqrt{DA}} L[X;z] = \frac{1}{D}(p_n - \nabla^2 L) + i\sqrt{\frac{A}{D}}\,(1 - \nu_s)\nabla^2 L$$

In dimensionless coordinates, the term $L[X;z]$ would be multiplied by the Reissner number which is large compared to unity. The PDE for X therefore has the structure of a singular perturbation problem as L is a second order differential operator while $\nabla^2\nabla^2$ is a fourth order differential operator.

3. Shallow Spherical Shells

For a shallow spherical shell frustum, we have $z' = \xi_0 r/r_0$, $r_i \le r \le r_0$. With $x = r/r_0$ so that $r_i/r_0 = x_i \le x \le 1$, the partial differential equations for X becomes

$$\Delta^2 X + i\beta^2 \Delta X = \frac{1}{D}[p_n - 2\frac{\xi_0}{r_0}L] + i\sqrt{\frac{A}{D}}\,(1 - \nu_s)L$$

where $\beta^2 = \xi_0 r_0/\sqrt{DA}$ and $\Delta(\ ) \equiv (\ )_{,xx} + x^{-1}(\ )_{,x} + x^{-2}(\ )_{,\theta\theta}$ is the two-dimensional Laplacian in dimensionless polar coordinates. The complementary solution X_c of the above equation may be written as a sum of X_0 and X_β :

$$X_c = X_0 + X_\beta$$

with

$$\Delta X_\alpha + i\alpha^2 X_\alpha = 0 \qquad (\alpha = 0, \beta) \quad .$$

For a shell frustum complete in the circumferential direction and subject to self-equilibrating surface and/or edge loads, it suffices to consider the case $X = \bar{X}(x)\cos(n\theta)$. In that case, we have $\bar{X} = \bar{X}_0(x) + \bar{X}_\beta(x)$ where

$$\bar{X}''_\alpha + \frac{1}{x}\bar{X}'_\alpha - \frac{n^2}{x^2}\bar{X}_\alpha + i\alpha^2\bar{X}_\alpha = 0 \qquad (\alpha = 0, \beta)$$

or

$$\bar{X}_0 = (A_r + iA_i)x^n + (B_r + iB_i)x^{-n}$$

$$\bar{X}_\beta = (C_r + iC_i)[\mathrm{ber}_n(\beta x) - i\,\mathrm{bei}_n(\beta x)]$$

$$+ (D_r + iD_i)[\mathrm{ker}_n(\beta x) - i\,\mathrm{kei}_n(\beta x)] \quad .$$

The eight real constants of integration A_r , A_i , B_r , B_i , etc. are to be determined by relevant boundary conditions at the circular edges of the shell frustum. Boundary value problems of this type have been treated in [19-24] and will not be discussed further. We only note here that exact solutions are also possible for shallow conical shells and shallow logarithmic shells (with $z' = \xi_0 r_i / r$) and that asymptotic methods of solution are applicable to cases where the Reissner number β is large compared to unity.

9. Spherical Shells

1. Governing Equations for Spherical Shells

As before, we let ξ be the slope angle between the tangent to a meridian and the horizon. Take the cylindrical coordinates of a point on a (nonshallow) spherical middle surface in the form

$$r = a\sin\xi \qquad z = -a\cos\xi \qquad (\xi_i \le \xi \le \xi_0)$$

with $0 \le \xi_i < \xi_0 \le \pi$. With $\xi_1 = \xi$ and $\xi_2 = \theta$, we have

$$r' = a\cos\xi \quad , \quad a' = a\sin\xi \quad , \quad r^{\bullet} = z^{\bullet} = 0$$

$$\alpha_1 = \sqrt{(r')^2 + (z')^2} = a \quad , \quad \alpha_2 = r = a\sin\xi$$

$$\frac{1}{R_{11}} = -\frac{\xi'}{\alpha_1} = -\frac{1}{a} \quad , \quad \frac{1}{R_{22}} = -\frac{\sin\xi}{r} = -\frac{1}{a} \quad , \quad \frac{1}{R_{12}} = \frac{1}{R_{21}} = 0$$

where $(\ )' \equiv \partial(\ )/\partial\xi_1 \equiv \partial(\ )/\partial\xi$, and $(\ )^{\bullet} = \partial(\ )/\partial\theta$.

For the surface coordinates (ξ,θ) , the strain displacement relations for a spherical shell take the form

$$\gamma_\xi = \phi_\xi + \frac{1}{a}(w' + u_\xi) \quad , \quad \gamma_\theta = \phi_\theta + \frac{1}{a}(\csc\xi w^{\bullet} + u_\theta)$$

$$\kappa_{\xi\xi} = \frac{1}{a}\phi_\xi' \quad , \quad \kappa_{\theta\theta} = \frac{1}{a}(\csc\xi\phi_\theta^{\bullet} + \cot\xi\phi_\xi)$$

$$\kappa_{\xi\theta} = \frac{1}{a}(\phi_\theta' + \omega) \quad , \quad \kappa_{\theta\xi} = \frac{1}{a}(\csc\xi\dot{\phi}_\xi - \cot\xi\phi_\theta - \omega)$$

$$\varepsilon_{\xi\xi} = \frac{1}{a}(u_\xi' - w) \quad , \quad \varepsilon_{\theta\theta} = \frac{1}{a}(\csc\xi\dot{u}_\theta + \cot\xi u_\xi - w)$$

$$\varepsilon_{\xi\theta} = \frac{1}{a}u_\theta' - \omega \quad , \quad \varepsilon_{\theta\xi} = \frac{1}{a}(\csc\xi\dot{u}_\xi - \cot\xi u_\theta) + \omega$$

$$\lambda_\xi = \frac{1}{a}(\omega' - \phi_\theta) \; , \quad \lambda_\theta = \frac{1}{a}(\csc\xi\dot{\omega} + \phi_\xi) \quad .$$

The equilibrium equations take the form

$$(rN_{\xi\xi})' + a\dot{N}_{\theta\xi} - r'N_{\theta\theta} - rQ_\xi + rap_\xi = 0$$

$$(rN_{\xi\theta})' + a\dot{N}_{\theta\theta} + r'N_{\theta\xi} - rQ_\theta + rap_\theta = 0$$

$$(rQ_\xi)' + a\dot{Q}_\theta + r(N_{\xi\xi} + N_{\theta\theta}) + rap_n = 0$$

$$(rM_{\xi\xi})' + a\dot{M}_{\theta\xi} - r'M_{\theta\theta} - raQ_\xi - rP_\theta + raq_\xi = 0$$

$$(rM_{\xi\theta})' + a\dot{M}_{\theta\theta} + r'M_{\theta\xi} - raQ_\theta + rP_\xi + raq_\theta = 0$$

$$(rP_\xi)' + a\dot{P}_\theta + ra(N_{\xi\theta} - N_{\theta\xi}) - r(M_{\xi\theta} - M_{\theta\xi}) + raq_n = 0 \quad .$$

The stress and strain measures are related by the system of stress-strain relations given in chapter 2. Supplemented by appropriate boundary conditions these equations define a well-posed boundary value problem in linear PDE. The solution of this BVP provides a description of the elastostatics of the spherical shell. For the solution process, additional relations such as compatibility equations and the stress function representation of the solution of equilibrium equations will also be useful. The compatibility equations are the static-geometric duals of the equilibrium equations while the stress function representation take the form

$$N_{\xi\xi} = -\frac{1}{a}(\csc\xi\Phi^{\cdot}_{\theta} + \cot\xi\Phi_{\xi}) + N_{\xi p} \quad , \quad N_{\theta\xi} = \frac{1}{a}(\Phi'_{\theta} + \Omega)$$

$$N_{\theta\theta} = -\frac{1}{a}\Phi'_{\xi} + N_{\theta p} \quad , \quad N_{\xi\theta} = \frac{1}{a}(\csc\xi\Phi^{\cdot}_{\xi} - \cot\xi\Phi_{\theta} - \Omega)$$

$$Q_{\xi} = \frac{1}{a}(\csc\xi\Omega^{\cdot} + \Phi_{\xi}) + Q_{\xi p} \quad , \quad \cdots\cdot$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \quad , \quad \cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad , \quad M_{\theta\theta} = \frac{1}{a}(U'_{\xi} - F)$$

where quantities with a subscript p denote any particular solution of the equilibrium equations for the given surface loads. If we have

$$p_{\xi} = \frac{\partial L}{\partial \xi} \quad , \quad p_{\theta} = \frac{1}{r}\frac{\partial L}{\partial \theta}$$

then it is not difficult to see that we may take

$$N_{\xi p} = N_{\theta p} = -L$$

as a part of the particular solution. Apart from the particular solution terms, the stress function solutions are the static-geometric duals of the strain-displacement relations.

2. Reduction for the Classical Theory

With $\gamma_\xi = \gamma_\theta = 0$, we have as usual

$$\phi_\xi = -\frac{1}{a}(w' + u_\xi) \quad , \quad \phi_\theta = -\frac{1}{a}(\csc\xi w^{\cdot} + u_\theta)$$

which in turn give $\kappa_{\xi\xi}$, $\kappa_{\xi\theta}$, $\kappa_{\theta\xi}$ and $\kappa_{\theta\theta}$ in terms of u_ξ , u_θ , w , and ω (and their derivatives). It turns out that for spherical shells, the expressions for the strain resultants ($\varepsilon_{\xi\xi}$, $\varepsilon_{\theta\xi}$, $\varepsilon_{\xi\theta}$, $\varepsilon_{\theta\theta}$) may be used to eliminate u_ξ , u_θ and ω from the expressions for the four strain couples to get

$$\kappa_{\xi\xi} = -\frac{1}{a^2}(w'' + w) - \frac{1}{a}\varepsilon_{\xi\xi} \quad ,$$

$$\kappa_{\theta\theta} = -\frac{1}{a^2}(\csc^2\xi w^{\cdot\cdot} + \cot\xi w' + w) - \frac{1}{a}\varepsilon_{\theta\theta} \quad ,$$

$$\{\kappa_{\xi\theta}, \kappa_{\theta\xi}\} = -\frac{1}{a^2}\csc\xi(w'^{\cdot} - \cot\xi w^{\cdot}) - \frac{1}{a}\{\varepsilon_{\xi\theta}, \varepsilon_{\theta\xi}\} \quad .$$

By the static-geometric dual development, we have also the following stress function representation for the stress resultants as consequences of $P_\xi = P_\theta = 0$ and the stress function solutions for the stress couples:

$$N_{\theta\theta} = \frac{1}{a^2}(F'' + F) + \frac{1}{a}(M_{\theta\theta} - M_{\theta p}) - L$$

$$N_{\xi\xi} = \frac{1}{a^2}(\csc^2\xi F^{\cdot\cdot} + \cot\xi F' + F) + \frac{1}{a}(M_{\xi\xi} - M_{\xi p}) - L$$

$$\{N_{\xi\theta}, N_{\theta\xi}\} = \frac{1}{a^2}\csc\xi(F'^{\cdot} - \cot\xi F^{\cdot}) + \frac{1}{a}\{M_{\xi\theta}, M_{\theta\xi}\} + N_p - \frac{1}{a}M_p \quad .$$

Next, we use the stress-strain relations to get $M_{\xi\xi}$, $M_{\theta\theta}$, $M_{\xi\theta}$, $M_{\theta\xi}$ and their dual strain measures in terms of F and w alone. From

$$M_{\xi\theta} = M_{\theta\xi} = \frac{1}{2}D(1-\nu_b)(\kappa_{\xi\theta}+\kappa_{\theta\xi}) \ , \quad \varepsilon_{\theta\xi} = \varepsilon_{\xi\theta} = \frac{1}{2}A(1+\nu_s)(N_{\xi\theta}+N_{\theta\xi}) \ ,$$

we get with the help of results already obtained in this section

$$M_{\xi\theta} + \frac{1}{a}D(1-\nu_b)\varepsilon_{\theta\xi} = -\frac{1}{a^2}D(1-\nu_b)\csc\xi(w'^{\cdot} - \cot\xi w^{\cdot})$$

$$\frac{1}{a}A(1+\nu_s)M_{\xi\theta} - \varepsilon_{\theta\xi} = \frac{1}{a^2}A(1+\nu_s)\{\csc\xi(F'^{\cdot} - \cot\xi F^{\cdot}) - a^2N_p + aM_p\} \quad .$$

These two equations may be solved for $M_{\xi\theta}$ and $\varepsilon_{\theta\xi}$ giving us

$$M_{\xi\theta} = M_{\theta\xi} = \frac{1}{1+\varepsilon^2}\{-\frac{1}{a^2}D(1-\nu_b)\csc\xi(w'^{\cdot} - \cot\xi w^{\cdot})$$
$$+ \frac{1}{a}\varepsilon^2\csc\xi(F'^{\cdot} - \cot\xi F^{\cdot}) - \varepsilon^2(aN_p - M_p)\}$$

$$\varepsilon_{\theta\xi} = \varepsilon_{\xi\theta} = \frac{1}{1+\varepsilon^2}\{\frac{1}{a^2}A(1+\nu_s)[\csc\xi(F'^{\cdot} - \cot\xi F^{\cdot}) - a^2N_p + aM_p]$$
$$+ \frac{1}{a}\varepsilon^2\csc\xi(w'^{\cdot} - \cot\xi w^{\cdot})\}$$

where $\varepsilon^2 = DA(1-\nu^2)/a^2 = h^2/12a^2 << 1$. Similarly, we get from the remaining four stress strain relations

$$(1+\varepsilon^2)M_{\xi\xi} = -\frac{1}{a^2}D[(w''+w) + \nu_b(\csc^2\xi w^{\cdot\cdot} + \cot\xi w' + w)]$$
$$-\frac{1}{a}\varepsilon^2[\csc^2\xi F^{\cdot\cdot} + \cot\xi F' + F] + \varepsilon^2(aL + M_{\xi p})$$

$$(1+\varepsilon^2)M_{\theta\theta} = -\frac{1}{a^2}D[\nu_b(w''+w) + (\csc^2\xi \ddot{w} + \cot\xi w' + w)]$$

$$-\frac{1}{a}\varepsilon^2[F'' + F] + \varepsilon^2(aL + M_{\theta p})$$

$$(1+\varepsilon^2)\varepsilon_{\theta\theta} = \frac{1}{a^2}A[(F'' + F) - \nu_s(\csc^2\xi \ddot{F} + \cot\xi F' + F)]$$

$$-\frac{1}{a}\varepsilon^2[\csc^2\xi \ddot{w} + \cot\xi w' + w] - A[(1-\nu_s)L + \frac{1}{a}D(1-\nu^2)\kappa_{\theta p}]$$

$$(1+\varepsilon^2)\varepsilon_{\xi\xi} = \frac{1}{a^2}A[(\csc^2\xi \ddot{F} + \cot\xi F' + F) - \nu_s(F'' + F)]$$

$$-\frac{1}{a}\varepsilon^2[w'' + w] - A[(1 - \nu_s)L + \frac{1}{a}D(1 - \nu^2)\kappa_{\xi p}]$$

where $\kappa_{\xi p} = (M_{\xi p} - \nu_b M_{\theta p})/D(1-\nu^2)$ and $\kappa_{\theta p} = (M_{\theta p} - \nu_b M_{\xi p})/D(1-\nu^2)$.

With $M_{\xi\theta} = M_{\theta\xi}$, the stress function solutions for $N_{\xi\theta}$ and $N_{\theta\xi}$ are identical, so that we have $N_{\xi\theta} = N_{\theta\xi}$. Given $q_n = 0$ and $P_\xi = P_\theta = 0$ in the classical theory, the sixth equilibrium equation is satisfied identically. Similarly, we have from $\varepsilon_{\xi\theta} = \varepsilon_{\theta\xi}$ the symmetry condition $\kappa_{\xi\theta} = \kappa_{\theta\xi}$ so that the sixth compatibility equation is also satisfied identically. The remaining two moment equilibrium equations (with $P_\xi = P_\theta = q_\xi = q_\theta = 0$) give Q_ξ and Q_θ in terms of w and F and surface load terms. Remarkably, all F terms cancel out in these expressions for the case of uniform shell thickness and material properties leaving us with

$$(1 + \varepsilon^2)Q_\xi = -\frac{1}{a^3}D[\nabla^2 w + 2w]' + \varepsilon^2[L' + \frac{1}{a}M'_{\xi p} + \frac{1}{a}\cot\xi(M_{\xi p} - M_{\theta p})$$

$$+ \frac{1}{a}\csc\xi(\dot{M}_p - a\dot{N}_p)]$$

$$(1+\varepsilon^2)Q_\theta = -\frac{1}{a^3}\csc\xi D[\nabla^2 w + 2w]^{\cdot} + \varepsilon^2[\frac{1}{a}(M_p' - aN_p') + \frac{2}{a}\cot\xi(M_p - aN_p)$$

$$+ \frac{1}{a}\csc\xi(aL^{\cdot} + \dot{M}_{\theta p})]$$

where ∇^2 is the dimensionless Laplace's operator in surface coordinates (ξ,θ) :

$$\nabla^2(\ ) = (\ )'' + \cot\xi(\ )' + \csc^2\xi(\ )^{\cdot\cdot} \quad .$$

The expressions for strain resultants and stress couples obtained above can also be used in the expressions for the strain couples and stress resultants to get these quantities in terms of w and F alone:

$$(1+\varepsilon^2)N_{\xi\xi} = \frac{1}{a^2}(\csc^2\xi F^{\cdot\cdot} + \cot\xi F' + F) - (L + \frac{1}{a}M_{\xi p})$$

$$- \frac{1}{a^3}D[(w'' + w) + \nu_b(\csc^2\xi w^{\cdot\cdot} + \cot\xi w' + w)]$$

$$(1+\varepsilon^2)N_{\theta\theta} = \frac{1}{a^2}(F'' + F) - (L + \frac{1}{a}M_{\theta p})$$

$$- \frac{1}{a^3}D[\nu_b(w'' + w) + (\csc^2\xi w^{\cdot\cdot} + \cot\xi w' + w)]$$

$$(1+\varepsilon^2)\kappa_{\xi\xi} = -\frac{1}{a^2}(w'' + w) + \frac{A}{a}[(1-\nu_s)L + \frac{1}{a}D(1-\nu^2)\kappa_{\xi p}]$$

$$- \frac{A}{a^3}[(\csc^2\xi F^{\cdot\cdot} + \cot\xi F' + F) - \nu_s(F'' + F)]$$

$$(1+\varepsilon^2)\kappa_{\theta\theta} = -\frac{1}{a^2}(\csc^2\xi w^{\cdot\cdot} + \cot\xi w + w) + \frac{A}{a}[(1-\nu_s)L + \frac{1}{a}D(1-\nu^2)\kappa_{\theta p}]$$

$$- \frac{A}{a^3}[(F'' + F) - \nu_s(\csc^2\xi F^{\cdot\cdot} + \cot\xi F' + F)]$$

With Q_ξ , Q_θ , $N_{\xi\xi}$ and $N_{\theta\theta}$ all expressed in terms of w and F , the third (force) equilibrium equation now becomes a PDE for w and F . In the absence of surface loads, this equation takes the form

$$(\nabla^2 + 2)\{[\nabla^2 + (1 + \nu_b)]w - \frac{a}{D}F\} = 0 \quad .$$

By the static-geometric duality, we have from the third compatibility equation a second PDE for w and F . Assuming again the absence of surface loads for simplicity, this second equation is evidently the static-geometric dual of the first equation:

$$(\nabla^2 + 2)\{[\nabla^2 + (1 - \nu_s)]F + \frac{a}{A}w\} = 0 \quad .$$

The appearance of Poisson's ratio notwithstanding, these two fourth order PDEs for w and F can be combined into a single fourth order complex PDE for a complex potential $X = w + \lambda F$. It can be readily seen from this complex equation that, for spherical shells complete in the circumferential direction, the solution for X (and therefore for w and F) may be obtained in terms of associated Legendre functions of real and complex degrees. Furthermore, the results are consistent with previously obtained results for axisymmetrically and laterally loaded spherical shells (provided we include suitably multivalued stress functions which lead to axial force, axial torque, side force and tilting moment). It is also possible to eliminate the Poisson's ratio term from both PDEs for w and F by using a modified stress function $\tilde{F} = F + D(1 - \nu_b)w/a$ instead of F. The elimination is achieved at a price, namely, a loss of the static-geometric duality between the two governing PDEs. The details of these observations will be left as exercises.

10 - Circular Cylindrical and Conical Shells

1. Equations for Higher Harmonics of the Shell Response

To the extent that we have completed the analysis for shells of revolution with axisymmetric loads and with lateral loads, it suffices to consider only the solution process for a typical higher harmonic in the Fourier decomposition of the shell response mentioned in Chapter 1 of these notes. For $n \geq 2$, the n-th Fourier component of the load distribution is self-equilibrating and there exists no first integral to reduce the order of the shell equations, as there are for the $n = 0$ case and the $n = 1$ case previously analyzed. Thus, it is appropriate to seek for the classical theory a reduction of the shell equations to two simultaneous fourth order differential equations for the transverse displacement component and its dual stress function analogous to those obtained in the last two chapters for shallow shells and for (non-shallow) spherical shells, respectively.

For the purpose of the contemplated reduction, we choose to satisfy all six equilibrium equations by way of six stress functions as indicated in chapter 2. Along with their dual strain displacement relations, the stress function representations transform the stress strain relations of Chapter 2 into twelve first order differential equations for the stress functions and their dual displacement components. The contemplated reduction can then be accomplished by using ten of the twelve equations to eliminate all but the transverse displacement component and its dual stress function from the remaining two equations. This process is simplified somewhat by the following observations for the classical theory. The conditions of vanishing transverse shearing strains, $\gamma_\xi = \gamma_\theta = 0$, immediately allow us to express ϕ_ξ and ϕ_θ in terms of

u_ξ, u_θ and w while the symmetry condition $\varepsilon_{\xi\theta} = \varepsilon_{\theta\xi}$ determines ω in terms of the same three midsurface displacement components. Static-geometric dual considerations give the stress functions Φ_ξ, Φ_θ and Ω in terms of U_ξ, U_θ and F (as well as the prescribed surface loads). These results effectively leave us with six (stress-strain) relations for u_ξ, u_θ, w and their dual stress functions.

For a typical harmonic of the shell response of the form

$$\{u_\xi, w, U_\xi, F\} = \{u_n(\xi_1),\ w_n(\xi_1),\ U_n(\xi_1),\ F_n(\xi_1)\}\cos(n\theta),$$

$$\{u_\theta,\ U_\theta\} \equiv \{v_n(\xi_1),\ V_n(\xi_1)\}\sin(n\theta), \qquad (n \geqslant 2)$$

four of the six remaining stress strain relations for u_ξ, u_θ, w and their duals may be re-arranged to give u_n', v_n', U_n' and V_n' separately in terms of u_n, v_n, U_n, V_n, v_n, w_n', F_n, and F_n':

$$u_n' = -\nu_s[\frac{r'}{r} u_n + \frac{n\alpha}{r} v_n] - [\frac{\alpha}{R_\xi} + \frac{\nu_s\alpha}{R_\theta}]w_n$$

$$-A(1-\nu_s^2)[\frac{r'}{rR_\xi} U_n + \frac{n\alpha}{rR_\theta} V_n + \frac{n^2\alpha}{r^2} F_n - \frac{r'}{r\alpha} F_n' + n_{\xi p}] \qquad (10.1)$$

$$\equiv L_u[u_n,\ v_n,\ U_n,\ V_n,\ w_n,\ F_n,\ F_n']$$

$$U_n' = -\nu_b[\frac{r'}{r} U_n + \frac{n\alpha}{r} V_n] - [\frac{\alpha}{R_\xi} - \frac{\nu_b\alpha}{R_\theta}]F_n + \nu_b m_{\xi p}$$

$$+D(1-\nu_b^2)[\frac{r'}{rR_\xi} u_n + \frac{n\alpha}{rR_\theta} v_n + \frac{n^2\alpha}{r^2} w_n - \frac{r'}{r\alpha} w_n'] \qquad (10.2)$$

$$\equiv L_U[U_n,\ V_n,\ u_n,\ v_n,\ F_n,\ w_n,\ w_n']$$

$$v_n' = \frac{1-\varepsilon_o^2}{1+\varepsilon_1^2}\frac{n\alpha}{r} u_n + \frac{r'}{r} v_n - \frac{4nD_SA_S}{r(1+\varepsilon_1^2)}\left(\frac{3}{R_\theta} - \frac{1}{R_\xi}\right)\left(w_n' - \frac{r'}{r} w_n\right)$$

$$+ \frac{4n\alpha\rho A_S}{r(1+\varepsilon_1^2)} V_n + \frac{4nA_S}{r(1+\varepsilon_1^2)}\left(F_n' - \frac{r'}{r} F_n\right) + \frac{4A_S}{1+\varepsilon_1^2} n_p \qquad (10.3)$$

$$\equiv L_v[u_n, v_n, V_n, w_n, w_n', F_n, F_n']$$

$$V_n' = \frac{1-\varepsilon_o^2}{1+\varepsilon_1^2}\frac{n\alpha}{r} U_n + \frac{r'}{r} V_n - \frac{4nD_SA_S}{r(1+\varepsilon_1^2)}\left(\frac{3}{R_\theta} - \frac{1}{R_\xi}\right)\left(F_n' - \frac{r'}{r} F_n\right)$$

$$- \frac{4n\alpha\rho D_S}{r(1+\varepsilon_1^2)} v_n - \frac{4nD_S}{r(1+\varepsilon_1^2)}\left(w_n' - \frac{r'}{r} w_n\right) + \frac{12D_SA_S}{R_\theta}\alpha n_p \qquad (10.4)$$

$$\equiv L_V[U_n, V_n, v_n, F_n, F_n', w_n, w_n']$$

where

$$\rho = \frac{1}{R_\theta} - \frac{1}{R_\xi}, \quad \varepsilon_o^2 = D_SA_S\left(\frac{3}{R_\theta} - \frac{1}{R_\xi}\right)\left(\frac{1}{R_\theta} - \frac{3}{R_\xi}\right), \quad \varepsilon_1^2 = D_SA_S\left(\frac{3}{R_\theta} - \frac{1}{R_\xi}\right)^2 \qquad (10.5)$$

$$n_{\xi p} = r^{-1}\int^{\xi_1}\left[r^{-3}\int^{\xi_1}(nr\bar{p}_\theta - n^2\bar{n}_{\theta p})r\alpha d\xi_1 - \left(\frac{1}{R_\xi}\bar{q}_{\xi p} - \frac{c}{r} n_{\theta p} + \bar{p}_\xi\right)\right]r\alpha d\xi_1$$

$$n_p = -r^{-2}\int^{\xi_1}(r\bar{p}_\theta - n\bar{n}_{\theta p})r\alpha d\xi_1, \quad n_{\theta p} = -sr\bar{p}_n$$

$$m_{\xi p} = -c^2r^{-1}\int^{\xi_1}\left[\int^{\xi_1}\bar{p}_n r\alpha d\xi_1\right]\alpha d\xi_1 \qquad (10.6)$$

$$q_{\xi p} = -r^{-1}\int^{\xi_1}c^2\bar{p}_n r\alpha d\xi_1$$

with $s \equiv \sin\xi = z'/\alpha$, $c \equiv \cos\xi = r'/\alpha$ and $(\ )' \equiv d(\ )/d\xi_1$ and with prescribed surface load distributions in the form

$$\{p_\xi, p_n\} = \{\bar{p}_\xi(\xi_1), \bar{p}_n(\xi_1)\}\cos(n\theta) \qquad p_\theta = \bar{p}_\theta(\xi_1)\sin(n\theta). \tag{10.7}$$

We now use (10.1) - (10.4) to eliminate u_n', v_n', U_n' and V_n' from the remaining two stress strain relations to get

$$\begin{aligned} w_n'' &= L_{w2}[u_n, v_n, w_n, w_n', U_n, V_n, F_n, F_n'] \\ F_n'' &= L_{F2}[U_n, V_n, F_n, F_n', u_n, v_n, w_n, w_n'] \end{aligned} \tag{10.8}$$

We will not write out the expressions $L_{W2}[\ldots]$ and $L_{F2}[\ldots]$ on the right hand side of (10.8) and merely note that they are linear combinations of the arguments. (See [14] and [15] for a more detailed discussion of the steps leading to (10.1) - (10.8)).

The remaining steps in the reduction process consist of (i) differentiating both sides of the expressions for w_n'' and F_n'' in (10.8) and using (10.1) - (10.4) to eliminate u_n', v_n', U_n' and V_n' from the resulting expressions to get

$$\begin{aligned} w_n''' &= L_{w3}[u_n, v_n, w_n, w_n', w_n'', U_n, V_n, F_n, F_n', F_n''], \\ F_n''' &= L_{F3}[U_n, V_n, F_n, F_n', F_n'', u_n, v_n, w_n, w_n', w_n''], \end{aligned} \tag{10.9}$$

(ii) differentiating both sides of (10.9) and again using (10.1) - (10.4) to eliminate u_n', v_n', U_n', and V_n', from the resulting expressions to get

$$\begin{aligned} w_n'''' &= L_{w4}[u_n, v_n, w_n, w_n', w_n'', w_n''', U_n, V_n, F_n, F_n'', F_n'''], \\ F_n'''' &= L_{F4}[U_n, V_n, F_n, F_n', F_n'', F_n''', u_n, v_n, w_n, w_n', w_n'', w_n'''], \end{aligned} \tag{10.10}$$

(iii) solving the four equations (10.8) and (10.9) (which are linear in U_n, V_n, u_n, v_n) for U_n, V_n, u_n, and v_n and using the results to eliminate these same four quantities from (10.10) to get two fourth order ODE's for w_n and F_n. The reduction process will be illustrated in the next section by applying it to the case of a circular cylindrical shell with uniform material properties and constant thickness.

Once we have w_n and F_n (and therefore u_n, v_n, U_n and V_n), the stress resultants and couple are determined by

$$N_{\xi\xi} = \{\frac{r'}{r\alpha^2} F_n' - \frac{n^2}{r} F_n - \frac{r'}{r\alpha R_\xi} U_n - \frac{n}{rR_\theta} V_n + n_{\xi p}\}\cos(n\theta)$$

$$N_{\theta\theta} = \{\frac{1}{\alpha^2} F_n'' - \frac{\alpha'}{\alpha^3} F_n' - \frac{1}{\alpha R_\xi} U_n' - \frac{1}{\alpha}(\frac{1}{R_\xi})' U_n + n_{\theta p}\}\cos(n\theta)$$

$$N_{\theta\xi} = \{\frac{n}{r\alpha} F_n' - \frac{nr'}{r^2\alpha} F_n + \frac{1}{2\alpha}(\frac{2}{R_\theta} - \frac{1}{R_\xi})(V_n' - \frac{r'}{r} V_n) - \frac{n}{2rR_\xi} U_n + n_p\}\sin(n\theta)$$

$$N_{\xi\theta} = \{\frac{n}{r\alpha} F_n' - \frac{nr'}{r^2\alpha} F_n + \frac{1}{2\alpha R_\theta}(V_n' - \frac{r'}{r} V_n) + \frac{n}{2r}(\frac{1}{R_\theta} - \frac{2}{R_\xi})U_n + n_p\}\sin(n\theta)$$

$$M_{\xi\xi} = \{\frac{n}{r} V_n + \frac{r'}{r\alpha} U_n + \frac{1}{R_\theta} F_n + m_{\xi p}\}\cos(n\theta)$$

$$M_{\theta\theta} = \{\frac{1}{\alpha} U_n' + \frac{1}{R_\xi} F_n\}\cos(n\theta), \qquad M_{\xi\theta} = M_{\theta\xi} = \frac{1}{2}\{\frac{1}{\alpha} V_n' - \frac{r'}{r\alpha} V_n - \frac{n}{r} U_n\}\sin(n\theta)$$

$$Q_\xi = \{\frac{1}{\alpha R_\theta} F_n' + \frac{n}{2r\alpha} V_n' + \frac{nr'}{2r^2\alpha} V_n + (\frac{n^2}{2\alpha r^2} - \frac{1}{\alpha R_\theta R_\xi})U_n + q_{\xi p}\}\cos(n\theta)$$

$$Q_\theta = \{-\frac{1}{\alpha}[\frac{1}{2\alpha} V_n' + \frac{r'}{2\alpha r} V_n + \frac{n}{2r} U_n]'$$

$$+ \frac{1}{R_\xi}[\frac{r'}{\alpha^2 r} F_n' - \frac{n^2}{r^2} F_n - \frac{r'}{r\alpha R_\xi} U_n - \frac{n}{rR_\theta} V_n]\}\sin(n\theta)$$

The strain measures can either be obtained from the stress measures by the stress strain relations (and compatibility equations) or from displacement components by the dual strain displacement relations.

2. The Classical Theory of Circular Cylindrical Shell

With $r(\xi_1) = a$ and $z(\xi_1) = a\xi_1$ for a circular cylindrical shell where the positive constant a is the radius of the circular cylindrical midsurface, we have

$$\alpha = a, \qquad \frac{1}{R_\theta} = -\frac{1}{a}, \qquad \frac{1}{R_\xi} = 0$$

Equations (10.1) - (10.4) simplify in this case to read

$$\begin{aligned}
u_n' &= -\frac{1}{a} A(1-\nu_s^2)[n^2F_n - nV_n] + \nu_s[w_n - nv_n] \\
v_n' &= -\frac{4nA_S}{a(1+\varepsilon_1^2)}[U_n - F_n'] + \frac{4n\varepsilon_1^2}{3(1+\varepsilon_1^2)} w_n' + \frac{n(3-\varepsilon_1^2)}{3(1+\varepsilon_1^2)} u_n \\
U_n' &= \frac{1}{a} D(1-\nu_b^2)[n^2w_n - nv_n] - \nu_b[F_n - nV_n] \\
V_n' &= \frac{4nD_S}{a(1+\varepsilon_1^2)}[u_n - w_n'] + \frac{4n\varepsilon_1^2}{3(1+\varepsilon_1^2)} F_n' + \frac{n(3-\varepsilon_1^2)}{3(1+\varepsilon_1^2)} U_n
\end{aligned} \tag{10.11}$$

where now

$$\varepsilon_1^2 = \frac{9}{a^2} D_S A_S = \frac{9}{48(1-\nu^2)} \frac{h^2}{a^2}$$

and where we have omitted all surface load terms to simplify the presentation. Appropriate equations for circular cylindrical shells including load terms may be obtained by specializing the results of the next section.

Equations (10.8) become

$$\begin{aligned}
w_n'' &= -\frac{a}{D}(nV_n - F_n) + \nu_b(n^2w_n - nv_n) \\
F_n'' &= \frac{a}{A}(nv_n - w_n) - \nu_s(n^2F_n - nV_n)
\end{aligned} \tag{10.12}$$

We now differentiate both sides of (10.12) to get

$$\begin{aligned}
w_n''' &= -\frac{a}{D}(nV_n' - F_n') + \nu_b(n^2w_n' - nv_n') \\
F_n''' &= \frac{a}{A}(nv_n' - w_n') - \nu_s(n^2F_n' - nV_n')
\end{aligned} \tag{10.13}$$

The expressions for V_n' and v_n' in (10.11) are then used to eliminate these quantities from (10.13) to give

$$w_n''' = n^2A_1^*(w_n' - u_n) + \frac{a}{D}(A_2^*F_n' - n^2A_3^*U_n),$$
$$F_n''' = n^2A_1(F_n' - U_n) - \frac{a}{A}(A_2w_n' - n^2A_3u_n). \tag{10.13'}$$

The expressions for A_1, A_2, A_3 and their duals A_1^*, A_2^* and A_3^* in terms of D, A, D_S, A_S and $\nu_b = \nu_s$ can be found in [14].

Next, we differentiate (10.13') and use (10.11) to eliminate u_n' and U_n' from the resulting expressions to get

$$w_n'''' = n^2(B_1^*w_n + B_2^*w_n'') - \frac{a}{D}(n^2B_3^*F_n + B_4^*F_n'') + n^2B_5^*v_n + \frac{an^3}{D}B_6^*V_n$$
$$F_n'''' = n^2(B_1F_n + B_2F_n'') - \frac{a}{D}(n^2B_3w_n + B_4w_n'') + n^2B_5V_n - \frac{an^3}{A}B_6v_n \tag{10.14}$$

The constants $B_1, \ldots, B_6$ and their duals $B_1^*, \ldots, B_6^*$ can also be found in [14].

We can then solve (10.12) for v_n and V_n and use the result to eliminate these two quantities from (10.14). In this way, we get

$$(1-\nu^2\varepsilon_2^2)nv_n = (1-n^2\nu^2\varepsilon_2^2)w_n + \nu_S\varepsilon_2^2w_n'' + \frac{1}{a}A[\nu_S(n^2-1)F_n + F_n''],$$
$$(1-\nu^2\varepsilon_2^2)nV_n = (1-n^2\nu^2\varepsilon_2^2)F_n - \nu_b\varepsilon_2^2F_n'' + \frac{1}{a}D[\nu_b(n^2-1)w_n + w_n''], \tag{10.15}$$

where $\varepsilon_2^2 = DA/a^2$, and

$$w_n'''' - c_2^*w_n'' + c_1^*w_n = \frac{a}{D}(c_4^*F_n'' + c_3^*F_n)$$
$$F_n'''' - c_2F_n'' + c_1F_n = -\frac{a}{A}(c_4w_n'' + c_3w_n) \tag{10.16}$$

where c_1, c_2, c_3, c_4 and their duals are given in [14].

It should be noted that the reduction described above is exact; it involves no approximation whatsoever. Similarly, the stress, strain and displacement measures of the shell can be calculated exactly by the auxiliary formulas given in the last section, specialized to the case of a circular cylindrical shell, once we have w_n and F_n. However, the exact equations for w_n and F_n, (10.16), contain many terms which are of magnitude no larger than the error inherent in shell theory (as an approximation of the exact solution of three dimensional elasticity) and may be omitted without loss of accuracy when viewed in the context of an approximate solution for the three dimensional problem. When these negligibly small terms are deleted, equations (10.16) simplify to read

$$\begin{aligned} w_n'''' - 2n^2 w_n'' + n^2(n^2-1)w_n &= \frac{a}{D} F_n'' \\ F_n'''' - 2n^2 F_n'' + n^2(n^2-1)F_n &= -\frac{a}{A} w_n'' \end{aligned} \qquad (10.16')$$

Note that the left hand side of the two equations in (10.16') corresponds to the biharmonic operator applied to $w = w_n(\xi_1)\cos(n\theta)$ and $F = F_n(\xi_1)\cos(n\theta)$, respectively. Therefore, we may also write these equations as

$$\nabla^2\nabla^2 w = \frac{a}{D} F'', \qquad \nabla^2\nabla^2 F = -\frac{a}{A} w'' \qquad (10.16'')$$

Equation (10.16") are identical to those obtained in [11] by a different method which also makes allowance for the inherent error of shell theory. In the form (10.16') or (10.16") the two governing equations for w and F may of course be further reduced to a single complex fourth order equation for a complex potential.

The exact reduction procedure for circular cylindrical shells leading to the two simultaneous equations for w_n and F_n given by (10.16) applies to any shell of revolution including those with inhomogeneous material properties and nonuniform thickness. The final two equations for w_n and F_n will normally contain many terms of the order of the error inherent in shell theory which may be deleted without affecting the accuracy of the final solution. It is generally possible to obtain the same two simplified (but accurate) equations for w_n and F_n by deleting the same type of small terms at the intermediate stages of the reduction process and thereby reduce the amount of calculations involved. This simplified reduction process has been applied to the case of conical shells with uniform material properties and constant thickness. The results will be reported in the next section.

3. Conical Shells and Generalized Hypergeometric Functions

For a conical shell, we may describe its middle surface by

$$r = r_0 + c\xi_1, \qquad z = z_0 + s\xi_1 \tag{10.17}$$

where r_0, z_0, c and s are fixed geometrical parameters and, in terms of the meridional slope angle ξ, $c = \cos\xi$ and $s = \sin\xi$. Correspondingly, we have

$$\alpha = 1, \qquad \frac{1}{R_\xi} = 0, \qquad \frac{1}{R_\theta} = -\frac{s}{r} \tag{10.18}$$

where R_ξ and R_θ are the two principal radii of curvature. Evidently, the independent variable ξ_1 is the meridional arclength measured from the end $r = r_0$ and $z = z_0$.

For our particular α, R_ξ and R_θ, the two equations (10.1) and (10.2) for u_n' and U_n' simplify to read:

$$\begin{aligned} u_n' &= -\nu_s r^{-1}(cu_n + nv_n) + A(1-\nu_s^2)nsr^{-2}V_n \\ &\quad + \nu_s sr^{-1}w_n + A(1-\nu_s^2)(cr^{-1}F_n' - n^2r^{-2}F_n + n_{\xi p}) \\ U_n' &= \nu_b r^{-1}(cU_n + nV_n) - D(1-{}_{b})nsr^{-2}v_n \\ &\quad - \nu_b sr^{-1}F_n - D(1-\nu_b^2)(cr^{-1}w_n' - n^2r^{-2}w_n) + \nu_b m_{\xi p} \end{aligned} \tag{10.19}$$

with the inhomogeneous terms $n_{\xi p}$, etc. given by (10.5), while (10.3) and(10.4) become

$$\begin{aligned} v_n' &= -4nsr^{-2}A_S U_n + cr^{-1}v_n + nr^{-1}u_n + 4nr^{-1}A_S(F_n' - cr^{-1}F_n) \\ &\quad + 12nsr^{-2} D_S A_S(w_n' - cr^{-1}w_n) + 4A_S n_p \\ V_n' &= 4nsr^{-2}D_S u_n + cr^{-1}V_n + nr^{-1}U_n - 4nr^{-1}D_S(w_n' - cr^{-1}w_n) \\ &\quad + 12nsr^{-2}D_S A_S(F_n' - cr^{-1}F_n) + 12D_S A_S sr^{-1}n_p. \end{aligned} \tag{10.20}$$

after terms of order $n^2\varepsilon = n^2h/R_\theta$ have been omitted in the presence of unity. (Again n is assumed to be not too large so that we have $n^2\varepsilon = O(1)$ at most.)

The two remaining stress strain relations may be re-arranged to read

$$cU_n + nV_n + \nu_b nsr^{-1}Dv_n = -Dr[w_n'' + \nu_b cr^{-1}w_n' - \nu_b n^2r^{-2}w_n] + sF_n - rm_{\xi p}$$

$$cu_n + nv_n + \nu_s nsr^{-1}AV_n = Ar[F_n'' - \nu_s cr^{-1}F_n' + \nu_s n^2r^{-2}F_n] + sw_n + rA(n_{\theta p} - \nu_s n_{\xi p})$$

where the inhomogeneous terms are again given by (10.5). Upon differentiating both sides with respective to ξ_1, eliminating u_n', v_n', U_n' and V_n', and then deleting terms of order $n^2\varepsilon^2$ compared to unity, we get

$$\begin{aligned}(n^2-c^2)rU_n + n^2s(2-\nu_b)Du_n - nsc(1+\nu_b)Dv_n \\ = -D\{r^3w_n''' - r[n^2(2-\nu_b) + c^2(1+\nu_b)]w_n' + 3cn^2w_n\} \\ + sr^2(F_n' - cr^{-1}F_n) - r^3m_{\xi p}' \\ (n^2-c^2)ru_n - n^2s(2+\nu_s)AU_n + nsc(1-\nu_s)AV_n \\ = A\{r^3F_n''' - r[n^2(2+\nu_s) + c^2(1-\nu_s)]F_n' + 3cn^2F_n\} \\ + sr^2(w_n' - cr^{-1}w_n) + r^2(re_{\theta p}' - ce_{\xi p} - 4ne_p)\end{aligned} \quad (10.22)$$

with $e_{\theta p} = A(n_{\theta p} - \nu_s n_{\xi p})$, $e_{\xi p} = A(n_{\xi p} - \nu_s n_{\theta p})$ and $e_p = 2A_S n_p$. The last four equations may be solved for u_n, v_n and their dual stress functions to get these quantities in terms of w_n, F_n and their derivatives with respect to ξ_1 up to third order as well as load terms. The resulting expressions, denoted by (10.21') and (10.22'), can be found in [15] and will not be listed here.

We now differentiate both sides of the two equations in (10.22) with respect to ξ_1 and use (10.19), (10.20), (10.21') and (10.22') to eliminate u_n, v_n, U_n, V_n and their first derivative. Upon deleting all terms of order $n^2\varepsilon^2$ in the presence of other O(1) terms, we get

$$w_n'''' + \frac{2c}{r} w_n''' - \frac{2n^2+c^2}{r^2} w_n'' + \frac{c(2n^2+c^2)(n^2-1)}{r^3(n^2-c^2)} w_n'$$

$$+ \frac{n^2(n^2-4c^2)(n^2-1)}{r^4(n^2-c^2)} w_n = \frac{s}{rD} F_n'' + \frac{c^2}{D} \bar{p}_n$$

$$F_n'''' + \frac{2c}{r} F_n''' - \frac{2n^2+c^2}{r^2} F_n'' + \frac{c(2n^2+c^2)(n^2-1)}{r^3(n^2-c^2)} F_n'$$

$$+ \frac{n^2(n^2-4c^2)(n^2-1)}{r^4(n^2-c^2)} F_n = - \frac{s}{rA} w_n'' + \frac{1}{r} \bar{p}$$

where, for the special case of $\bar{p}_\xi = 0$ and $\bar{p}_\theta = 0$,

$$\bar{p}(\xi_1) = s\{(r^2\bar{p}_n)'' - 2n^2\bar{p}_n + \frac{n^2c(2n^2+c^2)}{r^3(n^2-c^2)} \int^{\xi_1} \bar{p}_n r^2 d\xi_1$$

$$+ \frac{n^2(n^2-c^2)}{r^2} \int^{\xi_1} [\frac{1}{r^2} \int^{\xi_1} \bar{p}_n r^2 d\xi_1] d\xi_1\}$$

The expression for $\bar{p}$ when $\bar{p}_\xi$ or $\bar{p}_\theta$ does not vanish as well as the auxiliary formulas for stress and displacement measures can be found in [15]. We note here only that our two equations for w_n and F_n in (10.23) reduce correctly to the known results in two limiting cases. When $\xi = 0$ (so that $c = 1$ and $s = 0$), they reduce to the exact equation for plate bending and for generalized plane stress. When $\xi = \pi/2$ (so that $c = 0$ and $s = 1$), they reduce correctly to Simmonds' equations for circular cylindrical shell (also derived in the section (2) of this chapter). For the case of a shallow conical shell with $c \cong 1$, $s \cong \xi$ and $1 + O(s^2) \cong 1$ so that $(n^2-1)/(n^2-c^2) \cong 1$ within the shallow shell approximation, the two equations for w_n and F_n reduce to those for shallow shell theory obtained in Chapter 8. The differences in the inhomogeneous terms in the differential equations correspond to the different particular integrals used for the load terms in the equilibrium equations.

Since n^2 is effectively $\partial^2(\ )/\partial\theta^2$ for shells of revolution, the form of (10.23) suggests that a reduction of the equations for the linear theory of thin elastic shells to two simultaneous fourth order partial differential equations for F and w is generally not possible. The observation is consistent with the more general results of [12].

The two fourth order ODE's for w_n and F_n can again be combined into a single complex fourth order ODE for a complex potential $\Phi = w_n - i\sqrt{A/D}\, F_n$. For $c \neq 0$ so that the shell is not a flat plate, we introduce the dimensionless independent variable

$$\zeta = \frac{isr}{c^2\sqrt{DA}} \tag{10.25}$$

with $(\ )^{\bullet} \equiv d(\ )/d\zeta$ and write the homogeneous complex equation for Φ as

$$\Phi^{\bullet\bullet\bullet\bullet} + \frac{2}{\zeta}\Phi^{\bullet\bullet\bullet} - \frac{\sigma_2}{\zeta^2}\Phi^{\bullet\bullet} + \frac{\sigma_1}{\zeta^3}\Phi^{\bullet} + \frac{\sigma_0}{\zeta^4}\Phi = \frac{1}{\zeta}\Phi^{\bullet\bullet} \tag{10.26}$$

where

$$\sigma_2 = (2n^2+c^2)c^{-2}, \qquad \sigma_1 = (2n^2+1)(n^2-1)/[(n^2-c^2)c^2]$$

$$\sigma_0 = n^2(n^4-4c^2)(n^2-1)/[c^4(n^2-c^2)] \tag{10.27}$$

It can be shown (see [15]) that the general solution of (10.26) is

$$\Phi = \sum_{k=1}^{4} b_k\, \zeta^{\mu_k}\, \Phi_k(\zeta) \tag{10.28}$$

where μ_k, k = 1, 2, 3, and 4, are the four complex roots of the fourth degree polynomial

$$Q(\gamma) \equiv \gamma(\gamma-1)[\gamma^2-3\gamma-c^{-2}(2n^2-c^2)] + \sigma_1\gamma + \sigma_0 = 0, \tag{10.29}$$

b_k's are arbitrary complex constants of integration and ϕ_k, k = 1, 2, 3 and 4, are generalized hypergeometric functions ${}_2F_3$ with appropriate arguments [25]. For example, we have

$$\phi_4(\zeta) = {}_2F_3\begin{bmatrix}\mu_4, & \mu_4-1, & \zeta \\ \\ \mu_4-\mu_1+1, & \mu_4-\mu_2+1, & \mu_4-\mu_3+1\end{bmatrix} \tag{10.30}$$

keeping in mind that ζ is a complex (in fact, pure imaginary) argument.

References

1. H. Reissner, "Spannungen in Kugelschalen (Kuppelz)," Müller-Breslau Festschrift, 1912, 181-193.

2. E. Meissner, "Das Elastizitatsproblem fur dünne Schalen von Ringflachen-Kugel oder Kegelform," Physikalische Zeitschrift 14, 1913, 343-349.

3. E. Reissner, "On the theory of thin elastic shells," H. Reissner Anniversary Volume, J. W. Edwards, Ann Arbor, Mich., 1949, 230-247.

4. E. Reissner and F. Y. M. Wan, "Rotationally symmetric stress and strain in shells of revolution," Studies in Appl. Math. 48, 1969, 1-17.

5. V. S. Chernina, "On the system of differential equations of equilibrium of shells of revolution under bending loads," Prikl. Mat. Mech. (P.M.M.) 23, 1959, 258-265.

6. F. Y. M. Wan, "Circumferentially sinusoidal stress and strain in shells of revolution," Int'l. J. Solids & Structures (IJSS) 6, 1970, 959-973.

7. ________________, "Laterally loaded elastic shells of revolution," Ing. Arch. 42, 1973, 245-258.

8. K. Marguerre, "Zur Theorie der gekrümmten Platte grosser Formänderung," Proc. 5th Intern'l. Congr. Appl. Mech., 1938, 93-101.

9. J. G. Berry, "On thin hemispherical shells subjected to concentrated edge moments and forces," Proc. 2nd Midwest. Conf. on Solid Mech., 1955, 25-44.

10. W. T. Koiter, "A spherical shell under point loads at its poles," Progress in Appl. Mech. (The Prager Anniversary Volume). The Macmillan Co., New York, 1963, 155-169.

11. J. G. Simmonds, "A set of simple, accurate equations for circular cylindrical elastic shells," Intern'l. J. of Solids & Structures 2, 1966, 525-541.

12. A. L. Gol'denveiser, "Equations of the theory of shells in displacements and stress functions," P.M.M., 21, 1957, 801-814.

13. J. L. Sanders, "On the shell equations in complex form," Proc. 2nd IUTAM Shell Symp., Ed. by F. J. Niordson, Springer-Verlag, 1969, 135-160.

14. F. Y. M. Wan, "Exact reductions of the equations of linear theory of shells of revolution," Studies in Appl. Math. 48, 1969, 361-375.

15. ________________, "On the equations of the linear theory of elastic conical shells," Studies in Appl. Math. 49, 1970, 69-83.

16. ________________, "Lecture notes on theory of thin elastic shells," I.A.M.S. Tech. Rep. 83-11; Univ. British Columbia, Vancouver, Canada, July, 1983.

17. R. A. Clark, "On the theory of thin elastic toroidal shells," J. Math. & Phys. 24, 1950, 146-178.

18. F. B. Hildebrand, "On asymptotic integration in shell theory," Proc. Symp. in Appl. Math. Vol. III, A.M.S., Providence, 1950, 53-66.

19. F. Y. M. Wan, "Membrane and bending stresses in shallow spherical shells," Intern'l. J. Solids & Structures 3, 1967, 353-366.

20. ______________, "On the displacement boundary value problem of shallow spherical shells," Intern'l. J. Solids & Structures 4, 1968, 661-666.

21. E. Reissner, "On the transverse twisting of shallow spherical ring caps," J. Appl. Mech. 47, 1980, 101-105.

22. E. Reissner, "On the effect of a small circular hole on states of uniform membrane shear in spherical shells," J. Appl. Mech. 47, 1980, 430-431.

23. E. Reissner, "On the influence of a rigid circular inclusion on the twisting and shearing of a shallow spherical shell," J. Appl. Mech. 47, 1980, 586-588.

24. J. E. Reissner, "Effect of a circular hole on states of uniform twisting and shearing in shallow spherical shells," J. Appl. Mech. 48, 1981, 674-676.

25. A. Erdelyi et al, Higher transcendental functions, vol. I, McGraw-Hill Book Co., Inc., 1953.