NOTE: THE INSTITUTE OF APPLIED MATHEMATICS AND STATISTICS (IAMS) SHALL BE NOW KNOWN AS:

THE INSTITUTE OF APPLIED MATHEMATICS

LECTURE NOTES ON SHALLOW SHELL THEORY

F.Y.M. Wan

Technical Report No. 84-8
July, 1984

Cover photo courtesy of the U.B.C. Museum of Anthropology:

Haida totem pole; main figure, possibly bear, holding wolf between legs, frog in mouth, wolf between ears.

LECTURE NOTES ON SHALLOW SHELL THEORY

F.Y.M. Wan[(1)]
Institute of Applied Mathematics
and
Department of Mathematics
The University of British Columbia
Vancouver, B.C. V6T 1Y4
Canada

July 1984

(1) Currently on leave as Professor of Applied Mathematics and Mathematics at the University of Washington, Seattle, WA 98195.

Foreword

Over the years, this writer has taught courses on the theory of thin elastic shells at different institutions, mainly to students in applied mathematics. A set of lecture notes have been compiled from and for these courses. Portions of these notes are now published as technical reports of the Institute of Applied Mathematics (and Statistics, at one time) at the University of British Columbia. These technical reports have limited distribution but are deposited in the UBC Archive for future reference.

Because of time constraints, these shell courses usually focussed on the mathematical structure of the various special theories such as the linear theory, shallow shell theory, theory of shells of revolution, etc., and on the solution techniques these special structures induce for the relevant boundary value problems. Some discussion of the foundations of shell theory, i.e., the adequacy of shell theory as an approximate solution of a three-dimensional elasticity problem, is usually included to make the subject meaningful for engineering applications. For this part of the course, it is necessary for the audience to have some background on the three-dimensional linear theory of elasticity. To accommodate students in applied mathematics without the requisite background, the material in elasticity theory essential to the discussion of foundations of shell theory is outlined in reference [1] at the end of this Foreword (not to be confused with references of the main body of this report given at the end of the report).

The foundation problem for shells is, at this writing, not completely understood and is, in any event, far too complicated for students exposed to the subject for the first time. The writer has found it more effective to discuss the corresponding problem for linear plate theory as an illustration of the nature of the problem and the types of results attainable. For these reasons, the results of Friedrichs & Dressler and of E. Reissner are presented in reference [2]. It has always been the writer's intention to supplement [2] with a summary of his joint work with R.D. Gregory of the University of Manchester such as [3] and [4] to indicate a more attractive and practical solution of the foundation problem for plates. In principle, the same method of approach is also feasible for the corresponding shell problem though its implementation requires some fresh ideas. The compilation of the material on the foundation of plates, possibly including other results such as [5], will be the writer's next writing project.

To allow for a separate discussion of the structure and solution techniques of shell theory without any reference to three-dimensional elasticity theory, (and the foundation problem), the writer usually follows the approach of W. Gunther, E. Reissner and H. Schaefer, and develops shell theory as the mechanics of a two-dimensional deformable continuum. The constitutive relations will be adopted with little or no justifications whenever there is not sufficient time to discuss the foundations of theory. The fundamentals of linear shell theory is presented in this way in [6], though the relationship between shell theory and three-dimsional elasticity theory is briefly discussed by way of a class of transversely rigid thin elastic shells. The theory developed there is for general orthogonal surface coordinates. A tensorial treatment to allow for oblique surface coordinates is avoided to gain time for shell theory proper. Most shell designs encountered by this writer can in fact be analyzed by working with orthogonal surface coordinates. In any event, it is not needed for the present series of lecture notes.

The developments of two special classes of shell problems have had significant impact on the theory and applications of shell structures. For their importance in engineering designs alone, shallow shells deserve special attention. In this set of lecture notes, we summarize the salient features of the classical Marguerre's shallow shell theory and its modern extensions. Our discussion will be short on specific problems; there will be just a few to illustrate the main theoretical developments. In particular, the very important applications of shallow shell theory to shell buckling problems will be treated in a separate report. A similar summary for the linear theory of shells of revolution, also important in applications, is presented in [7].

References

1. F.Y.M. Wan, "Lecture Notes on Problems in Elasticity: I. Fundamentals of Linear Theory of Elasticity", I.A.M.S. Tech. Rep. No. 82-17, Univ. of British Columbia, Vancouver, Canada, November,1982.

2. __________, "Lecture Notes on Problems in Elasticity: II. Linear Plate Theory", I.A.M.S. Tech. Rep. No. 83-15, Univ. of British Columbia, Vancouver, Canada, September, 1983.

3. R.D. Gregory and F.Y.M. Wan, "Decaying States of Plane Strain in a Semi-infinite Strip and Boundary Conditions for Plate Theory", J. Elasticity 14, 1984, 27-64.

4. __________________________, "On Plate Theories and Saint-Venant's Principle", I.A.M.S. Tech. Rep. No. 83-12, Univ. of British Columbia, Vancouver, Canada, August, 1983.

5. J.G. Simmonds, "An Improved Estimate for the Error in the Classical, Linear Theory of Plate Bending", Quart. Appl. Math. 29, 1971, 439-447.

6. F.Y.M. Wan, "Lecture Notes on Theory of Thin Elastic Shells", I.A.M.S. Tech. Rep. No. 83-11, Univ. of British Columbia, Vancouver, Canada, July, 1983.

7. __________, "Lecture Notes on the Linear Theory of Thin Elastic Shells of Revolution", I.A.M.* Tech. Report No. 84-9, Univ. of British Columbia, Vancouver, Canada, to be published in August, 1984.

*The Institute of Applied Mathematics and Statistics of the University of British Columbia was formally re-named as the Institute of Applied Mathematics in March, 1984.

TABLE OF CONTENTS

1. Introduction

One of the most significant developments in shell theory is the formulation of shallow shell theory by K. Marguerre in 1938 [1]. The two restrictive assumptions of the theory are: (1) the tangential displacement components of the shell under the prescribed loading are an order of magnitude smaller than the normal displacement component, and (2) the effects of terms involving products of slope measures for both the undeformed and the deformed middle surface of the shell are negligible. A theory similar in mathematical structure to Marguerre's theory was formulated at about the same time [2-7], stipulating only the first assumption. This so-called "semi-shallow" shell theory has in principle a wider range of applications (including problems involving nonshallow shells) but does not reap the benefits of the second (i.e., the shallowness) assumption mentioned above. Hence, it is to be distinguished from the genuine shallow shell theory of Marguerre.

Because of its unusual combination of mathematical tractability and rich mechanical contents, Marguerre's shallow shell theory is a very attractive tool for structural analyses of shell designs. That it is widely used may be attributed to two different reasons. Many shell designs involve genuine shallow shells and it is naturally appropriate to use the shallow shell theory in these cases. For other designs involving nonshallow shells, the structural phenomenon to be analyzed may sometimes be local in nature and the far field behavior of the shell may have little or no influence on the phenomenon. Local buckling is an important example of this kind of phenomena. Even if the far field be= havior of the shell cannot be neglected, a local analysis often is still a part of the solution process, perhaps to be combined with the results of a far field study at a later stage.

Marguerre's shall shell theory (or its modern extension) also has a very attractive mathematical structure. With some re-arrangments, the classical Marguerre theory exploits the elegant static-geometric duality of shell theory to its limit, far beyond anything its creator might have had in mind. The presence of this duality in the theory provides a certain degree of mathematical symmetry which makes possible the development of time and labor saving schemes for analyses and computation. These added features in turn have made the theory even more attractive to the users.

The development leading to the final form of Marguerre's shallow shell theory undoubtedly benefitted from two groups of related earlier results: the Föppl-Hencky nonlinear membrane theory [3,4] and the von Kármán nonlinear plate theory [5]. In fact, we recover the latter from Marguerre's shallow shell theory if the rise z of the shallow shell's middle surface above the base plane is set to zero. At the same time, we get the Föppl-Hencky nonlinear membrane theory if we set the bending stiffness factor D to zero in the von Kármán plate theory. The development of the various theories of shallow shells in these notes takes advantage of these background relationships among the three groups of theories.

In this volume, we outline separately the formulation of the linear and nonlinear shallow shell theory without any reference to three dimensional elasticity theory (except for the implicit reference inherent in our choice of constitutive relations). Our linear theory is more general than the classical linear theory of Marguerre and contains a shallow shell theory with transverse shear deformation as a special case. For both the linear and nonlinear theory, the simpler case of cartesian coordinates is first treated with the results applied to problems for a pretwisted strip (or shallow hyperbolic paraboloidal shell). The corresponding theories in polar coordinates are formulated separately and applied to shallow shells of revolution as well as shallow helicoidal shells. Applications of shallow shell theories are so numerous that even the nontrivial ones cannot be all described in these notes. The few discussed in some details here are chosen for specific purposes and with the understanding that buckling problems will be treated in a separate volume.

2. The Linear Theory of Shallow Shells

1. Shallow Shell Approximations and Strain Displacement Relations

In cartesian coordinates $\{x_1, x_2, x_3\}$, a point on the middle surface of a thin shell is given by the position vector

$$\vec{r}(x_1,x_2) = x_1\vec{i}_1 + x_2\vec{i}_2 + z(x_1,x_2)\vec{i}_3$$

where $\vec{i}_1$, $\vec{i}_2$ and $\vec{i}_3$ are the unit vectors in the direction of the three cartesian axes. We denote by $(\ )'$ or $(\ )_{,1}$ a partial derivative with respect to x_1 and by $(\ )^{\bullet}$ or $(\ )_{,2}$ a partial derivative with respect to x_2. For a shell to be shallow, we must have $|z_{,k}| << 1$, $k = 1,2,$ and the shallow shell theory stipulates that product terms such as $z_{,k}z_{,j}$ are to be neglected. For example, we have from the expression for $\vec{r}$,

$$\vec{r}_{,1} = \vec{i}_1 + z_{,1}\vec{i}_3 \quad , \quad \vec{r}_{,2} = \vec{i}_2 + z_{,2}\vec{i}_3$$

and correspondingly,

$$\alpha_k^2 = 1 + z_{,k}^2 \cong 1 \quad , \quad \vec{r}_{,1}\cdot\vec{r}_{,2} = z_{,1}z_{,2} \cong 0$$

so that, consistent with the shallowness assumption, the surface coordinates are orthogonal. Similarly, the unit normal vector of the (undeformed) middle surface of the shell is given by

$$n = \frac{\vec{r}_{,1} \times \vec{r}_{,2}}{|\vec{r}_{,1} \times \vec{r}_{,2}|} = \frac{-z_{,1}\vec{i}_1 - z_{,2}\vec{i}_2 + \vec{i}_3}{\sqrt{1 + z_{,1}^2 + z_{,2}^2}} \cong -z_{,1}\vec{i}_1 - z_{,2}\vec{i}_2 + \vec{i}_3 \ .$$

Given that the surface coordinates are orthogonal within the framework of a shallow shell theory, the radii of curvature measures are given by

$$\frac{1}{R_{mm}} = - \frac{\vec{n}\cdot\vec{r},_{mm}}{\alpha_m^2} \cong -z,_{mm} \quad , \qquad (m = 1,2)$$

$$\frac{1}{R_{12}} = \frac{1}{R_{21}} = - \frac{\vec{n}\cdot\vec{r},_{mm}}{\alpha_1\alpha_2} \cong -z,_{12} \quad .$$

Hence, the surface coordinates are not line of curvature coordinates unless $z,_{12} \equiv 0$.

The shallow shell theory also stipulates that the tangential displacement components u_1 and u_2 (in the direction of $\vec{r},_1$ and $\vec{r},_2$, respectively) should be an order of magnitude smaller than the normal displacement component w , and that the rotation about the normal, ω , should be an order of magnitude smaller than the other two components of the rotation vector, ϕ_1 and ϕ_2 . More precisely, let $\xi_o \equiv \max[|z,_1|,|z,_2|]$, the theory stipulates

$$u_k = O(\xi_o w) \quad , \qquad \omega = O(\xi_o \phi_k) \quad .$$

The two assumptions (or stipulations) inherent in a shallow shell theory allow us to use the linear strain-displacement relations for orthogonal surface coordinates given in [11] and simplify them considerably. We list below the strain-displacement relations for a linear shallow shell theory in the order we will use them in the next section (with the same notations employed in [11]):

$$\gamma_k = \phi_k + w_{,k} \quad , \quad \kappa_{ij} = \phi_{j,i} \quad , \quad \varepsilon_{kk} = u_{k,k} - z_{,kk} w$$

$$\varepsilon_{12} = u_{2,1} - z_{,12} w - \omega \quad , \quad \varepsilon_{21} = u_{1,2} - z_{,12} w + \omega$$

$$\lambda_1 = \omega_{,1} + z_{,12}\phi_1 - z_{,11}\phi_2 \quad , \quad \lambda_2 = \omega_{,2} - z_{,12}\phi_2 + z_{,22}\phi_1 \ .$$

The twelve strain measures defined in terms of six displacement measures are not independent quantities. It is straightforward to verify that they satisfy the following six compatibility equations:

$$\kappa_{22,1} - \kappa_{12,2} = 0 \quad , \quad \kappa_{21,1} - \kappa_{11,2} = 0$$

$$\lambda_{2,1} - \lambda_{1,2} - z_{,11}\kappa_{22} - z_{,22}\kappa_{11} + z_{,12}(\kappa_{21} + \kappa_{12}) = 0$$

$$\varepsilon_{22,1} - \varepsilon_{12,2} - \lambda_2 - z_{,12}\gamma_2 + z_{,22}\gamma_1 = 0$$

$$-\varepsilon_{21,1} + \varepsilon_{11,2} + \lambda_1 + z_{,11}\gamma_2 - z_{,12}\gamma_1 = 0$$

$$\gamma_{2,1} - \gamma_{1,2} + \kappa_{21} - \kappa_{12} = 0 \ .$$

These equations are a simplified version of the general compatibility equations for linear shell theory given in [11]. Along with the various expected shallow shell simplifications (such as $\alpha_k \cong 1$, etc.), we see that terms of the form $z_{,ij}\lambda_k$ no longer appear in the first two equations. Evidently, they are considered to be negligibly small compared to $\kappa_{ij,m}$ terms in the context of a shallow shell theory.

2. Equilbrium Equations and Stress Strain Relations

Equilibrium equations for the stress resultants and couples of a shallow shell will be taken in a form consistent with the strain displacement relations of the theory for a virtual work principle. With the same notation used in [11], it is a straightforward calculation from the virtual work expression to get

$$N_{11},_1 + N_{21},_2 + p_1 = 0 \quad , \quad N_{12},_1 + N_{22},_2 + p_2 = 0$$

$$Q_1,_1 + Q_2,_2 + z,_{11}N_{11} + z,_{22}N_{22} + z,_{12}(N_{12} + N_{21}) + p_n = 0$$

$$M_{11},_1 + M_{21},_2 - Q_1 - z,_{12}P_1 - z,_{22}P_2 + q_1 = 0$$

$$M_{12},_1 + M_{22},_2 - Q_2 + z,_{11}P_1 + z,_{12}P_2 + q_2 = 0$$

$$P_1,_1 + P_2,_2 + N_{12} - N_{21} + q_n = 0 \quad .$$

This system of six equilibrium equations is evidently a simplified version of the general system for a linear shell theory given in [11]. Along with the various expected shallow shell simplifications such as $\alpha_k \cong 1$, etc., we see that terms of the for $z,_{ij}Q_k$ no longer appear in the first two equations. Evidently, they are considered to be negligibly small compared to $N_{ij},_k$ terms in the context of a shallow shell theory, so that we have in effect $Q_k = O(\xi_o N_{ij})$ for a shallow shell theory. Similarly, it follows from the 4th and 5th equations $z,_{ij}M_{mn} = O(\xi_o^2 N_{k\ell})$ and hence the sixth equation.

Consistent with the static-geometric duality of linear shell theory, [11], the six (homogeneous) equilibrium equations are the static-geometric duals of the six compatibility equations for shallow shells. In other words, the equilibrium equations with $p_1 = p_2 = p_n = q_1 = q_2 = q_n = 0$ can be obtained from the compatibility equations by replacing all strain measures in the latter by their dual stress measures according to the following table:

κ_{11}	κ_{22}	κ_{12}	κ_{21}	λ_1	λ_2
$-N_{22}$	$-N_{11}$	N_{21}	N_{12}	$-Q_2$	Q_1

ε_{11}	ε_{22}	ε_{12}	ε_{21}	γ_1	γ_2
M_{22}	M_{11}	$-M_{21}$	$-M_{12}$	$-P_2$	P_1

It is also well known that the equilibrium equations for thin shell theory may be satisfied identically by expressing all stress measures in terms of six stress functions, U_1, U_2, F, Φ_1, Φ_2 and Ω. In the absence of surface loads, the twelve stress function representations of the stress measures are known to be the static geometric duals of the strain displacement relations upon observing the duality between stress and strain measures given in the table above as well as the duality between displacement components and stress functions given in the table below

u_1	u_2	w	ϕ_1	ϕ_2	ω
U_1	U_2	F	Φ_1	Φ_2	Ω

As such, the stress function representations will not be listed here.

To complete the system of equations for a shallow shell theory, we still need to adopt a set of stress strain relations. Unless specifically stated otherwise, we will in the subsequent development take the stress strain relations in the form

$$\varepsilon_{11} = A(N_{11} - \nu_s N_{22}) \quad , \quad \varepsilon_{22} = A(N_{22} - \nu_s N_{11})$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}A(1 + \nu_s)(N_{12} + N_{21}) \quad , \quad \gamma_k = A_Q Q_k$$

$$M_{22} = D(\kappa_{22} + \nu_b \kappa_{11}) \quad , \quad M_{11} = D(\kappa_{11} + \nu_b \kappa_{22})$$

$$M_{21} = M_{12} = \frac{1}{2}D(1 - \nu_b)(\kappa_{21} + \kappa_{12}) \quad , \quad P_j = D_P \lambda_j \ .$$

With $\{A , \nu_s , A_Q\}$ taken as the dual of $\{-D , -\nu_b , -D_P\}$, respectively, the first half of the above stress strain relations is the static-geometric dual of the second half.

With suitably prescribed boundary conditions, the strain-displacement relations, stress strain relations and equilibrium equations define a well-posed boundary value problem in partial differential equations. From the general linear theory of shells [11], we know that, along an edge curve with a unit in-plane normal $\vec{\nu}$ and a unit in-plane tangent vector $\vec{t} = \vec{n} \times \vec{\nu}$, the appropriate stress boundary conditions are

$$\Delta N_{\nu\nu} = \Delta N_{\nu t} = \Delta Q_\nu = \Delta M_{\nu\nu} = \Delta M_{\nu t} = \Delta P_\nu = 0$$

where $\Delta f \equiv f - f^*$ and f^* is a prescribed quantity. The appropriate displacement boundary conditions along the same edge are

$$\Delta u_{\nu} = \Delta u_{t} = \Delta w = \Delta\phi_{\nu} = \Delta\phi_{t} = \Delta\omega = 0 \quad .$$

It is also known [11] that these displacement boundary conditions imply the following strain boundary conditions

$$\Delta\kappa_{tt} = \Delta\kappa_{t\nu} = \Delta\lambda_{t} = \Delta\varepsilon_{tt} = \Delta\varepsilon_{t\nu} = \Delta\gamma_{t} = 0 \quad .$$

Other physically realizable edge conditions call for different combinations of stress and displacement boundary conditions as well as conditions on different combinations of stress and displacement measures; they are too numerous to be listed here.

For the classical shell theory (with $\gamma_j \equiv P_k \equiv 0$ and $q_k \equiv q_n \equiv 0$) the appropriate (contracted) stress boundary conditions must be consistent with the virtual work principle. We know from [11] that there should be only four stress boundary conditions of the form

$$\Delta N_{\nu\nu} = \Delta N_{\nu t} = \Delta Q_{\nu}^{e} = \Delta M_{\nu\nu} = 0$$

with $Q_{\nu}^{e} = Q_{\nu} + \partial M_{\nu t}/\partial s$ where s is the arc length variable along the shell edge. Evidently, we have within the framework of a shallow shell theory, $\Delta N_{\nu\nu}^{e} \equiv (N_{\nu\nu} + M_{\nu t}/R_{\nu t}) \cong \Delta N_{\nu\nu}$ and $\Delta N_{\nu t}^{e} \equiv \Delta(N_{\nu t} + M_{t}/R_{tt}) \cong \Delta N_{\nu t}$. Similarly, the appropriate contracted strain boundary conditions are [11]

$$\Delta\kappa_{tt} = \Delta\kappa_{t\nu} = \Delta\lambda_{t}^{e} = \Delta\varepsilon_{tt} = 0$$

where $\lambda_{t}^{e} = \lambda_{t} - \partial\varepsilon_{t\nu}/\partial s$.

3. Reduction of the Classical Theory

With $\gamma_i \equiv 0$, we have from the strain-displacement relations

$$\phi_j = -w,_j \quad , \quad \kappa_{ij} = -w,_{ij}$$

and from the stress strain relations

$$M_{11} = -D(w,_{11} + \nu_b w,_{22}) \quad , \quad M_{22} = -D(w,_{22} + \nu_b w,_{11})$$

$$M_{12} = M_{21} = -D(1 - \nu_b)w,_{12} \ .$$

Two of the moment equilibrium equations with $P_j \equiv 0$ and $q_j \equiv 0$ (and with constant shell properties and uniform shell thickness) give

$$Q_j = -D(\nabla^2 w),_j$$

where $\nabla^2(\ ) = (\ ),_{11} + (\ ),_{22}$ is the two dimensional Laplace's operator in cartesian coordinates x_1 and x_2 . The remaining moment equilibrium equation (with $q_n \equiv 0$) reduces to $N_{12} = N_{21}$ so that we get from the stress strain relation for $\varepsilon_{12} = \varepsilon_{21}$

$$\varepsilon_{12} = \varepsilon_{21} = A(1 + \nu_s)N_{12} = A(1 + \nu_s)N_{21} \ .$$

The development up to this point is identical to those for the transverse bending of a flat plate. By the static-geometric duality, we expect a dual development identical to the theory of generalized plane stress.

For load components p_1 and p_2 which can be expressed in terms of a load potential $L(x_1,x_2)$ in the form

$$p_j = L,_j \qquad (j = 1,2) \quad ,$$

the first two force equilibrium equations are satisfied identically by the stress function representations

$$N_{11} = F,_{22} - L \quad , \quad N_{22} = F,_{11} - L \quad , \quad N_{12} = N_{21} = -F,_{12} \; .$$

The remaining force equilibrium equation becomes an equation for w and F :

$$D\nabla^4 w = L\{F,z\} + p_n - (\nabla^2 z)L \tag{2.1}$$

where $\nabla^4(\;) \equiv \nabla^2\nabla^2(\;)$ and

$$L\{f,g\} = g,_{11}f,_{22} + g,_{22}f,_{11} - 2g,_{12}f,_{12} \; . \tag{2.2}$$

Equation (2.1) reduces, as it should, to the governing PDE for the transverse bending of flat plates if $z \equiv 0$.

A second equation for w and F may be obtained from the dual consideration of the compatibility equations. It is a straightforward calculation to get from the stress strain relations

$$\varepsilon_{11} = A(F,_{22} - \nu_s F,_{11}) - A(1 - \nu_s)L$$

$$\varepsilon_{22} = A(F,_{11} - \nu_s F,_{22}) - A(1 - \nu_s)L$$

$$\varepsilon_{12} = \varepsilon_{21} = -A(1 + \nu_s)F,_{12}$$

and from the fourth and fifth compatibility equations (with $\gamma_j \equiv 0$)

$$\lambda_2 = A(\nabla^2 F),_1 - A(1 - \nu_s)L,_1 \quad , \quad -\lambda_1 = A(\nabla^2 F),_2 - A(1 - \nu_s)L,_2 \; .$$

The sixth compatibility equation with $\gamma_j \equiv 0$ reduces to $\kappa_{12} = \kappa_{21}$

which is trivially satisfied as $\kappa_{12} = \kappa_{21} = -w,_{12}$ follows as a consequence of $\gamma_j \equiv 0$ and the strain displacement relations. The first two compatibility equations are also satisfied by $\kappa_{ij} = -w,_{ij}$. The remaining compatibility equations may be written as an equation for w and F :

$$-A\nabla^4 F = L\{w,z\} - A(1 - \nu_s)\nabla^2 L \qquad (2.3)$$

which reduces, as it should, to the governing equation for the theory of generalized plane stress if $z \equiv 0$.

With suitably prescribed boundary conditions, the eighth order system of PDEs (2.1) and (2.3) determines F and w . All stress and strain measures may then be calculated from w and F and their various partial derivatives with the help of the various formulas obtained in this section [1,12]. With $\phi_k = -w,_k$ and

$$\varepsilon_{21} - \varepsilon_{12} = 2\omega + u_1,_2 - u_2,_1 = 0 \quad ,$$

all components of the rotation vector are known once we have u_1 and u_2 . These tangential displacement components are to be found by integrating the strain displacement relations

$$u_1,_1 = \varepsilon_{11} + z,_{11}w \quad , \qquad u_2,_2 = \varepsilon_{22} + z,_{22}w$$

$$u_1,_2 + u_2,_1 = \varepsilon_{12} + \varepsilon_{21} + 2z,_{12}w \quad .$$

From the first two equations, we have immediately

$$u_1 = \int^{x_1} (\varepsilon_{11} + z,_{11}w)dx_1 + f_1(x_2) \quad ,$$

$$u_2 = \int^{x_2} (\varepsilon_{22} + z,_{22}w)dx_2 + f_2(x_1) \quad ,$$

where f_1 and f_2 are arbitrary functions of their argument, and therewith

$$u_{1},_{2} + u_{2},_{1} = \int^{x_2}\int^{x_1} [(\varepsilon_{11} + z,_{11}w),_{22} + (\varepsilon_{22} + z,_{22}w),_{11}]dx_1dx_2 + f_{1},_{2} + f_{2},_{1} \quad .$$

With the help of the compatibility equations, we may write the above relation as

$$u_{1},_{2} + u_{2},_{1} = \varepsilon_{12} + \varepsilon_{21} + 2z,_{12}w,_{12} + f_{1},_{2} + f_{2},_{1} \quad .$$

For consistency with the strain displacement relations for ε_{12} and ε_{21}, we must have

$$f_{1},_{2} + f_{2},_{1} = 0$$

4. Reduction of the Theory for Transverse Shear-deformable Shells

If we have only $P_j \equiv 0$ (and $\gamma_j \not\equiv 0$), the reduction is more complicated but similar to that for flat plate with transverse shear deformations. The expression for γ_j may be written as

$$\phi_k = -w,_k + \gamma_k = -w,_k + A_Q Q_k$$

where we have made use of the stress strain relations to eliminate γ_k in favor of Q_k. The strain couples can then be written as

$$\kappa_{ij} = -w,_{ij} + (A_Q Q_j),_i \quad .$$

From the stress strain relations, we have for the stress couples:

$$M_{11} = -D(w,_{11} + \nu_b w,_{22}) + DA_Q(Q_1,_1 + \nu_b Q_2,_2)$$

$$M_{22} = -D(w,_{22} + \nu_b w,_{11}) + DA_Q(Q_2,_2 + \nu_b Q_1,_1)$$

$$M_{12} = M_{21} = -D(1 - \nu_b)w,_{12} + \frac{1}{2}D(1 - \nu_b)A_Q(Q_1,_2 + Q_2,_1)$$

where again we have assumed constant shell properties and uniform shell thickness.

Two of the moment equilibrium equations (with $P_j \equiv 0$ and $q_j \equiv 0$) now give

$$Q_1 = -D(\nabla^2 w),_1 + DA_Q\{\nabla^2 Q_1 + \frac{1}{2}(1 + \nu_b)\hat{X},_2\}$$

$$Q_2 = -D(\nabla^2 w),_2 + DA_Q\{\nabla^2 Q_2 - \frac{1}{2}(1 + \nu_b)\hat{X},_1\}$$

where

$$\hat{X} = Q_2,_1 - Q_1,_2 \quad .$$

It follows immediately from the expression for Q_1 and Q_2 that $\hat{X}$ satisfies the second order PDE

$$\frac{1}{2} DA_Q (1 - \nu_b) \nabla^2 \hat{X} - \hat{X} = 0 \quad . \tag{2.4}$$

With $P_j \equiv 0$, the sixth equilibrium equation (with $q_n \equiv 0$) reduces again to $N_{12} = N_{21}$ so that the stress strain relations for $\varepsilon_{12} = \varepsilon_{21}$ becomes

$$\varepsilon_{12} = \varepsilon_{21} = A(1 + \nu_s) N_{12} = A(1 + \nu_s) N_{21} \quad .$$

The first two force equilibrium equations admit the same stress function representations as in the classical theory. The remaining force equilibrium equation becomes

$$-D\nabla^4 w + DA_Q \nabla^2 (Q_{1,1} + Q_{2,2}) + L\{F,z\} + p_n - (\nabla^2 z) L = 0 \quad .$$

Upon using the same equilibrium equation to eliminate $Q_{1,1} + Q_{2,2}$, we get

$$D\nabla^4 w = [1 - DA_Q \nabla^2][L\{F,z\} + p_n - (\nabla^2 z) L] \tag{2.5}$$

which reduces to (2.1) for classical shell theory if $A_Q = 0$ and to the governing PDE for transverse bending of flat plates with transverse shear deformations if $z \equiv 0$.

For a second equation for F and w, we consider the correspon- ding development of the compatibility equations. Observing (2.4) and the expressions for κ_{ij}, the first, second and sixth compatibility equations are satisfied identically. The stress function representations for N_{ij} and for ε_{ij} are as in the classical theory; but with $\gamma_j \not\equiv 0$, the fourth and fifth compatibility equations now take the form

$$\lambda_2 = A(\nabla^2 F),_1 - A(1-\nu_s)L,_1 - A_Q(z,_{12}Q_2 - z,_{22}Q_1)$$

$$-\lambda_1 = A(\nabla^2 F),_2 - A(1-\nu_s)L,_2 + A_Q(z,_{11}Q_2 - z,_{12}Q_1) \quad .$$

These expressions along with the expressions for κ_{ij} may be substituted into the third compatibility equation. Remarkably, all terms involving Q_1 and Q_2 cancel out, leaving us with the same equation for w and F as the classical theory, namely, (2.3).

The two coupled fourth order PDEs (2.5) and (2.3) form again an eighth order system. In addition, we have to solve also the single second order PDE (2.4) as in the case of the transverse bending of flat plate with transverse shear deformations. The system (2.5), (2.3) and (2.4) are therefore to be supplemented by five boundary conditions along an edge of the shell. With the solution of boundary value problem for w, F and $\hat{X}$, we expect to be able to obtain all stress and strain measures of the shell as in the case of the classical theory, without solving another boundary value problem. However, we do not yet have auxiliary formulas which give Q_j explicitly in terms of w, $\hat{X}$ and F; evidently, some additional analysis is required to complete the solution process [12,13].

To get Q_j in terms of w, F and $\hat{X}$ alone, we need an expression for $\nabla^2 Q_j$. We get $\nabla^2 Q_1$ by writing the third force equilibrium equation as an expression for $Q_{1},_{1}$ and differentiation with respect to x_1 to get

$$Q_{1},_{11} = -Q_{2},_{21} - [L\{F,z\} + p_n - (\nabla^2 z)L],_1$$

and, after adding $Q_{1},_{22}$ to both sides of the above equation,

$$\nabla^2 Q_1 = -\hat{X},_2 - [L\{F,z\} + p_n - (\nabla^2 z)L],_1 \quad .$$

Upon writing the same force equilibrium equation as an expression for $Q_{2},_{2}$, we get in a similar way

$$\nabla^2 Q_2 = \hat{X},_1 - [L\{F,z\} + p_n - (\nabla^2 z)L],_2$$

With these expressions for $\nabla^2 Q_1$ and $\nabla^2 Q_2$, we can now write an expression for Q_j explicitly in terms of w , F and $\hat{X}$:

$$Q_1 = -D\{\nabla^2 w + A_Q[L\{F,z\} + p_n - (\nabla^2 z)L]\},_1 - \frac{1}{2}(1 - \nu_b)DA_Q\hat{X},_2$$

$$Q_2 = -D\{\nabla^2 w + A_Q[L\{F,z\} + p_n - (\nabla^2 z)L]\},_2 + \frac{1}{2}(1 - \nu_b)DA_Q\hat{X},_1 \quad .$$

From these expressions, we see that prescribing Q_ν along an edge of the shell (with unit in-plane normal vector $\vec{\nu}$) is equivalent to prescribing a linear combination of w , F (and their derivative up to third order) and the tangential derivative of $\hat{X}$.

We also have the stress couples, M_{ij} , in terms of $w,_{ij}$ and $Q_{i},_{j}$, with $Q_{i},_{j}$ themselves in terms of w , F and $\hat{X}$ (and their derivatives) alone. For example, we have

$$M_{12} = M_{21} = -D(1 - \nu_b)w,_{12} + \frac{1}{2}D(1 - \nu_b)A_Q(Q_1,_2 + Q_2,_1)$$

$$= -D(1 - \nu_b)\{w - DA_Q\nabla^2 w - DA_Q^2[L\{F,z\} + p_n - (\nabla^2 z)L]\},_{12}$$

$$-[\frac{1}{2}DA_Q(1 - \nu_b)]^2[\hat{X},_{22} - \hat{X},_{11}] \quad .$$

If M_{12} is prescribed along an $x_1 = x_1^*$ edge of the shell, we effectively have a boundary condition on a linear combination of w, F and $\hat{X}$ and their derivatives. The highest order derivative for w and F with respect to x_1 is third order. With $\hat{X}_{11}$ term eliminated by way of (2.4), no partial derivative with respect to x_1 appear in the boundary condition $M_{12} = M_{12}^*$. In general, normal derivatives of F and w higher than third order in a boundary condition involving M_{ij} should be eliminated with the help of the governing differential equations. Similarly, normal derivatives of $\hat{X}$ higher than first order should also be eliminated using (2.4).

5. Shallow Spherical Shells

By the static geometric duality, we know that equations for the general linear theory of shallow shells with $\gamma_j \equiv 0$ (but $P_j \not\equiv 0$) can also be reduced to two simultaneous fourth order PDEs for w and F and a second order PDE for the quantity $\lambda_{1},_{1} + \lambda_{2},_{2}$ alone. Except for surface load terms, these equations are just the static geometric dual of (2.1), (2.3) and (2.4). The derivation of these equations and the auxiliary formulas for the stress and strain measures is parallel to that given in the last section and will not be reported here.

The reduction of the shallow shell equations without the simplification of $\gamma_j \equiv 0$ or $P_j \equiv 0$ is much more complicated and not as complete as the previously discussed (more restricted) cases (see [12]). A complete reduction similar to the more restricted cases is possible for shallow spherical shells and will be described below.

We take $z = \frac{1}{2}\tau_o(x_1^2 + x_2^2)$ for shallow spherical shells with $1/\tau_o$ equal to spherical radius. The reduction proceeds as in section (5) giving M_{ij} in terms of $w,_{mn}$ and $Q_\ell,_k$. The fourth and fifth equilibrium equations may now be written as

$$Q_1 = -D(\nabla^2 w),_1 + DA_Q\{\nabla^2 Q_1 + \frac{1}{2}(1 + \nu_b)(Q_2,_1 - Q_1,_2),_2\} - \tau_o D_P \lambda_2$$

$$Q_2 = -D(\nabla^2 w),_2 + DA_Q\{\nabla^2 Q_2 - \frac{1}{2}(1 + \nu_b)(Q_2,_1 - Q_1,_2),_1\} + \tau_o D_P \lambda_1$$

and, upon forming the combinations $Q_1,_1 + Q_2,_2$ and $Q_2,_1 - Q_1,_2$, we get

$$X = -D\nabla^4 w + DA_Q \nabla^2 X - \tau_o D_P Y \tag{2.6}$$

$$\hat{X} = \frac{1}{2} DA_Q (1 - \nu_b) \nabla^2 \hat{X} + \tau_o D_P \hat{Y} \tag{2.7}$$

where

$$X = Q_{1\ 1} + Q_{2,2} \quad , \quad Y = \lambda_{2,1} - \lambda_{1,2}$$

$$\hat{X} = Q_{2,1} - Q_{1,2} \quad , \quad \hat{Y} = \lambda_{1,1} + \lambda_{2,2} \quad . \tag{2.8}$$

The static-geometric dual development with compatibility equations leads to the following expressions for the stress resultants N_{ij} and strain resultants ε_{ij} :

$$N_{11} = F,_{22} - L - (D_P\lambda_1),_2 \quad , \quad N_{22} = F,_{11} - L + (D_P\lambda_2),_1$$

$$N_{12} = -F,_{12} - (D_P\lambda_2),_2 \quad , \quad N_{21} = -F,_{12} + (D_P\lambda_1),_1$$

$$\varepsilon_{22} = A(F,_{11} - \nu_s F,_{22}) - A(1 - \nu_s)L + D_P A(\lambda_{2,1} + \nu_s \lambda_{1,2})$$

$$\varepsilon_{11} = A(F,_{22} - \nu_s F,_{11}) - A(1 - \nu_s)L - D_P A(\lambda_{1,2} + \nu_s \lambda_{2,1})$$

$$\varepsilon_{12} = \varepsilon_{21} = -A(1 + \nu_s)F,_{12} - \frac{1}{2}(1 + \nu_s) D_P A(\lambda_{2,2} - \lambda_{1,1}) \quad .$$

The fourth and fifth compatibility equations may then be written as

$$\lambda_2 = A(\nabla^2 F)_{,1} - A(1-\nu_s)L_{,1} + D_P A\{\nabla^2\lambda_2 - \frac{1}{2}(1-\nu_s)(\lambda_{1,1}+\lambda_{2,2})_{,2}\} + A_Q\tau_o Q_1$$

$$\lambda_1 = -A(\nabla^2 F)_{,2} + A(1-\nu_s)L_{,2} + D_P A\{\nabla^2\lambda_1 - \frac{1}{2}(1-\nu_s)(\lambda_{1,1}+\lambda_{2,2})_{,1}\} - A_Q\tau_o Q_2$$

and, upon forming the combinations $\lambda_{2,1} - \lambda_{1,2}$ and $\lambda_{1,1} + \lambda_{2,2}$, we get

$$Y = A\nabla^4 F - A(1-\nu_s)\nabla^2 L + D_P A\nabla^2 Y + \tau_o A_Q X \tag{2.9}$$

$$\hat{Y} = \frac{1}{2}D_P A(1+\nu_s)\nabla^2\hat{Y} - \tau_o A_Q\hat{X} \tag{2.10}$$

where X, Y, $\hat{X}$ and $\hat{Y}$ are as defined in (2.8)

With the expressions for N_{ij}, the third equilibrium equation may be written as

$$X + \tau_o D_P Y = -\tau_o(\nabla^2 F - 2L) - p_n \quad .$$

Similarly, the third compatibility equation may be written as

$$Y - \tau_o A_Q X = -\tau_o \nabla^2 w \quad .$$

The last two equations may be solved for X and Y in terms of F, w and the load terms to get

$$(1 + \tau_o^2 D_P A_Q)X = -\tau_o(\nabla^2 F - 2L) - p_n + \tau_o^2 D_P \nabla^2 w$$

$$(1 + \tau_o^2 D_P A_Q)Y = -\tau_o\nabla^2 w - \tau_o^2 A_Q(\nabla^2 F - 2L) - \tau_o A_Q p_n \quad .$$

These expressions are then used to eliminate X and Y from (2.6) and (2.7) to get two simultaneous PDEs for w and F :

$$D\nabla^4 w = [1 + \tau_o^2 D_P A_Q - DA_Q\nabla^2][\tau_o\nabla^2 F - 2\tau_o L + p_n] \qquad (2.11)$$

$$-A\nabla^4 F = [1 + \tau_o^2 D_P A_Q - D_P A\nabla^2][\tau_o\nabla^2 w] - A(1 - \nu^2)[1 + \tau_o^2 D_P A_Q]\nabla^2 L$$
$$+ \tau_o A_Q D_P A\nabla^2(2\tau_o L - p_n) \quad . \qquad (2.12)$$

For $D_P = 0$, equations (2.8) and (2.9) reduces to (2.5) and (2.3), respectively, with the latter specialized to the case of a shallow spherical shell. With $\tau_o = 0$, (2.8) reduces to one of the two governing equations for the transverse bending of a transverse shear-deformable flat plate, while (2.9) reduces to one of the two governing equations for Schaefer's theory of generalized plane stress including moment stress couple effects [14].

The eighth order system (2.11) and (2.12) is complemented by the fourth order system (2.7) and (2.10) for $\hat{X}$ and $\hat{Y}$. The two equations (2.7) and (2.10) uncouple for $A_Q = 0$, $D_P = 0$ or $\tau_o = 0$; they reduce correctly to known results for these special cases. We expect all stress and strain measures should be expressed in terms of w , F , $\hat{X}$ and $\hat{Y}$ without solving additional PDEs. For this purpose, we need expressions for $\nabla^2 Q_j$ and $\nabla^2\lambda_k$ in terms of the four functions w , F , $\hat{X}$ and $\hat{Y}$. Similar to the case of transverse shear-deformable shells, we write the third equilibrium equation in the form

$$\nabla^2 Q_1 + \tau_o D_P \nabla^2 \lambda_2 = -\hat{X},_2 + \tau_o D_P \hat{Y},_2 - [\tau_o(\nabla^2 F - L) + p_n],_1$$

and write the third compatibility equation in the form

$$\nabla^2 \lambda_2 - \tau_o A_Q \nabla^2 Q_1 = \hat{Y},_2 + \tau_o A_Q \hat{X},_2 - [\tau_o \nabla^2 w],_1 \quad .$$

These two equations may be solved for $\nabla^2 Q_1$ and $\nabla^2 \lambda_2$ in terms of $\hat{X}$, $\hat{Y}$, F and w. Similar manipulations with the third equilibrium and compatibility equation also give $\nabla^2 Q_2$ and $\nabla^2 \lambda_1$ in terms w, F, $\hat{X}$ and $\hat{Y}$. Next, we take fourth and fifth equilibrium and compatibility equations in the form

$$Q_1 + \tau_o D_P \lambda_2 = -D(\nabla^2 w),_1 + DA_Q\{\nabla^2 Q_1 + \frac{1}{2}(1 + \nu_b)\hat{X},_2\}$$

$$Q_2 - \tau_o D_P \lambda_1 = -D(\nabla^2 w),_2 + DA_Q\{\nabla^2 Q_2 - \frac{1}{2}(1 + \nu_b)\hat{X},_1\}$$

$$\lambda_2 - \tau_o A_Q Q_1 = A(\nabla^2 F),_1 - A(1 - \nu_s)L,_1 + D_P A\{\nabla^2 \lambda_2 - \frac{1}{2}(1 - \nu_s)\hat{Y},_2\}$$

$$\lambda_1 + \tau_o A_Q Q_2 = A(\nabla^2 F),_2 - A(1 - \nu_s)L,_2 + D_P A\{\nabla^2 \lambda_1 - \frac{1}{2}(1 - \nu_s)\hat{Y},_1\} \quad .$$

The right hand side of these equations can now be expressed in terms of F, w, $\hat{X}$ and $\hat{Y}$ alone. The first and third can be solved for Q_1 and λ_2 while the second and fourth can be solved for Q_2 and λ_1, all in terms of F, w, $\hat{Y}$ and $\hat{X}$.

6. Stretching, Twisting and Bending of Pretwisted Strips

The canonical form of two simultaneous equations for F and w (along with one or more auxiliary equations when $\gamma_j \not\equiv 0$ and/or $P_j \not\equiv 0$) is very useful in the solution process for many shallow shell problems. For other problems, the original or some intermediate form of the shallow shell equations may be more appropriate. To illustrate, we consider in this section a class of problems for a pretwisted strip (or shallow hyperbolic paraboloidal shell) using the well-known semi-inverse procedure of the theory of elasticity. The middle surface of the shell is given by $z = \tau_o x_1 x_2$, with $|x_1| \le a$, $|x_2| \le b$ $(b << a)$ and τ_o is the uniform pretwist of the strip. The shell is subject to no surface loads so that $p_j \equiv q_k \equiv 0$ and the two (long) edges $x_2 = \pm b$ are free of edge tractions, so that

$$x_2 = \pm b : \qquad N_{22} = N_{21} = Q_2 = M_{22} = M_{21} = P_2 = 0 \quad .$$

The ends $x_1 = \pm a$, are subject to edge resultants and couples resulting in equal and opposite axial forces of magnitude F_x , torques of magnitude T_x , plate-bending moments of magnitude M_p turning about the x_2 direction, and sheet-bending moments of magnitude M_s turning about the z direction. Expressed in terms $\vec{N}_1$ and $\vec{M}_1$, these overall end conditions take the form

$$x_1 = \pm a : \quad \begin{aligned} &\int_{-b}^{b} \vec{N}_1 dx_2 = F_x \vec{i}_1 \quad , \\ &\int_{-b}^{b} (\vec{M}_1 + \vec{r} \times \vec{N}_1) dx_2 = T_x \vec{i}_1 + M_p \vec{i}_2 + M_s \vec{i}_3 \quad . \end{aligned}$$

In scalar form, we have from these two vector relations (keeping in mind that $Q_j = O(\xi_o N_{mn})$ and hence $M_{ij} = O(\xi_o a N_{mn})$)

$$\int_{-b}^{b} N_{11} dx_2 = F_x \quad , \qquad \int_{-b}^{b} N_{12} dx_2 = 0 \quad ,$$

$$\int_{-b}^{b} (Q_1 + z,_1 N_{11} + z,_2 N_{12}) dx_2 = 0 \quad ,$$

$$\int_{-b}^{b} [M_{12} + z,_1 P_1 - x_2(Q_1 + z,_1 N_{11})] dx_2 = T_x \quad ,$$

$$\int_{-b}^{b} (M_{11} - z,_2 P_1 + z N_{11}) dx_2 = M_p \quad , \qquad \int_{-b}^{b} (P_1 - x_2 N_{11}) dx_2 = M_s$$

where we have used the first three conditions and $z,_2 x_2 = z$ for shallow hyperbolic paraboloid to simplify the last three conditions.

We seek a solution of the shallow shell equations for the pretwisted strip subject to the above stress boundary conditions along $x_2 = \pm b$ and the overall end conditions (without further details on the actual distributions of stress resultants and stress couples) along $x_1 = \pm a$. It is customary to invoke Saint Venant's principle (for shells) and conclude that the difference between the solution obtained for this Saint Venant type shell problem and the exact solution corresponding the actual edge distributions for resultants and couples should be confined to a narrow region of order b^2 in area adjacent to the edges $x_1 = \pm a$. In other words, this difference is expected to be a boundary layer phenomenon. For simplicity, we assume uniform shell thickness and material properties in the solution process although the restriction can be relaxed to allow the shell thickness and material properties to vary with x_2 but independent of x_1. We shall use an intrinsic formulation of the shell problem for our semi-inverse solution procedure and take equilibrium equations, stress strain relations and compatibility equations as our basic system of shallow shell equations.

To obtain a solution for our Saint Venant type problem, we recall the structure of the known solution for the corresponding problem of a flat strip and consider a solution of our shell equations with all stress resultants independent of x_1. In the absence of surface loads, this assumption simplifies the three force equilibrium equations to read

$$N_{21},_2 = 0 \quad , \quad N_{22},_2 = 0 \quad , \quad Q_2,_2 + \tau_o(N_{12} + N_{21}) = 0 \quad .$$

The first two of these together with the boundary conditions $N_{21} = N_{22} = 0$ along $x_2 = \pm b$ give

$$N_{21} \equiv 0 \quad , \quad N_{22} \equiv 0 \qquad (2.13a,b)$$

and the third reduces to

$$Q_2,_2 + \tau_o N_{12} = 0 \quad . \qquad (2.13c)$$

The assumption that N_{ij} and Q_j are independent of x_1 and the form of the stress strain relations also require that ε_{ij} and γ_j be independent of x_1. In that case, the last three compatibility equations become

$$\varepsilon_{12},_2 + \lambda_2 + \tau_o\gamma_2 = 0 \quad , \quad \varepsilon_{11},_2 + \lambda_1 - \tau_o\gamma_1 = 0 \qquad (2.14a,b)$$

$$\gamma_1,_2 + \kappa_{12} - \kappa_{21} = 0 \quad . \qquad (2.14c)$$

It follows that λ_j and $\kappa_{12} - \kappa_{21}$ are also independent of x_1. By the third compatibility equation, we have then κ_{12} and κ_{21} separately independent of x_1. The remaining two compatibility equations give

$$\kappa_{11} = \kappa_{11}(x_1) \quad , \quad \kappa_{22} = -\nu_b c_1 + x_1 \kappa_{12},_2(x_2)$$

where c_1 is an arbitrary constant.

The stress strain relations then imply that $M_{12} = M_{21}$ and P_j are all independent of x_1 so that the last two moment equilibrium equations become

$$M_{22},_2 - Q_2 + \tau_o P_2 = 0 \quad , \quad P_2,_2 + N_{12} = 0 \quad , \qquad (2.15a,b)$$

keeping in mind that we have $N_{21} \equiv 0$ from (2.13a). Differentiate (2.15a) with respect to x_2 and simplify the resulting equation with the help (2.13c) and (2.15b) to get

$$M_{22},_{22} = 0 \quad \text{or} \quad M_{22} = m_o(x_1) + m_1 x_2$$

where $m_o(x_1)$ is arbitrary function and m_1 is an unknown constant given that $M_{22},_2$ is independent of x_1. The boundary conditions $M_{22} = 0$ along $x_2 = \pm b$ require that $m_o(x_1) = m_1 = 0$. Therefore, we have

$$M_{22} \equiv 0 \quad , \quad Q_2 = \tau_o P_2 \qquad (2.15c,d)$$

and, from the stress strain relation for M_{22}, $\nu_b \kappa_{11} = -\kappa_{22} = \nu_b c_1 - x_1 \kappa_{12},_2(x_2)$. But κ_{11} is independent of x_2; therefore, we have $\kappa_{12},_2 = -\nu_b c_5$ and therewith

$$\kappa_{12} = -(c_3 + \nu_b c_5 x_2) \quad , \quad \kappa_{11} = c_1 + c_5 x_1 \quad , \quad \kappa_{22} = -\nu_b \kappa_{11} \qquad (2.16a,b,c)$$

where c_1, c_3 and c_5 are arbitrary constants. We can eliminate κ_{21} from the third compatibility equation with the help of the sixth to get $-\lambda_{1},_{2} + 2\tau_o\kappa_{12} + \tau_o\gamma_{1},_{2} = 0$ or

$$\lambda_1 - \tau_o\gamma_1 = 2[c_4 - \tau_o(c_3x_2 + \frac{1}{2}\nu_b c_5 x_2^2)] \qquad (2.16d)$$

where c_4 is a new constant of integration. The fifth compatibility equation and the stress strain relation for ε_{11} then gives

$$\varepsilon_{11} = c_2 - 2c_4x_2 + \tau_o(c_3x_2^2 + \frac{1}{3}\nu_b c_5 x_2^3)$$

$$= A(N_{11} - \nu_s N_{22}) = AN_{11} \qquad (2.16e)$$

where c_2 is another constant of integration.

It remains now to determine κ_{21}, N_{12}, γ_1 and P_2 (with (2.15d), (2.16d) and the stress-strain relations giving us the remaining unknown stress and strain measures). For N_{12} and P_2, we have from (2.14a), (2.15b), (2.15d), $P_2 = D_P\lambda_2$ and $\gamma_2 = A_Q Q_2$

$$D_P A_S \ddot{P}_2 - (1 + \tau_o^2 D_P A_Q)P_2 = 0 \quad ,$$

where $(\ )^{\bullet} \equiv (\ ),_2$ and $A_S = \frac{1}{2}A(1 + \nu_s)$. The second order ODE for P_2, supplemented by the boundary conditions $P_2 = 0$ along $x_2 = \pm b$, gives the trivial solution

$$P_2 \equiv 0 \quad , \quad Q_2 = \tau_o P_2 \equiv 0 \quad , \quad N_{12} = P_{2},_{2} \equiv 0 \qquad (2.17a)$$

and hence

$$\gamma_2 \equiv \varepsilon_{12} \equiv \varepsilon_{21} \equiv 0 \quad . \tag{2.17b}$$

For γ_1 and κ_{21}, we note that the fifth equilibrium equation and (2.14c) now take the form

$$M_{21,2} - Q_1 - \tau_o P_1 = -M_{11,1} = -D(1 - \nu_b^2)c_5 \quad ,$$

$$\gamma_{1,2} - \kappa_{21} = -\kappa_{12} = c_3 + \nu_b c_5 x_2 \quad ,$$

Along with (2.16d), $P_1 = D_P \lambda_1$ and $\gamma_1 = A_Q Q_1$, they give a second order ODE for M_{21}. Supplemented by the two boundary conditions $M_{21} = 0$ along $x_2 = \pm b$, this ODE determines M_{21} and hence all the remaining stress and strain measures with the five constants of integration $c_1, c_2, \ldots, c_5$, as parameters. These five constants are to be determined by the five non-trivial overall end conditions along $x_1 = \pm a$. (The second condition corresponding to no resultant force in the x_2 direction is trivially satisfied by $N_{12} \equiv 0$.) For example, it follows immediately from the last two end conditions

$$D(1 - \nu_b^2)\int_{-b}^{b} (c_1 + c_5 x_1)dx_2 - \tau_o x_1 \int_{-b}^{b} (P_1 - x_2 N_{11})dx_2 = M_p$$

or

$$2bD(1 - \nu_b^2)[c_1 + c_5 x_1] = M_p + \tau_o M_s x_1$$

so that

$$c_1 = \frac{M_p}{2bD(1-\nu_b^2)} \quad , \quad c_5 = \frac{\tau_o M_s}{2bD(1-\nu_b^2)} \quad . \tag{2.18}$$

Note that the two constants c_1 and c_5 are unaffected by the prescence or absence of transverse shear deformability and/or moment stress couples turning about the midsurface normal. In particular, the expressions in (2.18) remain unchanged for the classical shallow shell theory. The solution for the other three constants c_2 , c_3 and c_4 requires the solution of the boundary value problem for M_{21} . We will not carry out the straightforward solution process for the determination of M_{21} and the three parameters c_2 , c_3 and c_4 . The final results and some intermediate calculations can be found in [12].

3. Formulation in Polar Coordinates

1. Linear Theory in Polar Coordinates

In circular cylindrical coordinates (r,θ,z), the position vector of a point on the middle surface of a thin surface is given by

$$\vec{r} = r\vec{i}_r + z(r,\theta)\vec{i}_z$$

where $\vec{i}_r$ and $\vec{i}_z$ are unit vector in the radial and axial direction, respectively. With $\vec{i}_\theta = \vec{i}_z \times \vec{i}_r$, we have the surface coordinate tangent vectors

$$\vec{r},_1 \equiv \vec{r}' = \vec{i}_r + z'\vec{i}_z \quad , \quad \vec{r},_2 \equiv \vec{r}^{\bullet} = r\vec{i}_\theta + z^{\bullet}\vec{i}_z \quad .$$

From

$$|\vec{r}'|^2 = 1 + (z')^2 \cong 1 \quad , \quad |\vec{r}^{\bullet}|^2 = r^2 + (z^{\bullet})^2 \cong r^2$$

(so that $\alpha_1 \cong 1$ and $\alpha_2 \cong r$) for shallow shells, we have the (approximately) unit tangent vectors

$$\vec{t}_1 \cong \vec{i}_r + z'\vec{i}_z \quad , \quad \vec{t}_2 = \vec{i}_\theta + \frac{z^{\bullet}}{r}\vec{i}_z \quad ,$$

with

$$\vec{t}_1 \cdot \vec{t}_2 = \frac{1}{r}z^{\bullet}z' \cong 0$$

so that the surface coordinates are orthogonal within the framework of a shallow shell theory. The corresponding unit normal to the middle surface is then

$$\vec{n} = -z'\vec{i}_r - \frac{1}{r}z^{\bullet}\vec{i}_\theta + \vec{i}_z$$

for a shallow shell. From the Weingarten formulas in differential geometry, we have the following shallow shell expressions for the curvature measures of the undeformed middle surface [1]:

$$\frac{1}{R_{11}} = -z'' \quad , \quad \frac{1}{R_{12}} = -\left(\frac{z^{\bullet}}{r}\right)' \quad , \quad \frac{1}{R_{22}} = -\left(\frac{1}{r}z' + \frac{1}{r^2}z^{\bullet\bullet}\right) \quad .$$

The various expressions for geometrical features of the undeformed middle surface obtained above and the order of magnitude relations among the displacement components simplify the general linear strain displacement relations given in [11] considerably. With $\xi_1 = r$ and $\xi_2 = \theta$, the resulting strain displacement relations for shallow shells are

$$\varepsilon_{11} = u_1' - z''w \quad , \quad \varepsilon_{22} = \frac{1}{r}u_2^{\bullet} + \frac{1}{r}u_1 - \left(\frac{1}{r^2}z^{\bullet\bullet} + \frac{1}{r}z'\right)w$$

$$\varepsilon_{12} = u_2' - \left(\frac{z^{\bullet}}{r}\right)'w - \omega \quad , \quad \varepsilon_{21} = \frac{1}{r}u_1^{\bullet} - \frac{1}{r}u_2 - \left(\frac{z^{\bullet}}{r}\right)'w + \omega$$

$$\gamma_1 = \phi_1 + w' \quad , \quad \gamma_2 = \phi_2 + \frac{1}{r}w^{\bullet}$$

$$\kappa_{11} = \phi_1' \quad , \quad \kappa_{22} = \frac{1}{r}(\phi_2^{\bullet} + \phi_1)$$

$$\kappa_{12} = \phi_2' \quad , \quad \kappa_{21} = \frac{1}{r}(\dot{\phi}_1 - \phi_2)$$

$$\lambda_1 = \omega' + (\frac{\dot{z}}{r})'\phi_1 - z''\phi_2 \quad , \quad \lambda_2 = \frac{1}{r}\dot{\omega} - (\frac{\dot{z}}{r})'\phi_2 + (\frac{1}{r}z' + \frac{1}{r^2}\ddot{z})\phi_1 \ .$$

To have a virtual work principle, the corresponding equilibrium equations should be taken in the form

$$(rN_{11})' + \dot{N}_{21} - N_{22} + rp_1 = 0 \quad , \quad (rN_{12})' + \dot{N}_{22} + N_{21} + rp_2 = 0$$

$$(rQ_1)' + \dot{Q}_2 + z''(rN_{11}) + (\frac{1}{r}z' + \frac{1}{r^2}\ddot{z})(rN_{22}) + (\frac{\dot{z}}{r})'[r(N_{12} + N_{21})] + rp_n = 0$$

$$(rM_{11})' + \dot{M}_{21} - M_{22} - rQ_1 - (\frac{\dot{z}}{r})'(rP_1) - (\frac{1}{r}z' + \frac{1}{r^2}\ddot{z})(rP_2) = 0$$

$$(rM_{12})' + \dot{M}_{22} - M_{21} - rQ_2 + z''(rP_1) + (\frac{\dot{z}}{r})'(rP_2) = 0$$

$$(rP_1)' + \dot{P}_2 + r(N_{12} - N_{21}) = 0$$

where we have assumed the absence of surface moment loads $(q_1 \equiv q_2 \equiv q_n \equiv 0)$ for simplicity.

We complete the system of shallow shell equations in polar coordinates by adopting the same set of stress strain relations used in chapter 2. It is not difficult to repeat the reduction of the resulting complete system of shell equations for the various special cases treated there.

2. Classical Theory

For shells with $\gamma_j \equiv P_j \equiv 0$ and with uniform thickness and material properties, the systems of shell equations stated in the last section may be reduced to two coupled fourth order PDEs for w and F similar to the case of cartesian coordinates. With $q_1 \equiv q_2 \equiv q_n \equiv 0$, these two equations take the form

$$D\nabla^4 w = L\{F,z\} + p_n - (\nabla^2 z)L \tag{3.1a}$$

$$-A\nabla^4 F = L\{w,z\} - A(1-\nu_s)\nabla^2 L \tag{3.1b}$$

where

$$\nabla^2(\ ) = (\ )'' + \frac{1}{r}(\ )' + \frac{1}{r^2}(\ )^{\cdot\cdot} \quad , \quad \nabla^4(\ ) = \nabla^2\nabla^2(\ )$$

$$L\{f,g\} = g''\left(\frac{1}{r}f' + \frac{1}{r^2}f^{\cdot\cdot}\right) + f''\left(\frac{1}{r}g' + \frac{1}{r}g^{\cdot\cdot}\right) - 2\left(\frac{g^{\cdot}}{r}\right)'\left(\frac{f^{\cdot}}{r}\right)' \tag{3.1c}$$

and L is a load potential with

$$p_1 = L' \quad , \quad p_2 = \frac{1}{r}L^{\cdot} \quad . \tag{3.1d}$$

In terms of F and w, we have the following auxiliary formulas for the stress measures

$$N_{11} = \frac{1}{r}F' + \frac{1}{r^2}F^{\cdot\cdot} - L \quad , \quad N_{22} = F'' - L \quad , \quad N_{12} = N_{21} = -\left(\frac{F^{\cdot}}{r}\right)'$$

$$M_{11} = -D[w'' + \nu_b(\frac{1}{r}w' + \frac{1}{r}w^{\cdot\cdot})] \quad , \quad M_{22} = -D[\nu_b w'' + (\frac{1}{r}w' + \frac{1}{r^2}w^{\cdot\cdot})]$$

$$M_{12} = M_{21} = -D(1 - \nu_b)(\frac{w^{\cdot}}{r})'$$

$$Q_1 = -D(\nabla^2 w)' \quad , \quad Q_2 = -\frac{1}{r}D(\nabla^2 w)^{\cdot} \quad .$$

The corresponding strain measures can be obtained from the stress strain relations and compatibility equations. Except for load terms, they are the static geometric duals of the above stress measures. The two tangential displacement components may be obtained by integrating the strain displacement relations as discussed at the end of section (3) of chapter 2.

As in the cartesian coordinates case, the two fourth order PDEs (3.1a) and (3.1b) can be combined into a single fourth order complex PDE for a complex potential $X = w - i\sqrt{A/D}\,F$:

$$\nabla^4 X = \frac{i}{\sqrt{DA}} L\{X,z\} + \frac{1}{D}[p_n - (\nabla^2 z)L] - i\sqrt{\frac{A}{D}}(1 - \nu_s)\nabla^2 L \quad .$$

With $\sqrt{DA} = 0(h)$, an asymptotic solution for X is appropriate if the thickness-to-rise ratio is small compared to unity.

For the special case of a shallow spherical shell with $z = \xi_o r^2/2r_o$, the complex equation for X becomes

$$\nabla^4 X = \frac{i\xi_o}{r_o\sqrt{DA}}\nabla^2 X + \frac{1}{D}[p_n - 2\frac{\xi_o}{r_o}L] - i\sqrt{\frac{A}{D}}(1 - \nu_s)\nabla^2 L \quad .$$

The complementary solution for this complex PDE may be written as

$$X = X_I + X_E$$

with

$$\nabla^2 X_I = 0 \quad , \quad \nabla^2 X_E - \frac{i\xi_o}{r_o\sqrt{DA}} X_E = 0 \quad .$$

For a shell frustum (which is complete in the circumferential direction), Fourier decomposition of the solution in θ is possible, and complementary solutions of the complex equation for X may be sought in the form of $X = X_n(r)\cos n\theta$, where n is positive integer. We have in that case

$$X_{In}(r) = c_{1n} r^n + c_{2n} r^{-n}$$

$$X_{En}(r) = c_{3n} I_n(\sqrt{i}\beta r/r_o) + c_{4n} K_n(\sqrt{i}\beta r/r_o)$$

where

$$\beta^2 = \frac{\xi_o r_o}{\sqrt{DA}} \quad .$$

$$I_n(\sqrt{i}\,x) = \mathrm{ber}_n(x) + i\,\mathrm{bei}_n(x) \quad , \quad K_n(\sqrt{i}\,x) = \mathrm{ker}_n(x) + i\,\mathrm{kei}_n(x)$$

and c_{jn} , $j = 1,\dots,4$, are arbitrary complex constants. Among the notable problems solved by the above solution, we mention only two. One is the phenomenon of stress concentration around a small circular hole centered at the apex of the shallow spherical shell under the transverse twisting action of a self-equilibrating system of four concentrated forces applied at the four corners of the square base planform (or under uniform membrane shear along the four shell edges whose projections on the base plane are the four edges of a square). Only the solution for $n = 2$ is needed for this class of problems [15,16]. The other is the general stress boundary value problem of a shallow spherical cap (closed at the apex) under self-equilibrating edge loads $(n \geq 2)$. The solution for this second problem demonstrated for the first time that the dominant stress state of a shell away from its edge(s) is not necessarily always a membrane state (see [17,18] and references therein).

Other shallow shell shapes for which an exact complementary solution of the shallow shell equations in polar coordinates is possible in terms of known functions, include logarithmic shells (with $z = \xi_o r_o \ell n(r/r_o)$) and the conical shells (with $z = \xi_o r$). More generally, we have for all shells of revolution $z^{\bullet} \equiv 0$ so that

$$L\{f,z\} = z''\left(\frac{1}{r}f' + \frac{1}{r^2}f^{\bullet\bullet}\right) + \frac{1}{r}z'f''$$

and a Fourier decomposition of the exact solution is possible. This decomposition allows for the reduction of the PDE for X to and ODE for $X_n(r)$ by focussing consideration to a solution of the form $X = X_n(r)\cos n\theta$. However, we know from well known results for plate bending and for generalized plane stress that solutions which are non-periodic in θ are sometimes required for problems involving axisymmetric loads and lateral loads. At the same time, it is also known that consideration of these nonperiodic functions may be avoided if use is made of the existence of four first integrals of the equilibrium and compatibility equations to reduce the order of the governing system of shell equations (see [19]). We will examine in later sections some lower order system shell equations for these two special load cases.

3. Transverse Shear-deformable Shallow Shells

In this section, we carry out the reduction of the transverse shear deformable shallow shell equations in polar coordinates to a canonical form similar to that for the cartesian coordinates in chapter 2. We begin as before by writing the expressions for γ_j as

$$\phi_1 = A_Q Q_1 - w' \quad , \quad \phi_2 = A_Q Q_2 - \frac{1}{r} w^{\bullet}$$

and the expressions for κ_{ij} as

$$\kappa_{11} = -w'' + A_Q Q_1' \quad , \quad \kappa_{22} = -\left(\frac{1}{r} w' + \frac{1}{r^2} w^{\bullet\bullet}\right) + \frac{A_Q}{r}(Q_2^{\bullet} + Q_1)$$

$$\kappa_{12} = -\left(\frac{w^{\bullet}}{r}\right)' + A_Q Q_2' \quad , \quad \kappa_{21} = -\left(\frac{w^{\bullet}}{r}\right)' - \frac{A_Q}{r}(Q_1^{\bullet} - Q_2) \quad . \tag{3.2}$$

For simplicity, we assume uniform shell thickness and material properties throughout this section.

The stress strain relations then give M_{ij} in terms of w , Q_1 and Q_2 . We do not list these expressions for M_{ij} here but use them in the fourth and fifth equilibrium equations in conjunction with the axial force equilibrium equation to get

$$Q_1 = -D[\nabla^2 w + A_Q\{L[F,z] + p_n - (\nabla^2 z)L\}]' - \frac{1}{2} D A_Q (1 - \nu_b) r^{-1} \hat{X}^{\bullet}$$

$$Q_2 = -D r^{-1} [\nabla^2 w + A_Q\{L[F,z] + p_n - (\nabla^2 z)L\}]^{\bullet} + \frac{1}{2} D A_Q (1 - \nu_b) \hat{X}' \tag{3.3}$$

where

$$\nabla^2(\ ) \equiv (\ )'' + r^{-1}(\ )' + r^{-2}(\ )^{\cdot\cdot}$$

$$\hat{X} \equiv r^{-1}[(rQ_2)' - Q_1^{\cdot}]$$

$$L[F,z] = z''(r^{-1}F' + r^{-2}F^{\cdot\cdot}) + (r^{-1}z' + r^{-2}z^{\cdot\cdot})F'' - 2(r^{-1}z^{\cdot})'(r^{-1}F^{\cdot})' \tag{3.4}$$

and where we have made use of the stress function solution

$$N_{11} = r^{-1}F' + r^{-2}F^{\cdot\cdot} - L \quad , \quad N_{22} = F'' - L \quad , \quad N_{12} = N_{21} = -(r^{-1}F^{\cdot})'$$

for the first, second and sixth equilibrium equations, assuming

$$p_1 = L' \qquad , \qquad p_2 = r^{-1}L^{\cdot}$$

(and, of course, $q_1 \equiv q_2 \equiv q_n \equiv 0$ as stated previously).

We now insert the expression for Q_1 and Q_2 into the axial force equilibrium equation to get one equation for w and F :

$$D\nabla^4 w = [1 - DA_Q\nabla^2]\{L[F,z] + p_n - [\nabla^2 z]L\} \quad . \tag{3.5a}$$

A different combination of the two expressions in (3.3), namely,

$r^{-1}[(rQ_2)' - Q_1^{\bullet}]$, gives an equation for $\hat{X}$:

$$\frac{1}{2}DA_Q(1 - \nu_b)\nabla^2\hat{X} = \hat{X} \quad . \tag{3.5b}$$

With stress function representation for N_{ij} given above, it is straightforward to obtain from the fourth and fifth compatibility equations

$$\lambda_2 = A(\nabla^2 F)' - A(1 - \nu_s)L' - (r^{-1}z^{\bullet})'\gamma_2 + (r^{-1}z' + r^{-2}z^{\bullet\bullet})\gamma_1$$

$$-\lambda_1 = Ar^{-1}(\nabla^2 F)^{\bullet} - A(1 - \nu_s)r^{-1}L^{\bullet} + z''\gamma_2 - (r^{-1}z^{\bullet})'\gamma_1 \quad . \tag{3.6}$$

The third compatibility equation then gives

$$-A\nabla^4 F = L[w,z] - A(1 - \nu_s)\nabla^2 L \tag{3.5c}$$

as all terms involving γ_1 and γ_2 (rather remarkably) cancel out just as in the cartesian coordinate case.

Equations (3.5a)-(3.5c), supplemented by suitable boundary conditions, determine w, F and $\hat{X}$. All other stress and strain measures can then be obtained from w , F and $\hat{X}$ without solving another boundary value problem.

4. Axisymmetric Bending of Shallow Transverse Shear-deformable Shells of Revolution

The existence of four first integrals for the equilibrium and compatibility equations ensures a lower order system of governing equations for problems involving axisymmetric stress distributions. For a shell theory with transverse shear deformation but not moment stress couples $(P_j \equiv 0)$, the governing equations are expected to be sixth order. It is known that we need only to deal with a fourth order system and a second order system separately as the problem of bending and stretching is uncoupled from the problem of torsion and twisting. The general analysis leading to this conclusion and the final results of the analysis can be found in [19] which also gives the results and analysis for the general theory with $P_j \not\equiv 0$. In this section, we limit ourselves to obtaining a lower order system of governing equations for the axisymmetric bending (and stretching) problem directly from the general results obtained in the last section.

For shells of revolution, we have $z^{\bullet} \equiv 0$ and for axisymmetric bending (and stretching), we take $\hat{X} \equiv 0$ and $w^{\bullet} \equiv F^{\bullet} \equiv 0$ so that (3.5a) and (3.5c) simplify to read

$$D\Delta^2 w = [1 - DA_Q\Delta]\{z''r^{-1}F' + r^{-1}z'F'' + p_n - (\Delta z)L\} \tag{3.5a'}$$

$$-A\Delta^2 F = \{z''r^{-1}w' + r^{-1}z'w'' - A(1 - \nu_s)\Delta L\} \tag{3.5c'}$$

where $\Delta(\ ) \equiv (\ )'' + r^{-1}(\ )'$. Upon setting $w' = \phi$ and $F' = \Phi$, (3.5a') and (3.5c') may be written as

$$\Delta_1[\phi] - [1 - DA_Q\Delta_1][\frac{z'}{rD}\phi] = \frac{1}{rD}\int^r p_v r dr - \frac{z'}{D}L - A_Q\{p_v - r^{-1}(rz'L)'\}' \qquad (3.7a)$$

$$\Delta_1[\Phi] + \frac{z'}{rA}\phi = \frac{c_B}{rA} + (1 - \nu_s)p_1 \qquad (3.7b)$$

where

$$\Delta_1(\ ) \equiv (\ )'' + r^{-1}(\ )' - r^{-2}(\ ) \quad , \quad p_v \equiv p_n + z'p_1 \quad , \qquad (3.7c)$$

and c_B is a constant of integration which can be shown to be associated with a certain nonperiodic displacement field [19].

The auxiliary formulas for stress measures may now be simplified to read:

$$N_{11} = r^{-1}\Phi - L \quad , \quad N_{22} = \Phi' - L \quad ,$$

$$Q_1 = -\frac{z'}{r}\phi - \frac{1}{r}\int^r p_v r dr + z'L$$

$$M_{11} = -D(\phi' + \nu_b r^{-1}\phi) + DA_Q(Q_1' + \nu_b r^{-1}Q_1)$$

$$M_{22} = -D(\nu_b\phi' + r^{-1}\phi) + DA_Q(\nu_b Q_1' + r^{-1}Q_1)$$

while all other stress measures vanish identically for the axisymmetric bending problem. The simplified expression for Q_1 is obtained from the general expression of (3.3) after terms involving $\phi \equiv w'$ are eliminated with the help of (3.7a).

It should be noted that the above results can be brought into the more conventional (and more convenient) form (see [19]) by setting

$$\phi_B = \phi - A_Q Q_1 = w' + A_Q\left[\frac{z'}{r}\Phi + \frac{1}{r}\int^r p_v r dr - \frac{(z'rL)}{r}\right]$$

$$\Phi_B = \Phi - rL$$

Then, the auxiliary formulas become

$$N_{11} = \frac{1}{r}\Phi_B \quad , \quad N_{22} = \Phi_B' + rp_r \quad ,$$

$$Q_1 = -\frac{z'}{r}\Phi_B - \frac{1}{r}\int^r p_v r dr$$

$$M_{11} = -D(\phi_B' + \nu_b r^{-1}\phi_B) \quad , \quad M_{22} = -D(\nu_b\phi_B' + r^{-1}\phi_B)$$

while the two ODEs for ϕ and Φ become

$$\Delta_1[\phi_B] - \frac{z'}{rD}\Phi_B = \frac{1}{rD}\int^r p_v r dr$$

$$\Delta_1[\Phi_B] - \frac{A_Q}{A}\left(\frac{z'}{r}\right)^2\Phi_B + \frac{z'}{rA}\phi_B = -[rp_r' + (2+\nu_s)p_r] + \frac{c_B}{rA} + \frac{A_Q}{A}\frac{z'}{r^2}\int^r p_v r dv \; .$$

Comparison with the result in [19] gives a physical meaning to the unknown constant c_B , which requires that c_B be set to zero for a shell complete in the circumferential direction.

5. Laterally Loaded Shallow Transverse Shear-deformable Shallow Shells of Revolution

The existence of four first integrals for the equilibrium and compatibility equations for laterally loaded shells of revolution also ensure a lower order system of governing equations for this class of problems. The actual reduction of the shell equations has only been accomplished for the classical shell theory [19]. As expected, the reduced system of shell equations for this case is a fourth order system, and the bending action and torsion action of the shell are not uncoupled as in the axi-symmetric problem. In this section, we obtain a lower order system of governing equations for laterally loaded transverse shear-deformable shallow shells of revolution directly from the results in section (3) of this chapter.

For shells of revolution under lateral loads, we have $z^{\bullet} \equiv 0$ and take $\{w,F\} = \{\bar{w}(r),\bar{F}(r)\}\cos\theta$, and $\hat{X} = \bar{X}\sin\theta$ so that (3.5a)-(3.5c) simplify to read

$$D\Delta_1^2\bar{w} = [1 - DA_Q\Delta_1][z''(r^{-1}\bar{F}' - r^{-2}\bar{F}) + r^{-1}z'F'' + \bar{p}_n - r^{-1}(rz')'\bar{L} \qquad (3.5a'')$$

$$-A\Delta_1^2\bar{F} = z''(r^{-1}\bar{w}' - r^{-2}\bar{w}) + r^{-1}z'\bar{w}'' - A(1-\nu_s)\Delta_1\bar{L} \qquad (3.5c'')$$

$$\frac{1}{2}DA_Q(1-\nu_b)\Delta_1\bar{X} = \bar{X} \qquad (3.5b'')$$

where $L(r,\theta) = \bar{L}(r)\cos\theta$, $p_n(r,\theta) = \bar{p}_n(r)\cos\theta$, etc., and, as before, $\Delta_1(\ ) \equiv (\ )'' + r^{-1}(\ )' - r^{-2}(\ )$. As we wish to find one or more first integrals of these two equations, we note that $z''(r^{-1}f' - r^{-2}f) + r^{-1}z'f''$ can be written as the derivative of $[z'r^2(f/r)]'$, namely,

$$z''(r^{-1}f' - r^{-2}f) + r^{-1}z'f'' = r^{-2}[z'r^2(r^{-1}f)']' \quad .$$

As we also have

$$\Delta_1[g] \equiv g'' + r^{-1}g - r^{-2}g = r^{-2}[r^3(r^{-1}g)']' \quad ,$$

equations (3.5a") and (3.5c") can be integrated once immediately to get

$$\Delta_2[\hat{\phi}] - [1 - DA_Q\Delta_2][\frac{z'}{rD}\hat{\Phi}] = -rA_Q\{\frac{1}{r}[\bar{p}_v - \frac{1}{r}(rz'\bar{L})']\}'$$
$$+ \frac{1}{r^2D}\int^r [\bar{p}_v - \frac{1}{r}(rz'\bar{L})']r^2dr$$

$$\Delta_2[\hat{\Phi}] + \frac{z'}{rA}\hat{\phi} = r(1 - \nu_s)(\frac{\bar{L}}{r})' - \frac{c_2}{r^2A}$$

where

$$\hat{\phi} = r(r^{-1}\bar{w})' \quad , \qquad \hat{\Phi} = r(r^{-1}\bar{F})'$$

$$\bar{p}_v = \bar{p}_n + z'\bar{L}' = \bar{p}_n + z'\bar{p}_1 \quad ,$$

$$\Delta_2(\ ) \equiv (\ )'' + r^{-1}(\ )' - 4r^{-2}(\ ) \quad ,$$

and c_2 is a constant of integration which can be shown to be associated with a certain class of nonperiodic displacement fields [19].

The two simultaneous ODEs for $\hat{\phi}$ and $\hat{\Phi}$ may be put in a more familiar form by setting

$$\Phi = \hat{\Phi} \quad , \quad \phi = \hat{\phi} + A_Q z' r^{-1} \hat{\Phi}$$

and write the ODEs in terms of ϕ and Φ instead. The results are

$$\Delta_2[\phi] - \frac{z'}{rD}\Phi = \frac{1}{r^2 D}\int^r [\bar{p}_v - \frac{1}{r}(rz'\bar{L})']r^2 dr - rA_Q\{\frac{1}{r}[\bar{p}_v - \frac{1}{r}(rz'\bar{L})']\}'$$

$$\Delta_2[\Phi] - \frac{A_Q}{A}(\frac{z'}{r})^2\Phi + \frac{z'}{rA}\phi = r(1 - \nu_s)\bar{p}_\theta' - \frac{c_2}{r^2 A}$$

which reduce to known results for the classical theory $(A_Q = 0)$ [19].

The auxiliary formulas for the relevant stress and strain measures may also be simplified with the help of the fourth order system for Φ and ϕ to read:

$$\bar{N}_{11} = \frac{1}{r}\Phi - \bar{L} \quad , \quad \bar{N}_{22} = \Phi' + \frac{1}{r}\Phi - \bar{L} \quad , \quad N_{12} = N_{21} = \frac{1}{r}\Phi$$

$$Q_1 = -\{\frac{z'}{r}\Phi + \frac{D}{r}(\phi' + \frac{2}{r}\phi) + \frac{1}{r^2}\int^r [\bar{p}_v - \frac{1}{r}(rz'\bar{L})']r^2 dr\} - \frac{1}{r}D_s A_Q \bar{X}$$

$$Q_2 = \frac{D}{r}\{\phi' + \frac{2}{r}\phi + A_Q[\bar{p}_v - \frac{1}{r}(rz'\bar{L})']\} + D_s A_Q \bar{X}' \quad .$$

4. Nonlinear Theory

1. Equilibrium Equations for a Finite Rotation and Small Strain Theory

In essence, a nonlinear theory of shells differs from the linear theory in that it relates the correct stress measures to the strain measures in the stress strain relations. The correct stress measures are those experienced by the deformed shell, and the correct stress components are in directions tangent and normal to the middle surface of the deformed shell. In other words, we have for a nonlinear theory

$$\vec{N}_j = N_{j1}\vec{T}_1 + N_{j2}\vec{T}_2 + Q_j\vec{N}$$

$$\vec{M}_j = \vec{N} \times (M_{j1}\vec{T}_1 + M_{j2}\vec{T}_2) + P_j\vec{N}$$

where $\vec{T}_1$ and $\vec{T}_2$ are the unit tangent vectors along the coordinate curves, $\xi_2(=x_2)$ = constant and $\xi_1(=x_1)$ = constant curves, respectively, of the deformed middle surface, and $\vec{N}$ is the corresponding middle surface normal with $\vec{T}_1$, $\vec{T}_2$ and $\vec{N}$ forming a right-handed orthogonal triad. While N_{ij}, Q_j, M_{ij} and P_j are physically different from those introduced in the linear theory, we continue to use the same symbols as we will henceforth think of the linear theory as an approximation for the nonlinear theory.

Let $\vec{\rho} = \vec{r} + \vec{u}$ be the position vector of a point on the middle surface of the deformed shell. With $\vec{u} = \bar{u}_1\vec{i}_1 + \bar{u}_2\vec{i}_2 + \bar{w}\vec{i}_3$, partial derivatives of $\vec{\rho}$ with respect to the cartesian coordinates x_1 and x_2 are expressed in terms of $\bar{u}_1$, $\bar{u}_2$ and $\bar{w}$ by

$$\vec{\rho},_1 = (1 + \bar{u}_1,_1)\vec{i}_1 + \bar{u}_2,_1\vec{i}_2 + (z + \bar{w}),_1\vec{i}_3$$

$$\vec{\rho},_2 = \bar{u}_1,_2\vec{i}_1 + (1 + \bar{u}_2,_2)\vec{i}_2 + (z + \bar{w}),_2\vec{i}_3 \quad .$$

Note that $\bar{u}_k$ and $\bar{w}$ are the cartesian components of the displacement vector, not the components tangent and normal to the (undeformed) middle surface of the shell denoted by u_k and w in the linear theory. For the deformed shell to remain shallow, we stipulate that $\bar{w},_k = 0(\xi_o)$ with $|z,_j| \leq \xi_o$ as before. Also, we continue to expect the lateral displacement components $\bar{u}_1$ and $\bar{u}_2$ (and their gradients) to be small compared to the axial displacement $\bar{w}$ (and its gradient). More precisely, we stipulate for a nonlinear theory of shallow shells $\bar{u}_k = 0(\xi_o\bar{w})$ and $\bar{u}_k,_j = 0(\xi_o\bar{w},_m)$. Therefore, we have for a shallow shell theory

$$\vec{\rho},_k \cong \vec{i}_k + (z + \bar{w}),_k\vec{i}_3$$

with $\alpha_k^2 = |\vec{\rho},_k|^2 \cong 1$ and $\vec{\rho},_1\cdot\vec{\rho},_2 \cong 0$ so that the surface coordinates are (effectively) orthogonal. (In principle, it is possible to have a nonlinear theory for shells which are shallow before deformation without the stipulated restrictions on the displacement components. In that case, the deformed shell will not be shallow in general; and therefore it would not be appropriate to call the theory a shallow shell theory. Also, many of the nice features inherent in the structure of the linear shallow shell theory which carry over to the nonlinear shallow shell theory will be lost in this more general nonlinear theory.)

With $\vec{T}_k \cong \vec{\rho},_k$ and

$$\vec{N} = \frac{\vec{\rho},_1 \times \vec{\rho},_2}{|\vec{\rho},_1 \times \vec{\rho},_2|}$$

$$\cong -(z + \bar{w}),_1\vec{i}_1 - (z + \bar{w}),_2\vec{i}_2 + \vec{i}_3 = \vec{T}_1 \times \vec{T}_2 \quad ,$$

the vector triad $(\vec{T}_1, \vec{T}_2, \vec{N})$ of the deformed shell is obtained from the triad $(\vec{t}_1, \vec{t}_2, \vec{n})$ of the undeformed shell by replacing the axial coordinate z in the latter by $(z + \bar{w})$, the axial coordinate of the deformed shell. Hence the appropriate equilibrium equations for a nonlinear shallow shell theory are evidently the same as those for the linear theory with z replaced by $(z + \bar{w})$ wherever it appears. This result can also be verified by obtaining the scalar equilibrium equations directly from the vector force and moment equilibrium equations making use of the formulas for $\vec{T}_1$, $\vec{T}_2$ and $\vec{N}$ derived above, and keep in mind that, to be consistent with the structure of the linear shallow shell theory, we have also $Q_k = O(\xi_o N_{ij})$.

For the development of a nonlinear theory of shallow shells, it is more convenient to take the scalar force equilibrium equations in directions parallel and normal to base plane, i.e., the x_1, x_2 - plane. With $\vec{N}_k = \bar{N}_{k1}\vec{i}_1 + \bar{N}_{k2}\vec{i}_2 + \bar{Q}_k\vec{i}_3$ and $\vec{p} = \bar{p}_1\vec{i}_1 + \bar{p}_2\vec{i}_2 + \bar{p}_3\vec{i}_3$, these scalar equilibrium equations are

$$\bar{N}_{1k},_1 + \bar{N}_{2k},_2 + \bar{p}_k = 0 \quad , \qquad \bar{Q}_1,_1 + \bar{Q}_2,_2 + \bar{p}_3 = 0 \quad .$$

From the relations

$$\bar{N}_{jk} = N_{jk} - (z + \bar{w}),_k Q_j \cong N_{jk}$$

$$\bar{Q}_j = Q_j + (z + \bar{w}),_1 N_{j1} + (z + \bar{w}),_2 N_{j2} \quad ,$$

the scalar equations for the cartesian stress components may also be written as

$$N_{1k},_1 + N_{2k},_2 + \bar{p}_k = 0$$

$$Q_1,_1 + Q_2,_2 + [(z + \bar{w}),_1 N_{11} + (z + \bar{w}),_2 N_{12}],_1$$

$$+ [(z + \bar{w}),_1 N_{21} + (z + \bar{w}),_2 N_{22}],_2 + \bar{p}_3 = 0 \quad .$$

Thus, the first two force equilibrium equations may be interpreted as equilibrium in directions parallel to the base plane or tangent to the middle surface of the deformed shell, depending on whether we write $\bar{p}_k$ or p_k . The third can of course be rewritten as an equilibrium equation in the direction normal to the deformed middle surface if use is made of the first two equations.

2. The Classical Theory of Marguerre

The strain displacement relations for a nonlinear shallow shell theory will be taken in a form consistent with the equilibrium equations for a virtual work principle:

$$\sum_{i,j=1}^{2} \iint_S (N_{ij}\delta\varepsilon_{ij} + M_{ij}\delta\kappa_{ij} + Q_j\delta\gamma_j + P_j\delta\lambda_j)dx_1dx_2$$

$$= \int_C (\vec{N}_\nu^*\cdot\delta\vec{u} + \vec{M}_\nu^*\cdot\delta\vec{\phi})ds + \iint_S (\vec{p}\cdot\delta\vec{u} + \vec{q}\cdot\delta\vec{\phi})dx_1dx_2$$

where $(\ )^*$ is a prescribed edge stress measure, C is the portion of the edge of the shell with prescribed edge resultants and couples, and δf is to be taken as a first variation of f in the context of the calculus of variations. If we take the force equilibrium equations in directions parallel and normal to the base plane of the deformed middle surface, then the corresponding strain displacement relations take the form

$$\varepsilon_{11} = \bar{u}_{1,1} + (z_{,1} + \tfrac{1}{2}\bar{w}_{,1})\bar{w}_{,1} \quad , \quad \varepsilon_{22} = \ldots.$$

$$\varepsilon_{12} = \bar{u}_{2,1} + (z_{,2} + \tfrac{1}{2}\bar{w}_{,2})\bar{w}_{,1} - \omega \quad , \quad \varepsilon_{21} = \bar{u}_{1,2} + \ldots. + \omega$$

$$\gamma_k = \phi_k + \bar{w}_{,k} \quad , \quad \kappa_{ij} = \phi_{j,i}$$

$$\lambda_1 = \omega_{,1} + (z_{,12} - \tfrac{1}{2}\phi_{2,1})\phi_1 - (z_{,11} - \tfrac{1}{2}\phi_{1,1})\phi_2$$

$$\lambda_2 = \omega_{,2} + (z_{,22} - \tfrac{1}{2}\phi_{2,2})\phi_1 - (z_{,12} - \tfrac{1}{2}\phi_{1,2})\phi_2$$

where, for the present classical theory of (shallow) shells,

$$\phi_k = -\bar{w}_{,k} \qquad (k = 1,2)$$

which follow from $\gamma_k \equiv 0$. As in the general linear theory of thin shells, we have again started with the equilibrium equations and used the principle of virtual work to deduce the corresponding strain-displacement relations. For the linear theory of shallow shells, however, we did the opposite by starting with an appropriate set of strain-displacement equations and using the virtual work principle to get a consistent set of ~~strain-displacement~~ equilibrium equations.

In addition to $\gamma_j \equiv 0$ and $P_j \equiv 0$, the stress strain relations are again taken in the form used in chapter 2. These stress strain relations complete the system of nonlinear shallow shell equations for the classical theory of K. Marguerre [1]. Similar to the linear theory, this system can also be reduced to two simultaneous fourth order PDE for $\bar{w}$ and a stress function F . With $w = \bar{w} - (z_{,1} + \bar{w}_{,1})\bar{u}_1 - (z_{,2} + \bar{w}_{,2})\bar{u}_2 \cong \bar{w}$, we will not distinguish the axial and normal displacement component henceforth.

Starting with $\phi_k = -\bar{w}_{,k} \cong -w_{,k}$, we have again from the expression for κ_{ij} and the stress strain relations for M_{ij}

$$M_{11} = -D(w_{,11} + \nu_b w_{,22}) \quad , \quad M_{22} = -D(w_{,22} + \nu_b w_{,11})$$

$$M_{12} = M_{21} = -D(1 - \nu_b)w_{,12}$$

and from the moment equilibrium equations (with $q_1 \equiv q_2 \equiv q_n \equiv 0$ for simplicity)

$$Q_k = -D(\nabla^2 w),_k \quad , \quad N_{12} = N_{21} \ .$$

The first two equilibrium equations may be satisfied identically by setting

$$N_{11} = F,_{22} - L \ , \ N_{22} = F,_{11} - L \ , \ N_{12} = N_{21} = -F,_{12}$$

whenever $\bar{p}_1$ and $\bar{p}_2$ (or p_1 and p_2) can be expressed in terms of a load potential L by

$$\bar{p}_k = L,_k \ .$$

With these expressions for Q_j and N_{ij} , the remaining (axial force) equilibrium equation becomes an equation for w and F alone:

$$D\nabla^4 w = \mathcal{L}[F\ , z+w] + \bar{p}_3 - \{[(z+w),_1 L],_1 + [(z+w),_2 L],_2\} \qquad (4.1)$$

where the differential operator $\mathcal{L}$ is as defined in the linear theory (see equation (2.2)) and $\nabla^4 = \nabla^2\nabla^2$ is the biharmonic operator in cartesian coordinates.

From the stress strain relations for ε_{ij} , we get

$$\varepsilon_{11} = A[F,_{22} - \nu_s F,_{11} - (1-\nu_s)L] \quad , \quad \varepsilon_{22} = A[F,_{11} - \nu_s F,_{22} - (1-\nu_s)L]$$

$$\varepsilon_{12} = \varepsilon_{21} = -A(1+\nu_s)F,_{12} \quad .$$

The compatibility equations

$$\lambda_2 = \varepsilon_{22},_1 - \varepsilon_{12},_2 \qquad , \qquad \lambda_1 = \varepsilon_{21},_1 - \varepsilon_{11},_2$$

(which can be verified directly) then give

$$\lambda_2 = A(\nabla^2 F),_1 - A(1-\nu_s)L,_1 \quad , \quad \lambda_1 = -A(\nabla^2 F),_2 - A(1-\nu_s)L,_2 \quad .$$

Finally, the compatibility equation

$$\lambda_2,_1 - \lambda_1,_2 + [(z,_2 + \tfrac{1}{2}w,_2)w,_{11} - (z,_1 + \tfrac{1}{2}w,_1)w,_{12}],_2$$

$$+ [(z,_1 + \tfrac{1}{2}w,_1)w,_{22} - (z,_2 + \tfrac{1}{2}w,_2)w,_{12}],_1 = 0$$

(which can also be verified directly) gives us a second equation for w and F

$$-A\nabla^4 F = L\{w\ ,\ z + \tfrac{1}{2}w\} - A(1-\nu_s)\nabla^2 L \quad . \qquad (4.2)$$

The auxiliary formulas obtained in the intermediate steps of the above derivation allow us to calculate all stress measures once we have F and w. These auxiliary formulas are identical to those for the linear theory.

It should be noted that the two simultaneous PDEs for w and F reduce to the von Karman equations [10] for flat plates when $z \equiv 0$. For shells with no bending stiffness so that $D = 0$, these equations constitute an extension of the Föppl-Hencky nonlinear membrane theory [8,9] to allow for initially curved membranes.

3. Finite Twisting and Stretching of Pretwisted Strip

For a shallow hyperbolic paraboloidal shell (or pretwisted strip) with an undeformed middle surface described by $z = \beta x_1 x_2$, $|x_1| \le a$ and $|x_2| \le b$, the two governing partial differential equations for Marguerre's shallow shell theory, in the absence of surface loads, take the form

$$D\nabla^4 w = -2(\beta + w'^{\cdot})F'^{\cdot} + F^{\cdot\cdot}w'' + F''w^{\cdot\cdot}$$

$$A\nabla^4 F = 2(\beta + \tfrac{1}{2}w'^{\cdot})w'^{\cdot} - w''w^{\cdot\cdot}$$

where $(\ )' \equiv (\ )_{,1} \equiv \partial(\ )/\partial x_1$, $(\ )^{\cdot} \equiv (\ )_{,2} \equiv \partial(\ )/\partial x_2$, $\nabla^2(\ ) \equiv (\ )'' + (\ )^{\cdot\cdot}$ and $\nabla^4(\ ) \equiv \nabla^2\nabla^2(\ )$. We consider here a pretwisted strip free of edge tractions along $x_2 = \pm b$ so that

$$x_2 = \pm b : \qquad N_{22} = N_{21} = M_{22} = Q_2^e = 0$$

where $Q_2^e \equiv Q_2 + M'_{21}$. Along the ends $x_1 = \pm a$, the pretwisted strip is subject to equal and opposite resultant axial forces and torques so that

$$\int_{-b}^{b} N_{11}dx_2 = P \quad , \quad \int_{-b}^{b} N_{12}dx_2 = 0 \quad ,$$

$$x_1 = \pm a : \quad \int_{-b}^{b} V_1 dx_2 - [M_{12} + M_{21}]_{-b}^{b} = 0 \quad , \quad \int_{-b}^{b} (x_1 N_{12} - x_2 N_{11})dx_2 = 0 \quad ,$$

$$\int_{-b}^{b} [M_{11} + (z + w)N_{11} - x_1 V_1]dx_2 + [x_1(M_{12} + M_{21})]_{-b}^{b} = 0 \quad ,$$

$$\int_{-b}^{b} [x_2 V_1 - (z + w)N_{12}]dx_2 - [x_2(M_{12} + M_{21})]_{-b}^{b} = T \quad ,$$

where

$$V_1 \equiv Q_1^e + (z + w)'N_{11} + (z + w)^{\cdot} N_{12} \quad , \quad Q_1^e = Q_1 + M_{12}^{\cdot}$$

and where terms involving $M_{12} + M_{21}$ in these overall conditions represent the effect of fictitious corner forces associated with the assumption of vanishing transverse shear deformation [25].

Given the solution for the corresponding linear problem discussed in section (6) of chapter (2), we use here a semi-inverse procedure and seek a solution for w and F in the form

$$w(x_1 x_2) = \psi x_1 x_2 \quad , \quad F(x_1, x_2) = F(x_2)$$

where ψ is an unknown constant. One of the two governing equations for w and F is satisfied identically while the other requires that $F(x_2)$ and ψ satisfy the fourth order ODE

$$F'''' = \frac{1}{A}\psi(2\beta + \psi)$$

which may be integrated to give

$$F(x_2) = \frac{1}{A}\{c_0 + c_1 x_2 + \frac{1}{2}kx_2^2 + \frac{1}{3!}c_3 x_2^3 + \frac{\psi}{4!}(2\beta + \psi)x_2^4\} \quad .$$

The four constants of integration c_0 , c_1 , k and c_3 are to be determined by the boundary conditions along $x_2 = \pm b$ and the overall conditions along $x_1 = \pm a$. With

$$N_{11} = F^{\cdot\cdot} = \frac{1}{A}\{k + c_3 x_2 + \frac{1}{2}\psi(2\beta + \psi)x_2^2\}$$

$$N_{22} = F'' = 0 \quad , \quad N_{12} = N_{21} = -F'^{\cdot} = 0$$

$$M_{11} = -D(w'' + \nu w^{\cdot\cdot}) = 0 \quad , \quad M_{22} = -D(w^{\cdot\cdot} + \nu w'') = 0$$

$$M_{12} = M_{21} = -D(1-\nu)w'^{\cdot} = -D(1-\nu)\psi$$

$$Q_1 = -D(\nabla^2 w)' = 0 \quad , \quad Q_2 = -D(\nabla^2 w)^{\cdot} = 0 \quad ,$$

we see that all four conditions along $x_2 = \pm b$ are trivially satisfied while the six overall conditions along $x_1 = \pm a$ require $c_3 = 0$,

$$kb + \frac{1}{3!}\psi(2\beta + \psi)b^3 = \frac{1}{2}PA \quad ,$$

$$\frac{1}{3}k(\beta + \psi)b^3 + \psi\{\frac{1}{5}(\beta + \psi)(\beta + \frac{1}{2}\psi)b^5 - DA(1-\nu)b\} = \frac{1}{2}TA \quad .$$

We may write the above two overall load-deformation relations for k and ψ in dimensionless form:

$$\frac{P}{2Ehb} = k + \frac{1}{3}\theta(\lambda + \frac{1}{2}\theta)$$

$$\frac{T}{2Ehb^2} = k(\lambda + \theta) + \theta\{\frac{1}{5}(\lambda + \theta)(\lambda + \frac{1}{2}\theta) - \frac{\mu^2}{6(1+\nu)}\}$$

where

$$\theta = \psi b \quad , \quad \mu = \frac{h}{b} \quad .$$

The two conditions determine k and ψ respectively in terms of P and T or conversely. Note that terms in F associated with the two undetermined constants c_0 and c_1 do not contribute to the stress and strain measures of the shell and may be set equal to zero.

The two overall load-deformation relations also gives P and T in terms of k and θ . It can be shown from the strain displacement relations

$$\varepsilon_{11} = u_1' + (\psi\beta + \frac{1}{2}\psi^2)x_2^2 = A(N_{11} - \nu N_{22}) = AF^{\cdot\cdot}$$

$$\varepsilon_{22} = u_2^{\cdot} + (\psi\beta + \frac{1}{2}\psi^2)x_1^2 = A(N_{22} - \nu N_{11}) = -\nu AF^{\cdot\cdot}$$

$$\varepsilon_{12} + \varepsilon_{21} = u_1^{\cdot} + u_2' + (2\beta\psi + \psi^2)x_1x_2 = 2(1 + \nu)AN_{12} = 0$$

that the displacement components u_1 and u_2 are given by

$$u_1(x_1,x_2) = c_5 + kx_1$$

$$u_2(x_1,x_2) = c_4 - \frac{1}{2}(z\beta\psi + \psi^2)(x_1^2x_2 + \frac{\nu}{3!}x_2^3) - \nu\, x_2$$

with $u_{1,1} = k$. Thus, $k = u_1'$ is the axial strain of the pretwisted strip when there is no twist, i.e., when $\psi = 0$.

4. Marguerre's Theory in Polar Coordinates

In cylindrical coordinates, the position vector, $\vec{p}$, of a point of the middle surface of the deformed shell may be taken in the form

$$\vec{\rho} = \vec{r} + \vec{u} = r\vec{i}_r + (\bar{u}_1\vec{i}_r + \bar{u}_2\vec{i}_\theta + \bar{w}\vec{i}_z)$$

where the displacement vector $\vec{u}$ has been resolved in the radial, circumferential and axial direction. Again, we stipulate for a shallow shell theory

$$z' \equiv \frac{\partial z}{\partial r} \leq \xi_0 \quad , \quad \frac{1}{r}z^{\cdot} \equiv \frac{1}{r}\frac{\partial z}{\partial \theta} \leq \xi_0$$

$$\bar{w}' = 0(\xi_0) \quad , \quad \frac{1}{r}\bar{w}^{\cdot} = 0(\xi_0) \quad , \quad \bar{u}_k = 0(\xi_0\bar{w})$$

$$\bar{u}_k' = 0(\xi_0\bar{w}') \quad , \quad \frac{1}{r}\bar{u}_k^{\cdot} = 0(\frac{\xi_0}{r}\bar{w}^{\cdot}) \quad .$$

It follows from these order of magnitude relations and $1+\xi_0^2 \cong 1$ that

$$\vec{\rho}' \cong \vec{i}_r + (z + \bar{w})'\vec{i}_z \cong \vec{T}_1$$

$$\frac{1}{r}\vec{\rho}^{\cdot} \cong \vec{i}_\theta + \frac{1}{r}(z + \bar{w})^{\cdot}\vec{i}_z \cong \vec{T}_2$$

$$\vec{N} = \frac{\vec{\rho}' \times \vec{\rho}^{\cdot}}{|\vec{\rho}' \times \vec{\rho}^{\cdot}|}$$

$$\cong -(z+\bar{w})'\vec{i}_r - \frac{1}{r}(z+\bar{w})^{\cdot}\vec{i}_\theta + \vec{i}_z \cong \vec{T}_1 \times \vec{T}_2$$

and the vector triad $(\vec{T}_1, \vec{T}_2, \vec{N})$ of the deformed middle surface form an orthogonal set within the shallow shell approximation.

From the expressions

$$w = \vec{N}\cdot\vec{u} = -(z + \bar{w})'\bar{u}_1 - \frac{1}{r}(z + \bar{w})^{\cdot}\bar{u}_2 + \bar{w} \cong \bar{w} \quad ,$$

$$u_1 = \vec{T}_1\cdot\vec{u} = \bar{u}_1 + (z + \bar{w})'\bar{w} \quad ,$$

$$u_2 = \vec{T}_2\cdot\vec{u} = \bar{u}_2 + \frac{1}{r}(z + \bar{w})^{\cdot}\bar{w} \quad ,$$

we see that the normal displacement component w is indistinguishable from the axial component $\bar{w}$ within the framework of shallow shell theory. We see also that the displacement components u_1 and u_2 tangent to the deformed middle surface are of $O(\xi_0\bar{w})$ and are therefore an order of magnitude smaller than the normal (or axial) displacement component.

The triad $(\vec{T}_1, \vec{T}_2, \vec{N})$ can be obtained from the corresponding triad $(\vec{t}_1, \vec{t}_2, \vec{n})$ of the undeformed middle surface (given in section (1) of chapter (3)) by replacing the axial coordinate z in the latter by $(z + \bar{w})$. Hence the appropriate equilibrium equations for Marguerre's nonlinear shallow shell theory in polar coordinates may be obtained from those for the corresponding linear theory in chapter (3) by writing $(z + \bar{w})$ for z in the latter.

We identify ξ_1 with r and ξ_2 with θ . Let

$$\vec{N}_k = \bar{N}_{k1}\vec{i}_r + \bar{N}_{k2}\vec{i}_\theta + V_k\vec{i}_z \quad , \qquad \vec{p} = \bar{p}_1\vec{i}_r + \bar{p}_2\vec{i}_\theta + \bar{p}_3\vec{i}_z \quad .$$

The scalar force equilibrium equations in the directions of $\vec{i}_r$, $\vec{i}_\theta$, and $\vec{i}_z$ may be taken in the form

$$(r\bar{N}_{11})' + \dot{\bar{N}}_{21} - \bar{N}_{22} + r\bar{p}_1 = 0 \quad , \quad (r\bar{N}_{12})' + \dot{\bar{N}}_{22} + \bar{N}_{21} + r\bar{p}_2 = 0 \quad ,$$

$$(rV_1)' + \dot{V}_2 + r\bar{p}_3 = 0 \quad .$$

Again, we have from $\vec{N}_k = N_{k1}\vec{T}_1 + N_{k2}\vec{T}_2 + Q_k\vec{N}$

$$\bar{N}_{11} = N_{11} - (z + \bar{w})'Q_1 \cong N_{11} \quad , \text{ etc.,}$$

so that $\bar{N}_{ij} \cong N_{ij}$ $(i,j = 1,2)$, and

$$V_1 = Q_1 + (z + \bar{w})'N_{11} + \frac{1}{r}(z + \bar{w})^{\cdot}N_{12}$$

$$V_2 = Q_2 + (z + \bar{w})'N_{21} + \frac{1}{r}(z + \bar{w})^{\cdot}N_{22} \quad .$$

The three scalar force equilibrium equations can then be written as

$$(rN_{11})' + \dot{N}_{21} - N_{22} + r\bar{p}_1 = 0 \quad , \quad (rN_{12})' + \dot{N}_{22} + N_{21} + r\bar{p}_2 = 0$$

$$(rQ_1)' + \dot{Q}_2 + [(z + \bar{w})'(rN_{11}) + \frac{1}{r}(z + \bar{w})^{\cdot}(rN_{12})]'$$

$$+ [(z + \bar{w})'N_{21} + \frac{1}{r}(z + \bar{w})^{\cdot}N_{22}]^{\cdot} + r\bar{p}_3 = 0 \quad .$$

Thus, the first two force equilibrium equations may be interpreted as equilibrium in the directions parallel to the base plane or tangent to the deformed middle surface, depending on whether we write $\bar{p}_k$ or p_k. The third can be rewritten as an equilibrium equation in the direction normal to deformed middle surface if use is made of the first two equations to eliminate $(rN_{11})'$, $(rN_{12})'$, $\dot{N}_{21}$, and $\dot{N}_{22}$ from the third equation.

The strain displacement relations in polar coordinates are now taken in a form consistent with the equilbirium equations for the principle of virtual work stated in section (2) of this chapter. If we take the force equilibrium equations for the classical theory with $\gamma_j \equiv p_k \equiv 0$ and $\vec{q}\cdot\vec{N} = 0$ in the directions parallel and normal to the base plane then the corresponding strain displacement relations may be written as

$$\varepsilon_{11} = \bar{u}_1' + (z + \tfrac{1}{2}\bar{w})'\bar{w}' \qquad \varepsilon_{22} = \frac{1}{r}(\dot{\bar{u}}_2 + \bar{u}_1) + \frac{1}{r}(z + \tfrac{1}{2}\bar{w})^{\cdot}(\frac{\dot{\bar{w}}}{r})$$

$$\varepsilon_{12} = \bar{u}_2' + \frac{1}{r}(z + \tfrac{1}{2}\bar{w})^{\cdot}\bar{w}' - \omega \quad , \quad \varepsilon_{21} = \frac{1}{r}(\dot{\bar{u}}_1 - \bar{u}_2) + (z + \tfrac{1}{2}\bar{w})'(\frac{\dot{\bar{w}}}{r}) + \omega \quad ,$$

$$\gamma_1 = \phi_1 + \bar{w}' = 0 \quad , \quad \gamma_2 = \phi_2 + \frac{1}{r}\dot{\bar{w}} = 0 \quad ,$$

$$\kappa_{11} = \phi_r' = -\bar{w}'' \quad , \quad \kappa_{22} = \frac{1}{r}(\dot{\phi}_2 + \phi_1) = -(\frac{1}{r}\bar{w}' + \frac{1}{r^2}\ddot{\bar{w}})$$

$$\kappa_{12} = \phi_\theta' = -(\frac{\dot{\bar{w}}}{r})' \quad , \quad \kappa_{21} = \frac{1}{r}(\dot{\phi}_2 - \phi_2) = -(\frac{\dot{\bar{w}}}{r})' \quad .$$

These strain displacement relations constitute a special case of the general relations of [20] in which allowance is made for nonvanishing p_j and γ_k.

It can be verified directly that the strain displacement relations for the polar coordinate formulation satisfy the following compatibility equations:

$$(r\kappa_{22})' - \dot{\kappa}_{12} - \kappa_{11} = 0 \quad , \quad (r\kappa_{21})' - \dot{\kappa}_{11} + \kappa_{12} = 0 \quad ,$$

$$(r\lambda_2)' - \dot{\lambda}_1 - [(z + \tfrac{1}{2}\bar{w})'(r\kappa_{22}) - \tfrac{1}{r}(z + \tfrac{1}{2}\bar{w})^{\cdot}(r\kappa_{21})]'$$

$$+ [(z + \tfrac{1}{2}\bar{w})'\kappa_{12} - \tfrac{1}{r}(z + \tfrac{1}{2}\bar{w})^{\cdot}\kappa_{11}]^{\cdot} = 0$$

with $\kappa_{12} = \kappa_{21}$ and

$$r\lambda_2 = (r\varepsilon_{22})' - \dot{\varepsilon}_{12} - \varepsilon_{11} \quad , \quad r\lambda_1 = (r\varepsilon_{21})' - \dot{\varepsilon}_{11} + \varepsilon_{12} \quad .$$

The system of shallow shell equations in polar coordinates can also be reduced to the canonical form of two simultaneous fourth order partial differential equations for w ($\cong \bar{w}$) and a stress function F as the first two equilibrium equations are satisfied identically by

$$\bar{N}_{11} = \frac{1}{r}F' + \frac{1}{r^2}\ddot{F} - L \quad , \quad \bar{N}_{22} = F'' - L \quad , \quad \bar{N}_{12} = \bar{N}_{21} = -(\frac{\dot{F}}{r})'$$

with

$$\bar{p}_1 = L' \quad , \quad \bar{p}_2 = \frac{1}{r}\dot{L} \quad .$$

The stress strain relations

$$\varepsilon_{11} = A(\bar{N}_{11} - \nu\bar{N}_{22}) \quad , \quad \varepsilon_{22} = A(\bar{N}_{22} - \nu\bar{N}_{11}) \quad , \quad \varepsilon_{12} = \varepsilon_{21} = A(1 + \nu)\bar{N}_{12}$$

may be used to express ε_{ij} and hence λ_j in terms of F :

$$\lambda_2 = A(\nabla^2 F)' \quad , \quad r\lambda_1 = -A(\nabla^2 F)^{\cdot}$$

where $\nabla^2(\ ) \equiv (\ )'' + r^{-1}(\ )' + r^{-2}(\ )^{\cdot\cdot}$ is the Laplacian in polar coordinates. With the above expression for λ_j , the third compatibility equation may now be written as

$$-A\nabla^2\nabla^2 F + A(1 - \nu)\nabla^2 L = L[w, z + \frac{1}{2}w]$$

where

$$L[f,g] \equiv f''(r^{-1}g' + r^{-2}g^{\cdot\cdot}) + g''(r^{-1}f' + r^{-2}f^{\cdot\cdot})$$

$$- 2(r^{-1}f^{\cdot})'(r^{-1}g^{\cdot})' \quad .$$

To get a second equation for F and w , we use the stress strain relations for the stress couples to write M_{ij} in terms of w :

$$M_{11} = -D[w'' + \nu(r^{-1}w' + r^{-2}w^{\cdot\cdot})] \quad , \quad M_{22} = \cdots\cdot$$

$$M_{12} = M_{21} = -D(1 - \nu)(r^{-1}w^{\cdot})' \quad .$$

From the moment equilibrium equations, we also have Q_j in terms of w :

$$Q_1 = -D(\nabla^2 w)' \qquad , \qquad rQ_2 = -D(\nabla^2 w)^{\cdot} \quad .$$

The equation for axial force equilibrium may then be written as

$$D\nabla^2\nabla^2 w = L[F , z + w] + \bar{p}_3 - [\nabla^2 z]L \quad .$$

5. Axisymmetric Finite Bending of Shallow Shells of Revolution

With $z_{,\theta} = w_{,\theta} = F_{,\theta} = 0$, the Marguerre equations for shallow shells of revolution simplify considerably. The equation of axial force equilibrium becomes

$$(rV_1)' + rp_3 = 0$$

and can be integrated immediately to get

$$rV_1 = -\int^r rp_3 dr \quad , \quad rQ_1 = -\int^r rp_3 dr - (z + w)'H \quad , \quad H \equiv rN_{11} \quad .$$

The equation of radial equilibrium may be used to express N_{22} in terms of H :

$$\bar{N}_{22} = H' + r\bar{p}_1 \quad , \quad \bar{N}_{11} = \frac{H}{r} \quad .$$

Similarly, the third compatibility equation can be integrated to give

$$r\lambda_2 = c_B \oplus (z' + \frac{1}{2}\phi)\phi \quad , \quad \phi \equiv w' \quad .$$

?

With

$$\varepsilon_{11} = A(\bar{N}_{11} - \nu\bar{N}_{22}) = A(\frac{1}{r}H - \nu H' - \nu r\bar{p}_1)$$

$$\varepsilon_{22} = A(\bar{N}_{22} - \nu\bar{N}_{11}) = A(H' + r\bar{p}_1 - \frac{\nu}{r}H)$$

the defining equation for λ_2 may be written as a second order ODE for ϕ and H :

$$A[H'' + \frac{1}{r}H' - \frac{1}{r^2}H] + \frac{1}{r}(z' + \frac{1}{2}\phi)\phi = -A[\nu\bar{p}_1 + \frac{1}{r}(r^2\bar{p}_1)'] + \frac{c_B}{r}$$

where we have taken A to be a constant. The constant of integration c_B is known to be associated with a circumferentially nonperiodic displacement field and should be omitted if the shell is complete in the circumferential direction and has no slit along a meridian.

For a second equation for ϕ and H, we write the stress strain relations for the stress couples as

$$M_{11} = -D[\phi' + \frac{\nu}{r}\phi] \quad , \quad M_{22} = -D[\nu\phi' + \frac{1}{r}\phi] \quad .$$

The only nontrivial moment equilbrium equation may then be written as

$$D[\phi'' + \frac{1}{r}\phi' - \frac{1}{r^2}\phi] - \frac{1}{r}(z + \phi)H = q_1 + \frac{1}{r}\int^r p_3 r\,dr \quad .$$

Once we have the solution of the fourth order system of two second order ODEs for ϕ and H, all the remaining quantities of interest can be calculated as they are all expressed in terms of H and ϕ and their first derivative. The only exception is the axial displacement $\bar{w}$ given by

$$\bar{w} \cong w = \int^{r} \phi \, dr \quad .$$

The radial displacement component on the other hand can be obtained from

$$\bar{u}_1 = r\varepsilon_{22} + A(rH' + r^2\bar{p}_1 - \nu H) \quad .$$

Applications of the results in this section to problems involving polar dimpling and rotating shells can be found in [21-27].

6. Finite Twisting and Bending of Shallow Helicoidal Shells

As an example of applications of the polar coordinate formulation of shallow shell theory to shells other than shells of revolution, we consider a shallow helicoidal shell with an undeformed middle surface described by $z + K\theta$ where K is a positive constant, $|\theta| \leq \theta_0$ and $a \leq r \leq b$. The shell is free of edge tractions along the two radial edges $r = a$ and $r = b$. For simplicity, we limit our discussion to the case $a > 0$. Along the two ends $\theta = \pm\theta_0$, the shell is subject to equal and opposite axial forces and torques.

For such a shell, the two governing equations for w and F of Marguerre's theory in polar coordinates simplify to read

$$A\nabla^4 F = 2[r^{-1}(K + \tfrac{1}{2}w^{\cdot})]'[r^{-1}w^{\cdot}]' - w''(\tfrac{1}{r}w' + \tfrac{1}{r^2}w^{\cdot\cdot})$$

$$D\nabla^4 w = -2[r^{-1}(K + w^{\cdot}]'[r^{-1}F^{\cdot}]' + w''(\tfrac{1}{r}F' + \tfrac{1}{r^2}F^{\cdot\cdot}) + F''(\tfrac{1}{r}w' + \tfrac{1}{r^2}w^{\cdot\cdot}) .$$

The stress free conditions along the edge $r = a$ and $r = b$ take the form

$$r = a,b : \quad N_{11} = N_{12} = M_{11} = Q_1^e = 0$$

where the effective transverse shear resultant Q_1^e is equal to $Q_1 + r^{-1}M_{12}^{\cdot}$. The conditions on resultant force and moment at the two ends $\theta = \pm\theta_0$ are

$$\int_a^b N_{22}dr = 0 \quad , \quad \int_a^b N_{21}dr = 0 \quad , \quad \int_a^b V_2 dr - [M_{12} + M_{21}]_a^b = P$$

$$\int_a^b [M_{22} + (z + w)N_{22}]dr = 0 \quad , \quad \int_a^b rN_{22}dr = T \quad ,$$

$$\int_a^b rV_2 dr - [r(M_{12} + M_{21})]_a^b = 0 \quad ,$$

where $V_2 = Q_2^e + (z + w)'N_{21} + r^{-1}(z + w)^{\cdot}N_{22}$, $Q_2^e = Q_2 + M_{21}'$ and terms involving $(M_{12} + M_{21})$ in these overall conditions represent the effect of fictitious corner forces associated with the assumption of vanishing transverse shearing strains [25].

We seek a solution of this problem in the form

$$w(r,\theta) = Kk_0\theta \quad , \quad F(r,\theta) = F(r)$$

where k_0 is an unknown constant. Such a solution satisfies one of two governing partial differential equations and reduces the other to

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 F = \frac{K^2}{Ar^4}k_0(2 + k_0) \quad .$$

The exact solution of the above fourth order ODE is

$$F = \frac{1}{4A}[K^2k_0(1 + \tfrac{1}{2}k_0)(\ell nr)^2 + c_0 + c_1\ell n(r) + c_2r^2 + c_3r^2\ell nr]$$

where c_0, c_1, c_2 and c_3 are unknown constants of integration. The corresponding expressions for stress resultants and couples are:

$$N_{11} = \frac{1}{r}F' = \frac{1}{4A}[K^2k_0(2+k_0)r^{-2}\ell nr + c_1r^{-2} + 2c_2 + c_3(1+2\ell nr)]$$

$$N_{22} = F'' = \frac{1}{4A}[K^2k_0(2+k_0)r^{-2}(1-\ell nr) - c_1r^{-2} + 2c_2 + c_3(3+2\ell nr)]$$

$$M_{12} = M_{21} = D(1-\nu)Kk_0r^{-2}$$

$$N_{12} = N_{21} = M_{11} = M_{22} = Q_1 = Q_2 = 0 \quad .$$

It follows that three of the four boundary conditions along $r = a$ and $r = b$ are trivially satisfied; the remaining condition $N_{11} = 0$ requires $F'(a) = F'(b) = 0$ or

$$c_1 + 2c_2a^2 + c_3a^2(1+2\ell na) + K^2k_0(2+k_0)\ell na = 0$$

$$c_1 + 2c_2b^2 + c_3b^2(1+2\ell nb) + K^2k_0(2+k_0)\ell nb = 0 \quad .$$

These two conditions determine c_1 and c_2 in terms of the other two unknown constants c_3 and k_0. Note that the c_0 term in F does not appear in the expressions for stress or strain measures; such a "stress free part" of any stress function may be omitted from further consideration.

Of the six overall conditions at the ends, the second is trivially satisfied; the first and fourth may be written as

$$\int_a^b N_{22}dr = \int_a^b F''dr = F'(b) - F'(a) \quad ,$$

$$\int_a^b [M_{22} + (z+w)N_{22}]dr = \pm(1+k_0)K\theta_0\int_a^b F''dr = \pm(1+k_0)K\theta_0\,[F'(b) - F'(a)] \quad ,$$

respectively and are hence also satisfied automatically given that $N_{11}(a) = N_{11}(b) = 0$ are equivalent to $F'(a) = F'(b) = 0$. The last condition may now be written as

$$\int_a^b r[M'_{21} + K(1+k_0)r^{-1}N_{22}]dr - [r(M_{12}+M_{21})]_a^b$$

$$= \int_a^b [-M_{21} + K(1+k_0)F'']dr - [rM_{12}]_a^b$$

$$= K(1+k_0)[F'(b) - F'(a)] + D(1-\nu)Kk_0[\frac{1}{r} - \frac{1}{r}]_a^b = 0$$

and is also met by our choice of solution (which satisfies $F'(b) = F'(a) = 0$). The remaining two overall conditions may be transformed into

$$f_P(k_0,c_3) = P \quad , \quad f_T(k_o,c_3) = T$$

where

$$f_P(k_0,c_3) = (1+k_0)K\int_a^b r^{-1}F''dr - [D(1-\nu)Kk_0r^{-2}]_a^b$$

$$f_T(k_0, c_3) = F(a) - F(b) \quad .$$

These are two equations for c_3 and k_0 in terms of the applied force P and torque T. Alternatively, given c_3 and k_0, they determine the amount of force and torque required to produce the specified magnitude of stresses and deformations.

It should be noted that the multiplicative factor c_3 of the $r^2 \ell n r$ term in the stress function F is associated with a nonperiodic circumferential displacement field similar to the situation in two-dimensional elasticity. We see this from the strain displacement relations of section (4) specialized to a shallow helicoidal shell:

$$\varepsilon_{11} = \bar{u}_1' = A[r^{-1}F' - \nu F'']$$

$$\varepsilon_{22} = r^{-1}(\dot{\bar{u}}_2 + \bar{u}_1) + r^{-2}K^2 k_0(1 + \tfrac{1}{2}k_0) = A[F'' - \nu r^{-1}F']$$

$$\varepsilon_{12} + \varepsilon_{21} = \bar{u}_2' + r^{-1}(\dot{\bar{u}}_1 - \bar{u}_2) = A(1-\nu)N_{12} = 0 \quad .$$

We have from the first two relations

$$\bar{u}_1 = A\int^r (r^{-1}F' - \nu F'')dr - \dot{U}_2(\theta)$$

$$\bar{u}_2 = \theta[A(rF'' - \nu F') - r^{-2}K^2 k_0(1 + \tfrac{1}{2}k_0) - A\int^r (r^{-1}F' - \nu F'')dr] + u_2(\theta)$$

$$= c_3 r\theta + U_2(\theta) \quad .$$

and from $\varepsilon_{12} + \varepsilon_{21} = 0$

$$\ddot{\bar{u}}_2 + \bar{u}_2 = 0 \quad .$$

We may omit further consideration of the rigid body displacement field $\bar{u}_2(\theta) = \{\cos\theta , \sin\theta\}$. At the same time, we may now identify c_3 with ψ_0 , where $2\pi\psi_0$ is the relative angular displacement of the ends for one winding of the shallow helicoidal shell. The two nontrivial conditions on resultant force and torque, written as

$$f_P(k_0,\psi_0) = P \quad , \quad f_T(k_0,\psi_0) = T \quad ,$$

are evidently two overall load-deformation relations relating the axial strain k_0 and twist angle ψ_0 to the applied force P and applied torque T .

References

1. K. Marguerre, "Zur Theorie der gekrummten Platte grosser Formänderung", Proc. 5th Int'l Congress Appl. Mech. (1938), Wiley, New York, 1939, 93-101.

2. Kh. M. Mushtari, "On the stability of cylindrical shells subject to torsion", Trudy Kaz. aviats. in-ta., 2, 1938.

3. ______________, "Certain generalizations of the theory of thin shells", Izv. fiz. mat. ob-va pri Kaz. Un-te. 11, 8, 1938.

4. L. Donnell, Stability of Thin-Walled Tubes Under Torsion, NACA Rep. No. 479, 1933.

5. __________, A discussion of thin shell theory", Proc. 5th Congr. for Appl. Mech., 1939.

6. V.Z. Vlasov, "The fundamental differential equations of the general theory of elastic shells", P.M.M. (Akad. Nauk. S.S.S.R.) 8, 2, 1944, [Also NACA T.M. 1241].

7. ___________, The General Theory of Shells and Its Industrial Applications, Gostekhizdat, 1949.

8. A. Föppl, Vorlesungen über technische Mechanik, Vol. III (§24), Teubner, Leipzig, 1907.

9. H. Hencky, "Über den Spannungsgustand im kreisrunden Platten mit verschwindender Biegungssteifigkeit", Z.f. Math. u. Phys. 63, 1915, 311-317.

10. Th. von Kármán, "Festigkeitsprobleme im Maschinenban", Enzykl. Math. Wiss, IV, 1910, 349.

11. F.Y.M. Wan, "Lecture Notes on Theory of Thin Elastic Shells", I.A.M.S. Tech. Report No. 83-11, University of British Columbia, Vancouver, Canada, July, 1983.

12. E. Reissner and F.Y.M. Wan, "On the equations of linear shallow shell theory", Studies in Appl. Math. 48, 1969, 133-145.

13. ___________________________, "A note on the linear theory of shallow shear-deformable shells", ZAMP 33, 1982, 425-427.

14. H. Schaefer, "Versuch einer Elastizitatstheorie des zweidimensionalen ebenen Cosserat-Kontinuum", Tollmien Anniversary Volume, Akademie-Verlag, Berlin, 1962, 277-292.

15. E. Reissner, "On the transverse twisting of shallow spherical ring caps", J. Appl. Mech. 47, 1980, 101-105.

16. __________, "On the effect of a small circular hole on states of uniform membrane shear in spherical shells", J. Appl. Mech. 47, 1980, 430-431.

17. F.Y.M. Wan, "Membrane and bending stress in shallow spherical shells", Int'l J. Solids & Structures 3, 1967, 353-366.

18. __________, "On the displacement boundary value problem of shallow spherical shells", Int'l J. Solids & Structures 4, 1968, 661-666.

19. __________, "Lecture Notes on the Linear Theory of Thin Elastic Shells of Revolution", I.A.M.S. Tech. Report No. 84-9, University of British Columbia, Vancouver, Canada, to be published in August, 1984.

20. E. Reissner, "On the equations of nonlinear shallow shell theory", Studies in Appl. Math. 48, 1969, 171-175.

21. E. Reissner and F.Y.M. Wan, "Rotating shallow elastic shells of revolution", J. of SIAM (now SIAM J. of Appl. Math.) 18, 1965, 333-352.

22. F.Y.M. Wan, "Polar dimpling of complete spherical shells", Theory of Shells (Proc. 3rd IUTAM Shell Symp., Tibilisi, USSR, August, 1978), Ed. W.T. Koiter & G.K. Mikhailov, North-Holland, 1979, 191-207.

23. D.F. Parker and F.Y.M. Wan, "Finite polar dimpling of shallow caps under sub-buckling pressure loading", SIAM J. of Appl. Math. 44, 1984, 301-326.

24. Y-H Lin and F.Y.M. Wan, "Asymptotic solutions of steadily spinning shallow shells of revolution under uniform pressure", Int'l J. Solids & Structures, to appear.

25. F.Y.M. Wan, "Lecture Notes on Problems in Elasticity: II. Linear Plate Theory", I.A.M.S. Tech. Rep. No. 83-15, University of British Columbia, Vancouver, Canada, September, 1983.

26. H.J. Weinitschke, "On the stability problem of shallow spherical shells", J. Math. & Phys. 38, 1960, 209-231.

27. J.R. Fitch, "The buckling and postbuckling behavior of spherical caps under concentrated load", Int'l J. Solids & Struc. 4, 1968, 421-446.