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LECTURE NOTES ON THEORY OF THIN ELASTIC SHELLS

Frederic Y.M. Wan

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Cover photo courtesy of the U.B.C. Museum of Anthropology:

Haida totem pole; main figure, possibly bear, holding wolf between legs, frog in mouth, wolf between ears.

LECTURE NOTES ON THEORY OF THIN ELASTIC SHELLS

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INTRODUCTION

An elastic shell is a body of elastic substance with a "thickness" dimension small compared to the other dimensions. It may also be described as a body consisting of a middle surface and extending in directions normal to the surface by a thin layer of elastic material on both sides of the surface. The special cases of a cylindrical shell and a spherical cap illustrate this special geometrical feature of a shell structure.

Figure (0.1)

The behavior of an elastic shell under the influence of external loads (forces, moments, temperature gradients, etc.) is of course governed by the Theory of Elasticity. To learn how the shell reacts to a given loading condition, we need only to solve the relevant boundary/initial value problem in this theory. Unfortunately, the solution of most such problems is very difficult even in a linear theory. To obtain numerical data (for stresses, displacements, etc.) for design purposes is practically a hopeless task except for some very simple situations. Various approximate methods of solution such as perturbations, finite difference, direct method of calculus of variations, and finite element, made more powerful by high speed computers, have improved the situation substantially; but we are still a long way from being able to get all the desired answers.

Fortunately, by taking advantage of the special geometrical feature common to all shells, we can reduce the complexity of the mathematical problem posed by the three dimensional theory of elasticity. More specifically, we can determine, "to a good approximation", how the solution of the elasticity theory depends on one of the three spatial variables, namely the thickness coordinate, and thereby reduce the relevant boundary/initial value problem from one involving partial differential equations in three spatial variables to one involving partial differential equations in only two spatial variables. Similar to plate theory, this reduction was historically first accomplished with the help of some plausible physical assumptions known as the Euler-Bernoulli-Kirchhoff-Love hypotheses. In recent years, it has been made a consequence of an asymptotic analysis far more complicated than that for plate theory described in Part III of these lecture notes.

Most of the time and in its narrowest sense, Shell Theory refers to the study of the structure and further simplifications (in special cases) of this simplified boundary/initial value problem. The step from three dimensional theory of elasticity to the simplified theory is then referred to as the Foundation of Shell Theory, which must also be well understood in order that shell theory be meaningful as a part of structural mechanics. To indicate the magnitude of the difficulty involved in this single step, we need only to mention that it took more than a hundred years of investigation before some general agreement of what constitutes an "adequate and consistent first approximation linear shell theory"[1] finally emerged. An adequate

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[1] By this, we mean the correct boundary/initial value problem whose solution is the leading term of the outer asymptotic expansion of the exact solution of the corresponding problem in the three dimensional theory of elasticity.

and usable nonlinear theory is still a current research topic.

In a broader sense, shell theory includes these two areas of investigation and more. For example, it includes the extraction of useful information from the relevant simplified boundary/initial value problem for a specific shell and a specific load condition. While such problems are not as formidable as the corresponding problems in the three dimensional elasticity theory, they are, nevertheless, very challenging and in many cases still unsolved. In these notes, we will be concerned mainly with the methods of solution for a wide range of shell problems after a brief description of the basic theory of shells. We will see that imagination, resourcefulness and a variety of techniques are required to deal with these problems. We will only touch lightly on the foundation of shell theory[(2)]. However, Koiter's two papers in the First (1959) and Second (1967) IUTAM Symposia on Thin Shells and references contained therein should eventually be read by serious students of the subject. A reasonably well organized discussion of the essentials of the Foundation can also be found in "Thin Elastic Shells" by H. Kraus.

To accomodate those of you who have no previous exposure to shell theory in the Günther-Reissner form, we will begin with a discussion of the differential equations and general boundary value problems of the linear theory without getting into three dimensional theory of elasticity. Even at this point, we will encounter new important concepts. From then on, our objective is to use appropriate techniques in applied mathematics

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[(2)] Some of you already had a glimpse of what is involved in this aspect of shell theory in our discussion of plate theory in Part III of these notes.

to reduce different classes of problems of linear and non-linear shell theory to a tractable form. Sometimes, this means a complete solution of the problem in question. More often, we will leave a specific problem once it becomes evident that what remains is only a straightforward application of standard methods.

Finally, it should be mentioned that even in cases where the problem can be handled by applying an approximate method such as "finite element" to three-dimensional elasticity theory, results in the form of relatively simple analytic expressions via shell theory may still be preferable. To decipher the effect of several geometrical and material parameters from a massive pile of numerical data is neither a pleasant experience nor an easy task.

Texts on Elastic Shells

1. W. Flügge, "Stresses in Shells", Springer-Verlag, 1962.

2. A.L. Goldenveiser, "Theory of Elastic Thin Shells", Pergamon Press, 1961.

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4. A.E.H. Love, "A Treatise on the Mathematical Theory of Elasticity", 4th Edition, Dover, 1944.

5. P.M. Naghdi, "Foundations of Elastic Shell Theory", Progress in Solid Mechanics IV (Edited by R. Hill and I.N. Sneddon) pp. 1-90, North-Holland, 1963.

6. V.V. Novozhilov, "Thin Shell Theory", 2nd augmented and revised edition, P. Noordhoff Ltd., 1964.

7. S. Timoshenko and S. Woinowsky-Krieger "Theory of Plates and Shells", McGraw-Hill, 1959.

8. P. Seide, "Small Elastic Deformations of Thin Shells", Noordhoff International Publishing, Leyden, The Netherlands, 1975.

9. G. Wempner, "Mechanics of Solids with Applications to Thin Bodies", Martinus Nijhoff Publishers, The Netherlands, 1982.

I - THE LINEAR THEORY OF THIN ELASTIC SHELLS

1. Statics

1. Forces and Moments

For the purpose of analysis, a shell is represented geometrically by a surface in space surrounded by an elastic substance which extends a distance $h/2$ in both directions normal to the surface. The thickness of the shell, h, may vary from point to point on the middle surface. The surface itself will be defined parametrically by a position vector $\vec{r}(\xi_1, \xi_2)$ which is a vector function of two curvilinear surface coordinates ξ_1 and ξ_2. The unit vectors

$$\vec{t}_1 = \vec{r}_{,1}/\alpha_1, \qquad \vec{t}_2 = \vec{r}_{,2}/\alpha_2 \tag{1.1a}$$

with

$$\alpha_i = |\vec{r}_{,i}|, \qquad (\ )_{,i} = \frac{\partial(\ )}{\partial \xi_i} \tag{1.1b}$$

are tangent to the coordinate curves of the middle surface and are called base vectors. The positive unit normal of the middle surface will henceforth be taken to be

$$\vec{n} = \frac{\vec{r}_{,1} \times \vec{r}_{,2}}{|\vec{r}_{,1} \times \vec{r}_{,2}|} \tag{1.1c}$$

The surface coordinates are orthogonal if $\vec{t}_1 \cdot \vec{t}_2 = 0$, in which case we have $\vec{n} = \vec{t}_1 \times \vec{t}_2$.

Figure (1.1)

While a shell is, strictly speaking, a three dimensional body, certain aspects of a simplified (two dimensional) theory governing the mechanical behavior of such a body may be developed formally without any reference to the three dimensional theory of elasticity[1]. To this end, we will think of a shell as merely a surface (the middle surface) endowed with certain mechanical properties. For instance, there is a surface force, $\Delta\vec{F}_s$, associated with an element of the middle surface and an edge force, $\Delta\vec{F}_e$, associated with an edge of such an element. The resultant force, $\vec{F}$, acting on the shell (the mechanical surface) is then the vector sum of all such forces associated with all the elemental surfaces obtained from some imaginary partition of the middle surface:

$$\vec{F} = \sum_k \Delta\vec{F}_{ek} + \sum_k \Delta\vec{F}_{sk}$$

Figure (1.2)

By the principle of action and reaction, the only terms which appear in the first sum are those associated with the boundary curve(s) of the middle surface.

In the limit as all elemental surface areas and elemental arc lengths tending to zero, we have

$$F = \sum_i \oint_{C_i} d\vec{F}_e + \iint_S d\vec{F}_s$$

where C_i's are the boundary curves of the middle surface. The integrals are to be considered as Stieltjes integrals so that they include the possibility of concentrated (point) forces as well as discontinuousy

[1] The relationship between such a formal development of shell theory and 3-dimensional elasticity must still be established if the theory is to be a part of structural mechanicals.

distributed forces.

The moment of the forces, $\vec{M}^{(F)}$, with respect to the origin of the cartesian coordinates for the space in which the surface is embedded is given by

$$\vec{M}^{(F)} = \sum \oint_{C_i} \vec{r} \times d\vec{F}_e + \iint_S \vec{r} \times d\vec{F}_s .$$

In order to make precise the meaning of the position vector $\vec{r}$ in these integrals, we assume that the forces pass through the centroid of the elemental surfaces dS_i and elemental arcs ds_j, respectively, and that $\vec{r}$ is the position vector of the appropriate centroid.

In addition to the surface and edge forces and the moments due to these forces, there is a surface moment (couple), $\Delta\vec{M}_s$, associated with an element of the middle surface and an edge moment (couple), $\Delta\vec{M}_e$, associated with an edge of such an element. The presence of these additional moments means that the choice of centroids of dS and ds as points of application of $d\vec{F}_s$ and $d\vec{F}_e$ needs not be considered a restrictive assumption.

Combination of $\vec{M}^{(F)}$ and the additional moments leads to a resultant moment, $\vec{M}$, acting on the shell in the form

$$\vec{M} = \oint_C (\vec{r} \times d\vec{F}_e + d\vec{M}_e) + \iint_S (\vec{r} \times d\vec{F}_s + d\vec{M}_s)$$

where it is understood that the first term means a sum of integrals each for a closed boundary curve of the middle surface.

2. Stress Resultants and Couples

At a point on the middle surface and a directed curve C on the surface passing through it, we have a triad of unit vectors, $\vec{\nu}$, $\vec{t}$, and $\vec{n}$ where $\vec{t}$ is tangent to C (positive in the direction of C), $\vec{n}$ is the unit normal to the middle surface and $\vec{\nu} = \vec{t} \times \vec{n}$. The vector $\vec{\nu}$ is therefore normal to the curve but tangent to the surface.

Now, C fictitiously cuts the surface into two parts. For that part with $\vec{\nu}$ pointing away from the surface, we define a stress resultant vector, $\vec{N}_\nu$, at P as the limit (if exists) of the ratio

$$\vec{N}_\nu = \lim_{\Delta s \to 0} \frac{\Delta \vec{F}_\nu}{\Delta s} \quad ,$$

where Δs is the arc length of an elemental arc of C with P as its centroid and $\Delta \vec{F}_\nu$ is the edge force acting on the arc. Since the ratio will depend on the particular curve passing through P, we have replaced the subscript e by ν to indicate a specific curve. Similarly, we define a stress couple (or moment resultant) vector $\vec{M}_\nu$, at P as

$$\vec{M}_\nu = \lim_{\Delta s \to 0} \frac{\Delta \vec{M}_\nu}{\Delta s} \quad .$$

On the other portion of the surface, we have correspondingly, $\vec{N}_{-\nu}$ and $\vec{M}_{-\nu}$. The principle of action and reaction requires that

$$\vec{N}_{-\nu} = -\vec{N}_\nu \qquad \text{and} \qquad \vec{M}_{-\nu} = -\vec{M}_\nu \quad .$$

In this development, we have no knowledge about the three dimensional theory of elasticity; therefore we need the following fundamental postulate:

On any imaginary closed curve C in the interior of the middle surface of a shell, a stress resultant vector field and a stress couple vector field can be defined in such a way that their actions on the material occupying the space interior to C is equipollent to the action of the exterior material on the material inside C.

Now, there are many curves on the surface passing through the point P , so there are many stress resultant and couple vectors associated with the point P . In the special case where C is an ξ_i = constant coordinate curve, we denote the corresponding stress resultant and couple vectors by $\vec{N}_i$ and $\vec{M}_i$ $(i = 1 \text{ or } 2)$ respectively. Just as in three dimensional elasticity, all stress resultant and couple vectors defined at the point P are not independent of each other. In fact, the vector $\vec{N}_\nu$ associated with an arbitrary curve C are related to $\vec{N}_1$ and $\vec{N}_2$ by a linear relation. The same is true for $\vec{M}_\nu$ with respect to $\vec{M}_1$ and $\vec{M}_2$. If the surface coordinates are orthogonal, these relations are

$$\vec{N}_\nu = \nu_1\vec{N}_1 + \nu_2\vec{N}_2 \,, \qquad \vec{M}_\nu = \nu_1\vec{M}_1 + \nu_2\vec{M}_2 \tag{1.2}$$

where $\nu_i = \vec{\nu} \cdot \vec{t}_i$ are the relevant directional cosines. The validity of these formulas is a consequence of equilibrium considerations in the same way as that of Cauchy's formula for three dimensional elasticity.

Surface force intensity vector $\vec{p}$, and surface moment intensity vector, $\vec{q}$, are defined at a point P by

$$\vec{p} = \lim_{\Delta S \to 0} \frac{\Delta \vec{F}_s}{\Delta S}$$

$$\vec{q} = \lim_{\Delta S \to 0} \frac{\Delta \vec{M}_s}{\Delta S}$$

where ΔS is an element of the middle surface with the point P as its centroid. For many problems, contributions to $\vec{p}$ and $\vec{q}$ come exclusively from the known external excitations acting on the shell. So they are often referred to simply as <u>surface loads</u>.

3. Equilibrium

For a body in static equilibrium, we must have vanishing resultant force and moment for the entire body as well as any of its (fictitiously) isolated parts. For our shell, this means

$$\oint_C \vec{N}_\nu ds + \iint_S \vec{p}\, dS = \vec{0}$$

$$\oint_C (\vec{M}_\nu + \vec{r} \times \vec{N}_\nu) ds + \iint_S (\vec{q} + \vec{r} \times \vec{p}) dS = \vec{0} ,$$

where S is an arbitrary portion or the whole of the middle surface and C is the boundary curve(s) of S .

With the help of our Cauchy type formulas for shells, these two conditions become

$$\oint_C (\nu_1 \vec{N}_1 + \nu_2 \vec{N}_2) ds + \iint_S \vec{p}\, dS = \vec{0}$$

$$\int_C [\vec{r} \times (\nu_1 \vec{N}_1 + \nu_2 \vec{N}_2) + (\nu_1 \vec{M}_1 + \nu_2 \vec{M}_2)] ds + \iint_S (\vec{q} + \vec{r} \times \vec{p}) dS = \vec{0} .$$

By Green's theorem in normal form (a two dimensional version of the divergence theorem), we convert the line integrals into surface integrals to get

$$\iint_S [(\alpha_2 \vec{N}_1)_{,1} + (\alpha_1 \vec{N}_2)_{,2} + \alpha_1 \alpha_2 \vec{p}] \frac{dS}{\alpha_1 \alpha_2} = \vec{0}$$

$$\iint_S \{[\alpha_2 (\vec{M}_1 + r \times N_1)]_{,1} + [\alpha_1 (\vec{M}_2 + \vec{r} \times \vec{N}_2)]_{,2} + \alpha_1 \alpha_2 (\vec{q} + \vec{r} \times \vec{p})\} \frac{dS}{\alpha_1 \alpha_2} = \vec{0} .$$

For simplicity, we have assumed an orthogonal system of surface coordinates. We will not be concerned with the case of oblique surface coordinates until much later.

If the two conditions involving surface integrals only are to hold for an arbitrary portion of the middle surface, the two integrands must vanish identically and we get two differential equations for $\vec{N}_i$, $\vec{M}_i$, $\vec{p}$ and $\vec{q}$. Upon using the first of these to simplify the second, they take the form

$$\begin{aligned} &(\alpha_2\vec{N}_1)_{,1} + (\alpha_1\vec{N}_2)_{,2} + \alpha_1\alpha_2\vec{p} = \vec{0} , \\ &(\alpha_2\vec{M}_1)_{,1} + (\alpha_1\vec{M}_2)_{,2} + \vec{r}_{,1} \times (\alpha_2\vec{N}_1) + \vec{r}_{,2} \times (\alpha_1\vec{N}_2) + \alpha_1\alpha_2\vec{q} = \vec{0} \end{aligned} \tag{1.3}$$

Note that we have given here only the barest outline of the derivation of the two differential equations of equilibrium in vector form and in orthogonal surface coordinates. For instance, our use of Green's theorem here is by no means straightforward. We have also implicitly assumed that the geometry of the surface and the relevant vector fields are sufficiently smooth so that the steps leading to the final two differential equations are justified.

4. A Statically Indeterminate Structure

Our daily experience suggests that an elastic shell will deform when subject to external loads such as hydrostatic pressure. The deformation will be resisted by internal reactions which we have formulated in terms of stress resultants and stress couples. Just as in the case of a helical spring, this resistance increases (up to a point) with deformation and the shell ceases to deform further when the internal reactions are in equilibrium with the external loads. This resistance of a shell (or any structure) to external loads makes it useful in many ways, as beer and soft drink cans, the main hull of space or sea crafts, radomes for the protection of radio-telescopes, reflector surfaces of radio-telescopes, just to name a few.

Internal reactions of a shell do not increase with deformation without limit. Beyond a certain critical point depending on the material and the geometry of the shell, some other process governing the shell behavior will take place. In extreme cases, the shell loses its resistance completely; it simply deforms until it collapses and therefore fails to serve its intended purpose. The objective of a shell analysis very often is to determine the internal reaction under a given loading to see whether a given design will serve its purpose, or to come up with the optimal design for a particular purpose.

A shell theorist is mainly concerned with a part of this broad objective. He/she is concerned with the formulation of the mathematical problem for the determination of design information and with the reduction of this problem to a tractable form so that the design data are reasonably accessible to the designers and engineers. Of equal importance to a shell theorist is an understanding of the qualitative behavior of a shell independent of specific applications.

Even if the surface force and moment intensities, $\vec{p}$ and $\vec{q}$, are known functions of position (or not present), we see that equilibrium considerations only give us two vectorial equations for the four unknown vector stress measures, $\vec{N}_i$ and $\vec{M}_i$, and do not allow us to determine the internal reactions completely. To get additional equations for these unknowns, we will have to explore the idea that the internal reactions in the elastic range depend somehow on the deformation of the shell. To do so, we will need a mathematical description of the deformation of the shell.

2. Deformation

1. Translational and Rotational Displacement

Recall that a point P on the undeformed mid-surface of a shell[(1)] is described by a known position vector $\vec{r}(\xi_1,\xi_2)$. In terms of a cartesian reference frame of the space in which the surface is embedded, we have

$$\vec{r} = x(\xi_1,\xi_2)\vec{i}_x + y(\xi_1,\xi_2)\vec{i}_y + z(\xi_1,\xi_2)\vec{i}_z$$

For example, we have for a parapolidal shell of revolution, $x = r\cos\theta$, $y = r\sin\theta$ and $z = r^2/4f$ where f is a known constant. For this shell, we may take $\xi_1 = r$ and $\xi_2 = \theta$. Upon the application of external loads, the shell will deform until it reaches an equilibrium configuration[(2)] which will be referred to as the deformed state. Suppose that the point P is now in a new position P' on the deformed middle surface. Let the position vector for P' (with respect to the same cartesian frame) be $\vec{r}\,'$ with

$$\vec{r}\,' = \vec{r}\,'(\xi_1,\xi_2) = x'(\xi_1,\xi_2)\vec{i}_x + y'(\xi_1,\xi_2)\vec{i}_y + z'(\xi_1,\xi_2)\vec{i}_z \ .$$

We define a translational (or linear) displacement vector $\vec{u}$ by

$$\vec{u} = \vec{r}\,' - \vec{r} = u_x\,\vec{i}_x + u_y\,\vec{i}_y + u_z\,\vec{i}_z$$

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(1) That is, the middle surface before any external load (including the weight of the shell) is applied to it.

(2) For a linear theory, the loads are assumed to be small in magnitude so that this is always realized.

so that

$$u_x = x' - x \ , \quad u_y = y' - y \ , \quad u_z = z' - z \ .$$

We know all about the deformation of the middle surface if we know $\vec{u}$.

But a shell is really a three dimensional body and the deformation of the middle surface alone does not give any information about the deformation of the substance surrounding it. To account for some of the three dimensional features of a shell structure, we associate a unit vector $\vec{\eta}(\xi_1,\xi_2)$ (not necessarily the unit normal) with every point on the middle surface of the undeformed shell and will call it the director following recent literatures on Cosserat media. After deformation, $\vec{\eta}$ becomes $\vec{\eta}'$ associated with the point P' . We define a rotational (or angular) displacement vector $\vec{\phi}$ by

$$\vec{\phi}(\xi_1,\xi_2) = \vec{\eta} \times \vec{\eta}'$$

We postulate that within the framework of our theory, $\vec{u}$ and $\vec{\phi}$ completely describe the deformation of the shell.

For the special case $\vec{u} = \vec{u}_o$ (a constant vector) and $\vec{\phi} = \vec{0}$, the shell simply undergoes a rigid body translation from one part of the physical space to another part. It did not deform in the ordinary sense of the word and the elastic property of the body plays no role throughout this change of position. Evidently, there is no internal reaction developed in the shell for this and other rigid body displacements.

A body experiences a genuine deformation and exhibits its elasticity only if there is a non-uniform or relative displacement of points in the body. In other words, there are points in the body which do not experience the same linear and or angular displacement in both magnitude and direction. In three dimensional elasticity, this leads us to the concept of strain. The internal reactions are functions of the strain measures rather than the displacement measures themselves.

2. Strain Resultants and Strain Couples

In three dimensional elasticity, we derive the strain measures in terms of the displacement measures by a geometrical consideration of the relative displacements of neighboring points and angle changes. While we can do the same for our shell model here, though not so simply, it would be much less painful (and no more abstruse) to take a somewhat indirect route.

In the three dimensional linear theory of elasticity, the stress, strain, displacement and external load measures together satisfy a certain integrated relation called the Principle of Virtual Work. For a shell theory which is adequate for very "small" deformations, we can take an appropriate version of this principle (which is a theorem in three dimensional elasticity) as a postulate and use it to derive expressions os strain measures in terms of the displacement vectors $\vec{u}$ and $\vec{\phi}$. To state this principle for our shell model, we associate a strain measure with every stress measure appeared in our theory. Thus, associated with $\vec{N}_1$ and $\vec{N}_2$ are strain resultant vectors $\vec{\varepsilon}_1$ and $\vec{\varepsilon}_2$, and with $\vec{M}_1$ and $\vec{M}_2$ are strain couple vectors $\vec{\kappa}_1$ and $\vec{\kappa}_2$. The principle of virtual work for our shell theory is simply an assertion of the validity of the following identity

$$\iint_S (\vec{N}_1\cdot\delta\vec{\varepsilon}_1 + \vec{N}_2\cdot\delta\vec{\varepsilon}_2 + \vec{M}_1\cdot\delta\vec{\kappa}_1 + \vec{M}_2\cdot\delta\vec{\kappa}_2)\,dS = \tag{2.1}$$

$$\iint_S (\vec{p}\cdot\delta\vec{u} + \vec{q}\cdot\delta\vec{\phi})\,dS + \oint_C (\vec{N}_\nu^*\cdot\delta\vec{u} + \vec{M}_\nu^*\cdot\delta\vec{\phi})\,ds$$

with the following explanation of the notations in this equation.

Let the deformed shell be in static equilibrium with specified surface loads $\vec{p}$ and $\vec{q}$ and specified edge loads (in the form edge resultant and couple vectors) $\vec{N}_\nu^*$ and $\vec{M}_\nu^*$. Let $\vec{u}$ and $\vec{\phi}$ be the displacement vectors which describe the deformation and $\vec{N}_i$ and $\vec{M}_i$ be the internal reactions developed in the shell. Consider another deformation described by $\vec{u} + \delta\vec{u}$ and $\phi + \delta\vec{\phi}$ with no restriction on the virtual displacement vectors $\delta\vec{u}$ and $\delta\vec{\phi}$ other than they be three times differentiable. Let the strain resultant and strain couple vectors associated with this new deformation $(\vec{u} + \delta\vec{u}$ and $\vec{\phi} + \delta\vec{\phi})$ be $\vec{\varepsilon}_i + \delta\vec{\varepsilon}$ and $\vec{\kappa}_i + \delta\vec{\kappa}_i$ where $\vec{\varepsilon}_i$ and $\vec{\kappa}_i$ are those associated with $\vec{u}$ and $\vec{\phi}$, $i = 1,2$. The principle asserts that the relation among the various integrals stated above holds for all such $\delta\vec{u}$ and $\delta\vec{\phi}$.

We will now use this postulate to deduce the expressions for $\vec{\varepsilon}_i$ and $\vec{\kappa}_i$ in terms of $\vec{u}$ and $\vec{\phi}$. To do this, we use the two vector differential equations of equilibrium to express $\vec{p}$ and $\vec{q}$ in terms of $\vec{N}_i$ and $\vec{M}_i$. We then use Green's theorem in normal form to integrate by parts, leaving us with

$$\iint_S \sum_{i=1}^{2} [\vec{N}_i \cdot (\delta\vec{\varepsilon}_i - \frac{\delta\vec{u},_i + \vec{r},_i \times \delta\vec{\phi}}{\alpha_i}) + \vec{M}_i \cdot (\delta\vec{\kappa}_i - \frac{\delta\vec{\phi},_i}{\alpha_i})] dS$$

$$+ \oint_C [(\vec{N}_\nu - \vec{N}_\nu^*) \cdot \delta\vec{u} + (\vec{M}_\nu - \vec{M}_\nu^*) \cdot \delta\vec{\phi}] ds = 0$$

For the shell to be in equilibrium, we must have

$$\boxed{\vec{N}_\nu = \vec{N}_\nu^* \quad \text{and} \quad \vec{M}_\nu = \vec{M}_\nu^*}$$

which incidentally are the appropriate set of boundary conditions for a boundary value problem involving prescribed edge stress resultants and couples. What remains in the virtual work equation suggests that the strain measures be defined in terms of $\vec{u}$ and $\vec{\phi}$ by the relations

$$\alpha_i \vec{\varepsilon}_i = \vec{u},_i + \vec{r},_i \times \vec{\phi} \, , \qquad \alpha_i \vec{\kappa}_i = \vec{\phi},_i \qquad i = 1,2 \tag{2.2a,b}$$

We use the word "suggest" instead of "imply" because $\vec{N}_i$ and $\vec{M}_i$ are not completely arbitrary; they are related by the two differential equations of equilibrium. It turns out that one can actually prove that the virtual work principle does imply the above strain-displacement relations (up to a rigid motion). We will sketch a proof of this later.

3. Elasticity

1. Component Representation of Vector Fields

The linear and angular displacements, strain resultants and strain couples introduced in Chapter 2 have led us to a quantitative description of the deformation of the shell. But they also introduce six additional unknown vector fields related by only four vector strain-displacement relations. So, we are in a worse position now than we were before we undertook the analysis of deformation insofar as the determination of the internal reactions of the shell is concerned. Since all purely statical and "geometrical" information has been exhausted (or so we postulated), the additional equations needed to complete our theory must come from the properties of the shell material.

In three dimensional elasticity theory[(1)], the elastic property of the substance is defined by equations relating scalar components of stress and strain measures at the same position and time without involving derivatives or anti-derivatives of either set of quantities[(2)]. We expect the same to be true for an elastic shell, and will presently introduce scalar (stress and strain) resultants and couples for a description of the elasticity of shells.

It turns out to be more appropriate to take component representation for the resultant and couple vector fields in the form

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[(1)] We will not be concerned with thermal, electro-magnetic and chemical effects here.

[(2)] This is what we mean by elasticity. The equations expressing the quantitative aspects of elasticity are called stress strain relations. They are a special form of the so-called constitutive equations which describe more general material properties.

$$\vec{N}_k = N_{k1}\vec{t}_1 + N_{k2}\vec{t}_2 + Q_k\vec{n} \quad , \quad \vec{M}_k = \vec{n} \times (M_{k1}\vec{t}_1 + M_{k2}\vec{t}_2) + P_k\vec{n}$$
$$\vec{\varepsilon}_k = \varepsilon_{k1}\vec{t}_1 + \varepsilon_{k2}\vec{t}_2 + \gamma_k\vec{n} \quad , \quad \vec{\kappa}_k = \vec{n} \times (\kappa_{k1}\vec{t}_1 + \kappa_{k2}\vec{t}_2) + \lambda_k\vec{n} \qquad (3.1)$$

where $\vec{t}_1$ and $\vec{t}_2$ are the unit tangent vectors along the two sets of orthogonal curvilinear surface coordinate curves and $\vec{n} = \vec{t}_1 \times \vec{t}_2$. The rather peculiar resolution for $\vec{M}_k$ and $\vec{\kappa}_j$ was dictated by the desire to conform with the accepted engineering notations. We have also the following terminology for the scalar components:

N_{ii}, ε_{ii} : in-plane normal stress and strain resultants
N_{ij}, ε_{ij} $(i \neq j)$: in-plane shear stress and strain resultants
Q_i, γ_i : transverse shear stress and strain resultants
M_{ii}, κ_{ii} : bending stress and strain couples
M_{ij}, κ_{ij} $(i \neq j)$: twisting stress and strain couples
P_i, λ_i : moment stress and strain couples

It should be kept in mind that our definition of the strain resultants and couples do not lend themselves to a geometrical interpretation of these quantities. The geometrical interpretation will come only after we establish contact with three dimensional elasticity theory. On the other hand, the stress resultants and stress couples do have the physical interpretation of forces and moments per unit arc length in directions described by the Figure (3-1). The difference in the direction of positive resultants and couples between the two ξ_2 = constant edges should be carefully studied in connection with the remark on action and reaction in Chapter 1.

While they are not needed for the discussion of shell elasticity, the following component representations of the displacement and load vectors will also be needed later:

$$\vec{u} = u_1\vec{t}_1 + u_2\vec{t}_2 + w\vec{n} \quad , \quad \vec{\phi} = \vec{n} \times (\phi_1\vec{t}_1 + \phi_2\vec{t}_2) + \omega\vec{n}$$

$$\vec{p} = p_1\vec{t}_1 + p_2\vec{t}_2 + p_n\vec{n} \quad , \quad \vec{q} = \vec{n} \times (q_1\vec{t}_1 + q_2\vec{t}_2) + q_n\vec{n} \qquad (3.2)$$

2. The Conventional Shell Theory

Until very recently, all analyses of elastic shells assume $P_j = 0$ and $\gamma_j = 0$ a priori. At first, the assumptions were based on plausible arguments suggested by insight to the physical problem. In the late fifties, W.T. Koiter showed that for a very wide range of external loads and for the usual isotropic materials, the effect of γ_j is of higher order compared to the other strain measures and should be neglected in a (first approximation) shell theory[3]. It is important to note that even if $\gamma_j = 0$, Q_j will still be needed in general to maintain equilibrium as we shall see later. All investigators recognized this, though some (notably J.L. Sanders at Harvard and his followers) quickly eliminate Q_j from the general theory once the differential equations of equilibrium have been established (see Kraus' book for instance).

In the late sixties, E. Reissner showed that only certain moment or couple stress components of the so-called Cosserat (or Generalized) media contribute to the moment stress couples P_j [4]. For a conventional elastic substance and a wide range of loading conditions, the effect of couple stresses is not noticeable at all and we may sefely set $P_j = 0$ in a shell theory for conventional structures. Very similar to the situation involving negligible transverse shearing strains, λ_j need not be zero even if we assume P_j to be absent. But the history of shell theory shows that λ_j have never been assigned an independent role parallel to that of Q_j . The failure to recognize the existence of λ_j

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[3] W.T. Koiter, Proc. IUTAM Symp. on the Theory of Thin Elastic Shells, (Delft, 1959), 12-33, 1960, Edited by W.T. Koiter.

[4] E. Reissner, Proc. 2nd IUTAM Symp. on The Theory of Thin Shells, (Copenhagen, 1967), 15-30, 1969, Edited by Niordson.

as legitimate independent state variables has resulted in many unnecessary conceptual and mathematical difficulties as well as a pedagogically unpalatable theory.

With λ_j treated as legitimate state variables, a shell theory with the built-in assumption of $P_j = \lambda_j = 0$ is apparently very attractive indeed. Obviously, such a theory contains fewer unknowns and therefore needs fewer equations. Not so obvious is the fact that the system of PDE for such a theory is an eighth order system while the system for a more general theory, not making the assumption $P_j = \gamma_j = 0$ a priori, is a twelfth order system. Nevertheless, our point of view here is that whether a shell can develop moment stress couples or experience transverse shearing strains is strictly a property of the shell material. Insofar as the dynamics and the kinematics of the shell are concerned they must be assumed present until the stress strain relations tell us otherwise.

It would seem that our point of view should lead us to a theory of elastic shells far more complicated than the conventional theory. In some ways, it does. But this more complicated (or at least higher order) theory in turn will make things simpler in many other ways.

3. Stress-Strain Relations for Elastic Shells

Since elasticity is a material property, it should in principle be determined by a set of suitable experiments (at least within the framework of continuum mechanics). But such experiments for shell models appear to be very difficult conceptually as well as in practice. Existing sets of equations describing the elasticity of shells were all derived from some accepted generalized Hooke's law of the three-dimensional elasticity theory. In this connection, it appears that all we have to do is to express an adopted set of three dimensional stress-strain relations in terms of our scalar resultants and couples. But this is a decidedly nontrivial step. Only a small part of our difficulty lies in the fact that we do not even know yet how to relate the stress and strain measures of our two dimensional theory to those of the three dimensional elasticity theory. We will not be concerned with this difficult problem related to the foundation of shell theory here. Instead, we will merely make some remarks about the form of general stress-strain relations for shells, and then state a specific and generally accepted set of such relations to be used in subsequent analyses. Derivation of stress-strain relations from three dimensional elasticity theory will be discussed in a later chapter.

A simple count of unknowns and available equations for them suggests that we need twelve such stress-strain relations. The most general form of these relations will evidently be

$$f_k(N_{11}, \ldots, P_2; \varepsilon_{11}, \ldots, \lambda_2) = 0 \qquad k = 1, \ldots, 12$$

However, except in trivial (or degenerate) cases no use has ever been made of such general implicit relations. Instead, stress-strain relations for shells are invariably given in the explicit form

$$N_{11} = g_1(\varepsilon_{11},\varepsilon_{12}, \ldots, \lambda_2), \ldots, Q_2 = g_6(\varepsilon_{11}, \ldots, \lambda_2) ,$$
$$M_{11} = g_1(\varepsilon_{11}, \ldots,\lambda_2), \ldots, P_2 = g_{12}(\varepsilon_{11},\varepsilon_{12}, \ldots,\lambda_2) \tag{3.3}$$

or, alternately, the inverted form

$$\varepsilon_{11} = G_1(N_{11},N_{12}, \ldots, P_2) , \ldots,$$
$$\lambda_2 = G_{12}(N_{11},N_{12}, \ldots, P_2)$$

For the purpose of establishing variational theorems, we will be particularly interested in those materials for which the stress-strain relations (3.3) can be expressed in terms of a scalar functional $S(\varepsilon_{11},\varepsilon_{12}, \ldots, \lambda_2)$ by the relations

$$N_{11} = \frac{\partial S}{\partial \varepsilon_{11}}, N_{12} = \frac{\partial S}{\partial \varepsilon_{12}}, \ldots, \ldots, Q_1 = \frac{\partial S}{\partial \gamma_1} , \ldots,$$
$$M_{11} = \frac{\partial S}{\partial \kappa_{11}}, M_{12} = \frac{\partial S}{\partial \kappa_{12}}, \ldots, \ldots, P_1 = \frac{\partial S}{\partial \lambda_i}, \ldots . \tag{3.4}$$

These scalar relations may be written more compactly as <u>four</u> vector equations:

$$\vec{N}_j = \frac{\partial S}{\partial \vec{\varepsilon}_j} , \qquad \vec{M}_j = \frac{\partial S}{\partial \vec{\kappa}_j} \qquad (j=1,2) \tag{3.4'}$$

where the vectorial differentiations are to mean

$$\frac{\partial S}{\partial \vec{\varepsilon}_j} = \frac{\partial S}{\partial \varepsilon_{j1}} \vec{t}_1 + \frac{\partial S}{\partial \varepsilon_{j2}} \vec{t}_2 + \frac{\partial S}{\partial \gamma_j} \vec{n} , \quad \frac{\partial S}{\partial \vec{\kappa}_j} = \vec{n} \times (\frac{\partial S}{\partial \kappa_{j1}} \vec{t}_1 + \frac{\partial S}{\partial \kappa_{j2}} \vec{t}_2) + \frac{\partial S}{\partial \lambda_j} \vec{n} \tag{3.5}$$

The quantity S is called the <u>strain energy (density) function</u> in engineering literature but is also called the stress potential (in strain space) herein.

If the stress strain relations are given in the inverted form instead, we will be particularly interested in those which are expressible in terms of a scalar strain potential (in stress space) or complementary energy (density) function, $C(N_{11}, N_{12}, \ldots, P_2)$, in the form

$$\varepsilon_{11} = \frac{\partial C}{\partial N_{11}}, \quad \ldots, \quad \ldots, \quad \lambda_2 = \frac{\partial C}{\partial P_2} \tag{3.6}$$

or in vector form

$$\vec{\varepsilon}_j = \frac{\partial C}{\partial \vec{N}_j}, \quad \vec{\kappa}_j = \frac{\partial C}{\partial \vec{M}_j} \quad (j=1,2) \tag{3.6'}$$

We will occasionally call shells with stress strain relations expressible either in terms of a stress potential or a strain potential hyperelastic shells. We do not know at this point the form of S or C for a particular class of shells (e.g. isotropic shells). But there are things which we can say immediately.

If the shell can not carry moment stress couples, we must have $P_1 = P_2 = 0$ as two of our stress strain relations. The stress potential S must therefore be independent of λ_1 and λ_2. Similarly, if the shell does not develop transverse shear strains, two of our stress strain relations must be $\gamma_1 = \gamma_2 = 0$. In this case, the strain potential C must be independent of Q_1 and Q_2.

4. Linear Stress Strain Relations for Isotropic Shells

It is possible to have a "small deformation" and "small strain" shell theory which involves linear equilibrium equations and linear strain displacement relations but nonlinear stress-strain relations. Such a theory is of course not a linear theory. For a linear theory of elastic shells, each one of the twelve stress-strain relations must be a linear relation. Evidently, the stress and strain potentials for linear stress strain relations are quadratic in their respective arguments (in the case of no initial stress and strain).

In the subsequent development, we will be interested mainly in shells whose material properties have no directional preference in the tangent plane of the middle surface. Such shells are called isotropic shells. In fact, we will be interested almost exclusively in a class of isotropic elastic shells whose elasticity is described by the system

$$\begin{cases} \varepsilon_{11} = A(N_{11}-\nu_N N_{22}), \quad \varepsilon_{22} = A(N_{22}-\nu_N N_{11}), \quad \gamma_j = A_Q Q_j \\ \varepsilon_{12} = A(1+\nu_N)[(1-t_N)N_{12}+t_N N_{21}], \; \varepsilon_{21} = A(1+\nu_N)[(1-t_N)N_{21}+t_N N_{12}] \end{cases} \tag{3.7}$$

$$\begin{cases} M_{11} = D(\kappa_{11}+\nu_M\kappa_{22}), \quad M_{22} = D(\kappa_{22}+\nu_M\kappa_{11}), \quad P_j = D_P\lambda_j \\ M_{12} = D(1-\nu_M)[(1+t_M)\kappa_{12}-t_M\kappa_{21}], \; M_{21} = D(1-\nu_M)[(1+t_M)\kappa_{21}-t_M\kappa_{12}] \end{cases} \tag{3.8}$$

where A, D, A_Q, D_P, ν_N, ν_M, t_N and t_M are independent parameters the values of which depend on the particular shell material and may be functions of the middle surface coordinates ξ_1 and ξ_2 .

We mention in passing that for (conventional) elastic material which is homogeneous across the shell thickness, we have

$$\nu_N = \nu_M = \nu, \qquad A = 1/Eh, \qquad D = Eh^3/12(1-\nu^2) \tag{3.9}$$

where E is <u>Young's modulus</u> (in planes tangent to the middle surface) and ν is (in-plane) <u>Poisson's ratio</u>. Insofar as the resultants and couples are unrelated by the stress-strain relations (3.7) and (3.8), these relations are called (bending-stretching) <u>uncoupled</u> stress-strain relations.

The relations (3.7) and (3.8) are not quite in the form (3.1) or the inverted form. To get them into the form (3.1), we simply solve (3.7) for the stress resultants in terms of the strain resultants to get

$$\begin{cases} N_{11} = (\varepsilon_{11}+\nu_N\varepsilon_{22})/A(1-\nu_N^2), \quad N_{22} = \ldots\ldots, \quad Q_j = \gamma_j/A_Q \\ N_{12} = \dfrac{(1-t_N)\varepsilon_{12} - t_N\varepsilon_{21}}{A(1+\nu_N)(1-2t_N)} \qquad N_{21} = \ldots\ldots\ldots \end{cases} \tag{3.7'}$$

Equations (3.7') and (3.8) are now in the form of (3.1) and they can be written in terms of a stress potential S in the form given in (3.4) with

$$S = \frac{(\varepsilon_{11}+\varepsilon_{22})^2 - 2(1-\nu_N)(\varepsilon_{11}\varepsilon_{22}-\varepsilon_{12}\varepsilon_{21}) + (1-\nu_N)(1-t_N)(\varepsilon_{12}-\varepsilon_{21})^2/(1-2t_N)}{2A(1-\nu_N^2)}$$

$$+ \frac{D}{2}[(\kappa_{11}+\kappa_{22})^2 - 2(1-\nu_M)(\kappa_{11}\kappa_{22}-\kappa_{12}\kappa_{21}) + (1+t_M)(1-\nu_M)(\kappa_{12}-\kappa_{21})^2]$$

$$+ \frac{\gamma_1^2+\gamma_2^2}{2A_Q} + \frac{D_p}{2}(\lambda_1^2+\lambda_2^2) \tag{3.10}$$

Thus, a shell whose elasticity is described by (3.7') and (3.8) is hyperelastic. We have written S in terms of some rather peculiar combinations of the strain resultants and couples. It can be verified that they are in fact the only scalar (uncoupled) invariant of ε_{ij}, γ_j, κ_{ij}, λ_j with respect to a rotation of the orthogonal middle surface coordinates. We will say a little more about this in the next section.

The inversion of (3.7) is not meaningful when $t_N = \frac{1}{2}$. For then, we really have only one relation for N_{12} and N_{21} while we must have $\varepsilon_{12} = \varepsilon_{21}$ to be consistent. We will return to this special case later when we discuss the foundation of shell theory and the Flugge-Lur'e-Byrne stress-strain relations[5].

Alternately, if we invert (3.8) to get

$$\kappa_{11} = \frac{M_{11} - \nu_M M_{22}}{D(1-\nu_M^2)}, \qquad \kappa_{22} = \ldots\ldots, \qquad \lambda_j = P_j/D_p,$$

$$\kappa_{12} = \frac{(1+t_M)M_{12} + t_M M_{21}}{D(1-\nu_M)(1+2t_M)}, \qquad \kappa_{21} = \ldots\ldots. \tag{3.8'}$$

Now, (3.7) and (3.8') can be written in terms of a strain potential C in the form given in (3.6) where

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(5) Readers are referred also to E. Reissner and F.Y.M. Wan, "A Note on Stress Strain Relations of the Linear Theory of Shells", J. Appl. Math. & Phys. (ZAMP) 17, 1966, 676-681.

$$C = \frac{A}{2}[(N_{11}+N_{22})^2 - 2(1+\nu_N)(N_{11}N_{22}-N_{12}N_{21})+(1-t_N)(1+\nu_N)(N_{12}-N_{21})^2]$$

$$+ \frac{(M_{11}+M_{22})^2 - 2(1+\nu_M)(M_{11}M_{22}-M_{12}M_{21})+(1+t_M)(1+\nu_M)(M_{12}-M_{21})^2/(1+2t_M)}{2D(1-\nu_M^2)}$$

$$+ \frac{A_Q}{2}(Q_1^2+Q_2^2) + \frac{P_1^2+P_2^2}{2D_P} \tag{3.11}$$

Similar to the inversion of (3.7), the inversion of (3.8) breaks down if $t_M = -\frac{1}{2}$ as we have only one relation for κ_{12} and κ_{21} and M_{12} must be equal to M_{21}. We can not avoid this rather singular situation since the generally accepted system of stress strain relations for a conventional theory of isotropic shell (without transverse shearing strains and moment stress couples) is in fact a special case of our more general system (3.7) and (3.8), namely the special system corresponding to $A_Q = D_P = 0$ and $t_N = -t_M = \frac{1}{2}$.

We can of course leave (3.7) and (3.8) as they are for this special case if we only want to use them to solve a specific problem. But to get them in terms of S or C for the purpose of establishing variational principles and uniqueness theorem, we will need the form (3.7') and (3.8) or (3.7) and (3.8') as an intermediate step. It turns out that the solution to this intermediate step offers no particular difficulty for the conventional stress strain relations cited above. The more difficult problem is to construct the corresponding S or C once we have obtained either set of the one sided stress strain relations. The problem of obtaining one sided relations for the conventional shell theory and constructing the corresponding scalar potentials was solved not long ago[6] and will be discussed in a later chapter of these notes.

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(6) E. Reissner and F.Y.M. Wan, "On the stress strain relations and strain displacement relations of the linear theory of shels", Recent Progress in Applied Mechanics (The Folke Odqvist Volume), Almqvist & Wiksell, Stockholm, 1967, 487-500.

5. Transformation Laws

Before leaving the topic of stress strain relations for shells, we emphasize that we have done very little other than singling out a rather special set of such relations. This set has the special features of being linear, isotropic, uncoupled and containing the conventional relations as a special case. We have made no effort to justify the form of these relations or their content up to this point. That they deserve our exclusive attention here is dictated by their practical importance (in the case of the conventional relations) and simplicity (in the case of the more general relations).

A few words should be said at this point in connection with the form of the stress and strain potentials. To do so, we will digress for a moment and talk about the transformation laws for the scalar resultants and couples.

Suppose that we decide to use a different set of orthogonal middle surface coordinates ξ_1' and ξ_2' instead of ξ_1 and ξ_2. Then the stress resultant and couple vectors $\vec{N}_j'$ and $\vec{M}_j'$, j = 1,2, associated with the new coordinate curves will in general be different from $\vec{N}_j$ and $\vec{M}_j$, respectively. In fact, the Cauchy type formulas tell us that

$$\vec{N}_j' = t_{j1}\vec{N}_1 + t_{j2}\vec{N}_2 \qquad t_{jk} = \vec{t}_j'\cdot\vec{t}_k \neq t_{kj}$$

where $\vec{t}_j'$ are the base vectors for the new surface coordinates. The unit normal is unaffected by the change of surface coordinates so that

$$\vec{n} = \vec{t}_1 \times \vec{t}_2 = \vec{t}_1' \times \vec{t}_2'$$

With the component representation

$$\vec{N}'_j = N'_{j1}\vec{t}'_1 + N'_{j2}\vec{t}'_2 + Q'_j\vec{n}$$

we have

$$N'_{jk} = \vec{N}'_j \cdot \vec{t}'_k = (t_{j1}\vec{N}_1 + t_{j2}\vec{N}_2)\cdot\vec{t}'_k$$

$$= t_{j1}t_{k1}N'_{11} + t_{j1}t_{k2}N_{12} + t_{j2}t_{k1}N_{21} + t_{j2}t_{k2}N_{22}$$

$$Q'_j = \vec{n}\cdot\vec{N}'_j = t_{j1}Q_1 + t_{j2}Q_2$$

If we introduce an angle ϕ defined by $\vec{t}'_1 \cdot \vec{t}_1 = \cos\phi$, then these formulas can be simplified considerably to

$$N'_{11} = \cos^2\phi N_{11} + \sin^2\phi N_{22} + \cos\phi\sin\phi(N_{12}+N_{21})$$

$$N'_{22} = \sin^2\phi N_{11} + \cos^2\phi N_{22} - \cos\phi\sin\phi(N_{12}+N_{21})$$

$$N'_{12} = \sin\phi\cos\phi(N_{22}-N_{11}) + \cos^2\phi N_{12} - \sin^2\phi N_{21} \qquad \underline{\text{Figure (3.2)}}$$

$$N'_{21} = \sin\phi\cos\phi(N_{22}-N_{11}) + \cos^2\phi N_{21} - \sin^2\phi N_{12}$$

$$Q'_1 = \cos\phi Q_1 + \sin\phi Q_2 \ , \qquad Q'_2 = -\sin\phi Q_1 + \cos\phi Q_2$$

It is not difficult to verify that the transformation laws for M_{ij} and P_j are exactly the same as those for N_{ij} and Q_j respectively. In short, Q_j and P_j transform like components of vectors, the N_{ij} and M_{ij} transform as components of (locally) cartesian tensors of rank two.

If a shell is isotropic in the tangent plane of its midsurface then the form of the stress-strain relations must be independent of the choice of ξ_1 and ξ_2 . This means, for instance, that the form of the strain potential C should remain the same for different sets of surface coordinates. In other words, C can only depend on the scalar invariants of N_{ij}, Q_j, M_{ij} and P_j with respect to a locally orthogonal transformation. It is then a straight-forward calculation to show that $(N_{11}+N_{22})$, $(N_{12}-N_{21})$, $(N_{11}N_{22}-N_{12}N_{21})$, $(Q_1^2+Q_2^2)$ and the corresponding combinations for M_{ij} and P_j are such invariants. Moreover, if we are only interested in the invariants of each of N_{ij}, M_{ij}, P_j, Q_j there is no other first and second order invariants. The requirement of a homogeneous quadratic form for C further fixes the form of C up to eight independent parameters (the elastic moduli A, ν_N, t_N, A_Q, etc.). In principle, these parameters can be determined by suitable experiments.

The corresponding transformation laws for the strain resultants and couples needed to discuss the stress potential S can be established similarly once we show that

$$\vec{\kappa}'_j = t_{j1}\vec{\kappa}_1 + t_{j2}\vec{\kappa}_2, \qquad \vec{\varepsilon}'_j = t_{j1}\vec{\varepsilon}_1 + t_{j2}\vec{\varepsilon}_2$$

The validity of these formulas follows from a known result in the vector calculus in curvilinear coordinates

$$\frac{1}{\alpha'_j}\ \frac{\partial\vec{v}}{\partial\xi'_j} = \frac{t_{j1}}{\alpha_1}\frac{\partial\vec{v}}{\partial\xi_1} + \frac{t_{j2}}{\alpha_2}\ \frac{\partial\vec{v}}{\partial\xi_2}$$

where $\vec{v}$ is any differentiable vector field.

4. Boundary Value Problems

1. Summary of Vector Differential Equations and Boundary Conditions

The governing equations for our linear theory of elastic shells developed in the last few sections are:

Strain-Displacement Relations:

$$\alpha_j \vec{\kappa}_j = \vec{\phi},_j \qquad \alpha_j \vec{\varepsilon}_j = \vec{u},_j + \vec{r},_j \times \vec{\phi} \qquad (j=1,2) \qquad (2.2a,b)$$

Stress Strain Relations (or Equations of Elasticity):

$$\vec{N}_j = \frac{\partial S}{\partial \vec{\varepsilon}_j}\,, \quad \vec{M}_j = \frac{\partial S}{\partial \vec{\kappa}_j} \qquad (j=1,2) \qquad (3.4')$$

Equilibrium Equations:

$$(\alpha_2 \vec{N}_1),_1 + (\alpha_1 \vec{N}_2),_2 + \alpha_1 \alpha_2 \vec{p} = \vec{0} \qquad (1.3a)$$

$$(\alpha_2 \vec{M}_1),_1 + (\alpha_1 \vec{M}_2),_2 + \vec{r},_1 \times (\alpha_2 \vec{N}_1) + \vec{r},_2 \times (\alpha_1 \vec{N}_2) + \alpha_1 \alpha_2 \vec{q} = \vec{0} \quad (1.3b)$$

where the surface load intensities, $\vec{p}$ and $\vec{q}$, are either completely prescribed or given linearly in terms of the unknowns and where the stress potential, $S(\varepsilon_1, \ldots, \lambda_2)$, is homogeneous quadratic in its arguments.

Altogether, they are <u>ten</u> vectorial equations for the <u>ten</u> vector unknowns $\vec{u}, \vec{\phi}, \vec{\kappa}_j, \vec{\varepsilon}_j, \vec{N}_j$, and $\vec{M}_j$, $j = 1,2$. With a set of appropriately prescribed boundary (and initial) conditions, we have then a boundary (/initial) value problem in partial differential equations.

For static problems in which the shell eventually settles into a final equilibrium configuration without further motion, we are interested in the distribution of stress, strain and displacement measures in the final deformed state of the shell. For such a problem, we may have prescribed displacement conditions at an edge C of the shell so that

$$\vec{u} = \vec{u}^*(\xi_s) \quad \text{and} \quad \vec{\phi} = \vec{\phi}^*(\xi_s) \qquad a \leq \xi_s \leq b \tag{4.1a,b}$$

where the starred quantities are known functions of an edge variable ξ_s. For example, the edge may be constrained so that no displacement and rotation is possible there, in which case $\vec{u}^* = \vec{\phi}^* = \vec{0}$ along C.

If we have prescribed distributions of stress resultants and couples at the edge instead, then the boundary conditions are

$$\vec{N}_\nu = \vec{N}_\nu^*(\xi_s) \quad \text{and} \quad \vec{M}_\nu = \vec{M}_\nu^*(\xi_s) \qquad (a \leq \xi_s \leq b) \tag{4.2a,b}$$

where, by the Cauchy type formulas,

$$\vec{N}_\nu = \nu_1\vec{N}_1 + \nu_2\vec{N}_2 , \qquad \vec{M}_\nu = \nu_1\vec{M}_1 + \nu_2\vec{M}_2 \tag{1.2}$$

For example, the shell may be free of any constraints (and therefore free of edge tractions) so that $\vec{N}_\nu^* = \vec{M}_\nu^* = \vec{0}$.

The <u>displacement</u> and the <u>stress boundary value problem</u> are typical of the kind of boundary value problems one comes across in shell theory. They will be referred to occasionally as the <u>first</u> and the <u>second fundamental problem</u> of shell theory, respectively. There are of course other physically realizable mixed edge conditions and we will have an opportunity to see them later.

The above formulation shell problems is also applicable to dynamical problems. Within the framework of a linear theory, we only have to appeal to d'Alembert's principle and set (with $(\;)_{,t} \equiv \partial(\,)/\partial t$)

$$\vec{p} = -\rho_s\vec{u}_{,tt} + \vec{p}\,'(\xi_1,\xi_2,t), \quad \vec{q} = -\vec{I}_s \times \vec{\phi}_{,tt} + \vec{q}\,'(\xi_1,\xi_2,t), \tag{4.3}$$

where ρ_s is the surface mass density and $\vec{I}_s$ is some sort of moment of inertia vector to be defined more carefully later. The boundary conditions will now be supplemented by the initial conditions at the reference time $t = 0$:

$$\begin{aligned} &\vec{u}(\xi_1,\xi_2,0) = \vec{s}(\xi_1,\xi_2)\ , \qquad \vec{\phi}(\xi_1,\xi_2,0) = \vec{\beta}(\xi_1,\xi_2) \\ &\vec{u},_t(\xi_1,\xi_2,0) = \vec{v}(\xi_1,\xi_2)\ , \qquad \vec{\phi},_t(\xi_1,\xi_2,0) = \vec{\Omega}(\xi_1,\xi_2) \end{aligned} \tag{4.4}$$

We will confine our discussion in the next few chapters to static problems. No mention of the dynamic problems will be made again until we get to the subject of vibration.

2. Uniqueness Theorems

While we have obtained enough differential equations and boundary conditions to determine the quantities describing the elastostatics of a shell under loadings, there is always the question whether our theory is an adequate mathematical model of the physical problem. The answer to this question depends ultimately on how well does our theory explains and predicts shell behavior. But we will have more confidence in a theory if we can at least show that the problem is well-posed as a mathematical problem in partial differential equations. One of the criteria of a well-posed problem is that the solution of the problem should be unique. We can prove such a theorem very simply for our fundamental problems if our homogeneous quadratic strain energy density function S is positive definite, i.e, S is non-negative if its arguments are real, and is zero only if all its arguments take on zero values. The proof is sketched below.

Suppose there are two distinct solutions $\vec{\phi}_1$, $\vec{u}_1$ and $\vec{\phi}_2$, $\vec{u}_2$ for one of two fundamental problems. Consider their differences $\vec{u} = \vec{u}_1 - \vec{u}_2$ and $\vec{\phi} = \vec{\phi}_1 - \vec{\phi}_2$ and let $\vec{N}_j$, $\vec{M}_j$, $\vec{\varepsilon}_j$ and $\vec{\kappa}_j$ be the resultants and couples corresponding these differences. Note that $\vec{N}_j$ and $\vec{M}_j$ satisfy the homogeneous equilibrium equations so that

$$(\alpha_2 \frac{\partial S}{\partial \vec{\varepsilon}_1})_{,1} + (\alpha_1 \frac{\partial S}{\partial \vec{\varepsilon}_2})_{,2} = \vec{0} ,$$

$$(\alpha_2 \frac{\partial S}{\partial \vec{\kappa}_1})_{,1} + (\alpha_1 \frac{\partial S}{\partial \vec{\kappa}_2})_{,2} + \vec{r}_{,1} \times (\alpha_2 \frac{\partial S}{\partial \vec{\varepsilon}_1}) + \vec{r}_{,2} \times (\alpha_2 \frac{\partial S}{\partial \vec{\varepsilon}_2}) = 0 .$$

From these, we get

$$[(\alpha_2 \frac{\partial S}{\partial \vec{\varepsilon}_1}),_1 + (\alpha_1 \frac{\partial S}{\partial \vec{\varepsilon}_2}),_2] \cdot \frac{\vec{u}}{\alpha_1\alpha_2} + [(\alpha_2 \frac{\partial S}{\partial \vec{\kappa}_1}),_1 + \cdots] \cdot \frac{\vec{\phi}}{\alpha_1\alpha_2} = 0$$

in the interior of the shell. Integrate over the entire surface and use Green's theorem in normal form. We then get

$$0 = \oint_C \{[\nu_1 \frac{\partial S}{\partial \vec{\varepsilon}_1} + \nu_2 \frac{\partial S}{\partial \vec{\varepsilon}_2}] \cdot \vec{u} + [\nu_1 \frac{\partial S}{\partial \vec{\kappa}_1} + \nu_2 \frac{\partial S}{\partial \vec{\kappa}_2}] \cdot \vec{\phi}\} ds$$

$$- \iint \{\frac{\vec{u},_1}{\alpha_1} \frac{\partial S}{\partial \vec{\varepsilon}_1} + \frac{\vec{u},_2}{\alpha_2} \frac{\partial S}{\partial \vec{\varepsilon}_2} + \frac{\vec{\phi},_1}{\alpha_1} \frac{\partial S}{\partial \vec{\kappa}_1} + \frac{\vec{r},_1 \times \vec{\phi}}{\alpha_1} \frac{\partial S}{\partial \vec{\varepsilon}_1} + \frac{\vec{r},_2 \times \vec{\phi}}{\alpha_2} \frac{\partial S}{\partial \vec{\varepsilon} 2}\} d\xi_1 d\xi_2$$

or

$$\oint_C \{\vec{N}_\nu \cdot \vec{u} + \vec{M}_\nu \cdot \vec{\phi}\} ds - \iint \{\vec{\varepsilon}_1 \cdot \frac{\partial S}{\partial \vec{\varepsilon}_1} + \vec{\varepsilon}_1 \cdot \frac{\partial S}{\partial \vec{\varepsilon}_2} + \vec{\kappa}_1 \cdot \frac{\partial S}{\partial \vec{\kappa}_1} + \vec{\kappa}_2 \cdot \frac{\partial S}{\partial \vec{\kappa}_2}\} d\xi_1 d\xi_2 = 0 \; .$$

Now, we have either $\vec{N}_\nu = \vec{M}_\nu = \vec{0}$ or $\vec{u} = \vec{\phi}$ for our "difference solution". At the same time,

$$\vec{\varepsilon}_j \cdot \frac{\partial S}{\partial \vec{\varepsilon}_j} + \vec{\kappa}_j \cdot \frac{\partial S}{\partial \vec{\kappa}_j} = 2S$$

since S is homogeneous quadratic in its arguments. Altogether, we are left with

$$2\iint S \; d\xi_1 d\xi_2 = 0$$

S is always non-negative, so we must have $S = 0$ throughout the midsurface. But S is positive definite, so its arguments must be zero on the entire surface, i.e. $\vec{\varepsilon}_j \equiv \vec{\kappa}_j \equiv \vec{0}$, $j = 1,2$. The stress strain relations then give us $\vec{N}_j \equiv \vec{M}_j \equiv \vec{0}$.

For the second fundamental (stress boundary value) problem, this is all we can say, a situation very similar to that of the Neumann problem in potential theory. There is some non-uniqueness in the displacement vectors which amounts to a rigid displacement and rotation of the shell and can be eliminated by fixing one point in the shell (against both translational and rotational displacement).

For the first fundamental problem, the strain-displacement relations and the homogeneous displacement edge conditions form a Dirichlet type homogeneous boundary value problem for $\vec{u}$ and $\vec{\phi}$ which has only a trivial solution. Therefore, the two distinct solutions $\vec{u}_i$ and $\vec{\phi}_i$ are in fact the same solution.

3. The Displacement Method

Having gained some confidence in the correctness of our boundary value problems, we can now embark on a program to obtain their solutions. A method often used by investigators of shells is to express the stress measures in terms of the two displacement vectors by substituting the strain-displacement relations into the stress- strain relations. The resulting stress-displacement relations are then substituted into the equilibrium equations giving us two vector equations for the two vector unknowns $\vec{u}$ and $\vec{\phi}$. These two equations, equivalent to the Navier equations in three dimensional elasticity theory, and the appropriate edge conditions are then solved to determine the two displacement vectors. Having $\vec{u}$ and $\vec{\phi}$, we can then calculate the strain measures by way of the strain-displacement relations and the stress measures by way of the stress-strain relations.

Note that the stress-displacement relations also allow us to express the stress boundary conditions in terms of the displacements and their first derivatives. Evidently, the displacement boundary value problem, when the displacement method is employed, is analogous to the Dirichlet problem in potential theory, while the stress boundary value problem, using the same method, is analogous to the mixed problem.

The vector equations discussed so far have been useful for a compact presentation of the general theory of shells in a manner similar to the so-called tensorial formulation of shell theory. But the solution of specific boundary value problems is usually obtained via the corresponding scalar differential equations for the corresponding scalar resultants, couples and displacement components. The reason is that, unlike the equilibrium equations and strain-displacement relations, the stress strain relations are not really vectorial relations but only symbolically so. As a first step toward the solution of our boundary value problems, we will have to convert the vector equilibrium equations and strain displacement relations into scalar equations. For this purpose, we will need formulas expressing the various partial derivatives of the base vectors, $\vec{t}_1$ and $\vec{t}_2$, and the unit normal, $\vec{n}$, in terms of the same triad of vectors. These formulas are the so-called <u>Gauss-Weingarten</u> equations in classical differential geometry.

4. Geometry of the Middle Surface

Given the position vector $\vec{r}(\xi_1,\xi_2)$ for points on the middle surface of the shell, we can compute $\vec{t}_i$ and $\vec{n}$ as well as the α_i . $i = 1,2$, by formulas in chapter (1). For example, the middle surface of a shell of revolution is given by the position vector

$$\vec{r} = r\cos\theta\vec{i}_x + r\sin\theta\vec{i}_y + z\vec{i}_z = r\vec{i}_r + z\vec{i}_z$$

where r and z are given parametrically in terms of a meridional independent variable ξ (which may be r or z itself in special cases). We can now take $\xi_1 = \theta$ and $\xi_2 = \xi$. Then, with prime indicating differentiation with respect to ξ , we have

$$\vec{r}_{,1} = r(-\sin\theta\vec{i}_x + \cos\theta\vec{i}_y) = r\vec{i}_\theta \;, \qquad \vec{r}_{,2} = r'\vec{i}_r + z'\vec{i}_z$$

and correspondingly,

$$\alpha_1 = r \;, \qquad \alpha_2 = (r'^2 + z'^2)^{1/2} \;,$$

$$\vec{t}_1 = \vec{i}_\theta \;, \quad \vec{t}_2 = (r'\vec{i}_r + z'\vec{i}_z)/\alpha_2 \;,$$

Since $\vec{t}_1 \cdot \vec{t}_2 = 0$, the mid-surface coordinates are orthogonal and therefore

$$\vec{n} = \vec{t}_1 \times \vec{t}_2 = (z'\vec{i}_r - r'\vec{i}_z)/\alpha_2.$$

Returning to general shells (but still using only orthogonal surface coordinates), we begin our derivation of the differentiation formulas for the triad of vectors by writing

$$\frac{\vec{n}_{,k}}{\alpha_k} = \frac{\vec{t}_1}{R_{k1}} + \frac{\vec{t}_2}{R_{k2}} \qquad (k = 1,2) \tag{4.5a}$$

Note that the derivative of any unit vector $\vec{u}$ is orthogonal to the unit vector itself since $\vec{u}\cdot\vec{u} = 1$ and

$$2\vec{u}\cdot\vec{u}_{,i} = (\vec{u}\cdot\vec{u})_{,i} = (1)_{,i} = 0$$

Also $\vec{u}_{,i}$ will no longer be a unit vector in general. The explicit expression for the radii of curvature, R_{kj}, can be obtained by taking appropriate dot products. To get them directly in terms of $\vec{r}$ we observe that

$$\frac{1}{R_{kj}} = \frac{\vec{n}_{,k}}{\alpha_k}\cdot\vec{t}_j = \frac{(\vec{n}\cdot\vec{t}_j)_{,k} - \vec{n}\cdot\vec{t}_{j,k}}{\alpha_k} = -\frac{\vec{n}}{\alpha_k}\cdot\left(\frac{\vec{r}_{,j}}{\alpha_j}\right)_{,k} \tag{4.5b}$$

$$= -\frac{\vec{n}\cdot\vec{r}_{,jk}}{\alpha_j\alpha_k} - \frac{\vec{n}\cdot\vec{r}_{,j}}{\alpha_k}\left(\frac{1}{\alpha_j}\right)_{,k} = -\frac{\vec{n}\cdot\vec{r}_{,jk}}{\alpha_j\alpha_k}$$

We will assume that $\vec{r}$ is twice continuously differentiable so that $R_{jk} = R_{kj}$.

For the derivative of the base vectors, we set

$$\frac{\vec{t}_{1,1}}{\alpha_1} = C_1\vec{t}_2 + C_2\vec{n}\ , \qquad \frac{\vec{t}_{1,2}}{\alpha_2} = C_3\vec{t}_2 + C_4\vec{n}$$

and get immediately

$$C_2 = \frac{\vec{n}\cdot\vec{t}_{1,1}}{\alpha_1} = \frac{(\vec{n}\cdot\vec{t}_1)_{,1} - \vec{n}_{,1}\cdot\vec{t}_1}{\alpha_1} = -\frac{1}{R_{11}}$$

$$C_4 = \frac{\vec{n}\cdot\vec{t}_{1,2}}{\alpha_2} = \frac{(\vec{n}\cdot\vec{t}_1)_{,2} - \vec{n}_{,2}\cdot\vec{t}_1}{\alpha_2} = -\frac{1}{R_{21}}$$

To get C_1 and C_3, we observe that

$$C_1 = \frac{\vec{t}_{1,1}\cdot\vec{t}_2}{\alpha_1} = \frac{(\vec{t}_{1,1})\cdot(\alpha_2\vec{t}_2)}{\alpha_1\alpha_2} = \frac{(\vec{t}_1\cdot\alpha_2\vec{t}_2)_{,1} - \vec{t}_1\cdot(\vec{r}_{,2})_{,1}}{\alpha_1\alpha_2}$$

$$= -\frac{\vec{t}_1\cdot(\vec{r}_{,1})_{,2}}{\alpha_1\alpha_2} = -\frac{\vec{t}_1\cdot\alpha_{1,2}\vec{t}_1 + \vec{t}_1\cdot(\alpha_1\vec{t}_{1,2})}{\alpha_1\alpha_2} = -\frac{\alpha_{1,2}}{\alpha_1\alpha_2}$$

$$C_3 = \frac{\vec{t}_{1,2}\cdot\vec{t}_2}{\alpha_2} = \frac{\vec{t}_2\cdot(\vec{r}_{,1}/\alpha_1)_{,2}}{\alpha_2} = \frac{\vec{t}_2\cdot\vec{r}_{,12}}{\alpha_1\alpha_2} = \frac{\vec{t}_2\cdot(\alpha_2\vec{t}_2)_{,1}}{\alpha_1\alpha_2}$$

$$= \frac{\alpha_{2,1}}{\alpha_1\alpha_2}$$

Corresponding formulas for $\vec{t}_{2,1}$ and $\vec{t}_{2,2}$ are obtained similarly. Altogether, we have

$$\frac{\vec{t}_{1,1}}{\alpha_1} = -\frac{\alpha_{1,2}}{\alpha_1\alpha_2}\vec{t}_2 - \frac{1}{R_{11}}\vec{n}\ , \quad \frac{\vec{t}_{2,2}}{\alpha_2} = -\frac{\alpha_{2,1}}{\alpha_1\alpha_2}\vec{t}_1 - \frac{1}{R_{22}}\vec{n}\ , \tag{4.6}$$

$$\frac{\vec{t}_{1,2}}{\alpha_2} = \frac{\alpha_{2,1}}{\alpha_1\alpha_2}\vec{t}_2 - \frac{1}{R_{12}}\vec{n}\ , \quad \frac{\vec{t}_{2,1}}{\alpha_1} = \frac{\alpha_{1,2}}{\alpha_1\alpha_2}\vec{t}_1 - \frac{1}{R_{12}}\vec{n}\ , \tag{4.7}$$

$$\frac{\vec{n}_{,k}}{\alpha_k} = \frac{1}{R_{k1}}\vec{t}_1 + \frac{1}{R_{k2}}\vec{t}_2\ , \quad \frac{1}{R_{kj}} = \frac{1}{R_{jk}} = \frac{-\vec{n}\cdot\vec{r}_{,jk}}{\alpha_j\alpha_k}\ , \tag{4.8}$$

If $1/R_{12} = 0$, ξ_1 and ξ_2 are called the <u>lines of curvature coordinates</u> of the surface, the directions of $\vec{t}_i$ are called the <u>principal directions</u> and the corresponding $R_{11} \equiv R_1$ and $R_{22} \equiv R_2$ are the <u>principal radii of curvature</u>. If $\vec{r}$ is twice continuously differentiable, there is always a set of lines of curvature coordinates covering the entire surface (see texts on classical differential geometry).

5. Scalar Differential Equations

With the help of the differentiation formulas obtained in the last section, we get from the two vector differential equations of equilibrium the following six scalar equations:

$$\frac{(\alpha_2 N_{11})_{,1} + (\alpha_1 N_{21})_{,2} + \alpha_{1,2} N_{12} - \alpha_{2,1} N_{22}}{\alpha_1 \alpha_2} + \frac{Q_1}{R_{11}} + \frac{Q_2}{R_{12}} + p_1 = 0 ,$$

$$\frac{(\alpha_2 N_{12})_{,1} + (\alpha_1 N_{22})_{,2} + \alpha_{2,1} N_{21} - \alpha_{1,2} N_{11}}{\alpha_1 \alpha_2} + \frac{Q_1}{R_{12}} + \frac{Q_2}{R_{22}} + p_2 = 0 ,$$

$$\frac{(\alpha_2 Q_1)_{,1} + (\alpha_1 Q_2)_{,2}}{\alpha_1 \alpha_2} - \frac{N_{11}}{R_{11}} - \frac{N_{22}}{R_{22}} - \frac{N_{12} + N_{21}}{R_{12}} + p_n = 0 , \tag{4.9}$$

$$\frac{(\alpha_2 M_{11})_{,1} + (\alpha_1 M_{21})_{,2} + \alpha_{1,2} M_{12} - \alpha_{2,1} M_{22}}{\alpha_1 \alpha_2} - Q_1 + \frac{P_1}{R_{12}} + \frac{P_2}{R_{22}} + q_1 = 0 ,$$

$$\frac{(\alpha_2 M_{12})_{,1} + (\alpha_1 M_{22})_{,2} + \alpha_{2,1} M_{21} - \alpha_{1,2} M_{11}}{\alpha_1 \alpha_2} - Q_2 - \frac{P_1}{R_{11}} - \frac{P_2}{R_{12}} + q_2 = 0 ,$$

$$\frac{(\alpha_2 P_1)_{,1} + (\alpha_1 P_2)_{,2}}{\alpha_1 \alpha_2} + N_{12} - N_{21} + \frac{M_{12}}{R_{11}} - \frac{M_{21}}{R_{22}} + \frac{M_{22} - M_{11}}{R_{12}} + q_n = 0 .$$

For shells which cannot carry stress couples about the midsurface normal so that $P_1 = P_2 = 0$, the <u>sixth</u> (scalar) equilibrium equation reduces to an algebraic equation and the resulting system or shell equations will be of lower order. Associated with the reduction of the order of the system is a reduction of the number of boundary conditions at each edge of the shell from six to five. This poses no particular problem when stress boundary conditions are prescribed; we simply omit the condition on $P_\nu = \vec{M}_\nu \cdot \vec{n}$ since we must have $P_\nu^* = 0$

to be consistent with the assumption that the shell cannot carry moment stress couples. On the other hand, it is not clear at this point how we should handle the reduction of the number of boundary conditions if displacement conditions are prescribed at an edge of the shell.

From the four vector strain-displacement relations, we get the following twelve scalar equations:

$$\begin{aligned}
\epsilon_{11} &= \frac{u_{1,1}}{\alpha_1} + \frac{\alpha_{1,2}u_2}{\alpha_1\alpha_2} + \frac{w}{R_{11}} , & \epsilon_{22} &= \frac{u_{2,2}}{\alpha_2} + \frac{\alpha_{2,1}u_1}{\alpha_1\alpha_2} + \frac{w}{R_{22}} \\
\epsilon_{12} &= \frac{u_{2,1}}{\alpha_1} - \frac{\alpha_{1,2}u_1}{\alpha_1\alpha_2} + \frac{w}{R_{12}} - \omega , & \epsilon_{21} &= \frac{u_{1,2}}{\alpha_2} - \frac{\alpha_{2,1}u_2}{\alpha_1\alpha_2} + \frac{w}{R_{12}} + \omega , \\
\gamma_1 &= \phi_1 + \frac{w,_1}{\alpha_1} - \frac{u_1}{R_{11}} - \frac{u_2}{R_{12}} , & \gamma_2 &= \phi_2 + \frac{w,_2}{\alpha_2} - \frac{u_2}{R_{22}} - \frac{u_1}{R_{12}} , \\
\kappa_{11} &= \frac{\phi_{1,1}}{\alpha_1} + \frac{\alpha_{1,2}\phi_2}{\alpha_1\alpha_2} + \frac{\omega}{R_{12}} , & \kappa_{22} &= \frac{\phi_{2,2}}{\alpha_2} + \frac{\alpha_{2,1}\phi_1}{\alpha_1\alpha_2} - \frac{\omega}{R_{12}} , \\
\kappa_{12} &= \frac{\phi_{2,1}}{\alpha_1} - \frac{\alpha_{1,2}\phi_1}{\alpha_1\alpha_2} - \frac{\omega}{R_{11}} , & \kappa_{21} &= \frac{\phi_{1,2}}{\alpha_2} - \frac{\alpha_{2,1}\phi_2}{\alpha_1\alpha_2} + \frac{\omega}{R_{22}} \\
\lambda_1 &= \frac{\omega,_1}{\alpha_1} - \frac{\phi_1}{R_{12}} + \frac{\phi_2}{R_{11}} , & \lambda_2 &= \frac{\omega,_2}{\alpha_2} + \frac{\phi_2}{R_{12}} - \frac{\phi_1}{R_{22}} .
\end{aligned} \tag{4.10}$$

For a theory (e.g. the conventional shell theory) which does not allow transverse shearing strains so that $\gamma_1 \equiv \gamma_2 \equiv 0$ throughout the shell a priori, the above scalar strain displacement relations tell us that ϕ_1 and ϕ_2 are no longer independent unknowns since

$$\phi_1 = -\frac{w,_1}{\alpha_1} + \frac{u_1}{R_{11}} + \frac{u_2}{R_{12}} , \quad \phi_2 = -\frac{w,_2}{\alpha_2} + \frac{u_1}{R_{12}} + \frac{u_2}{R_{22}}$$

In this case, it appears that we can no longer satisfy all six independent scalar displacement conditions at an edge of the shell. Is there a reduction in the order of the system of differential equations even though no derivative was obliterated by $\gamma_1 \equiv \gamma_2 \equiv 0$? Unlike a theory with $P_1 \equiv P_2 \equiv 0$, the answer to this question is not obvious. If there is such a reduction of the order of the system, how do we reduce the number of stress boundary conditions in this case? We can not insist $Q_j \equiv 0$ through out the shell since $A_Q = 0$ in this case and in any event we need Q_j in general to maintain equilibrium. We will consider all these questions more carefully later after we have a better understanding of the structure of our theory.

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II - THE STRUCTURE OF THE LINEAR THEORY

5. The Stress Function Method

1. Exact Solution of Equilibrium Equations

The two vector differential equations of equilibrium are four vector unknowns $\vec{N}_j$ and $\vec{M}_j$, $j = 1,2$. Evidently, we can solve for two of them in terms of the other two. Equivalently, we can express all four of them in terms of two arbitrary vector (stress) functions, $\vec{K}$ and $\vec{H}$, in such a way that the equilibrium equations are identically satisfied. An exact solution in the case $\vec{p} = \vec{q} = \vec{0}$ is easily seen to be

$$\alpha_2\vec{N}_1 = \vec{K},_2 \qquad \alpha_1\vec{N}_2 = -\vec{K},_1 \tag{5.1a,b}$$

$$\alpha_2\vec{M}_1 = \vec{H},_2 + \vec{r},_2 \times \vec{K} , \quad \alpha_1\vec{M}_2 = -\vec{H},_1 - \vec{r},_1 \times \vec{K} \tag{5.1c,d}$$

For nonvanishing $\vec{p}$ and $\vec{q}$, which are prescribed functions of position and do not involve the unknowns we need only to add to the above <u>stress function representation</u> of the stress resultants and couples certain particular integrals of $\vec{p}$ and $\vec{q}$. Since an appropriate choice of these integrals often depends on the specific problem under consideration, we will restrict our discussion here only to the case $\vec{p} = \vec{q} = \vec{0}$.

The vector stress functions $\vec{K}$ and $\vec{H}$ are related to the resultant force and moment. Recall that the resultant force $\vec{F}$ is given by

$$\vec{F} = \oint_C \vec{N}_\nu ds + \iint_S \vec{p}\, dS = \oint_C (\nu_1\vec{N}_1 + \nu_2\vec{N}_2) ds + \iint_S \vec{p}\, dS$$

where the line integral is to be interpreted as $\sum_i \oint_{C_i}$ if the shell has more than one edge. Now, for $\vec{p} = \vec{q} = \vec{0}$, we have

$$\nu_1\vec{N}_1 + \nu_2\vec{N}_2 = \nu_1\vec{K},_2/\alpha_2 - \nu_2\vec{K},_1/\alpha_1 = \vec{K},_s/\alpha_s$$

where $(\ ),_s = \partial(\ )/\partial\xi_s$ and $\alpha_s d\xi_s = ds$. Therefore,

$$\vec{F} = \oint_C \vec{K},_s \, d\xi_s = [\vec{K}]_C \tag{5.2}$$

For a shell in static equilibrium, we have $\vec{F} = 0$. It follows that $\vec{K}$ must be a single-valued function of ξ_s is the shell is simply connected. If the shell is not simply connected, $\vec{K}$ needs not be single-valued as long as $\sum_i [\vec{K}]_{C_i} = 0$. (Keep in mind that $\vec{p} = \vec{0}$ in this discussion). Similarly, when $\vec{q} = 0$, we have

$$\vec{M} = \oint_C (\vec{M}_\nu + \vec{r} \times \vec{N}\,)\,ds = [\vec{H} + \vec{r} \times \vec{K}]_C \tag{5.3}$$

With the component representation

$$\vec{H} = H_1\vec{t}_1 + H_2\vec{t}_2 + F\vec{n}\ , \qquad \vec{K} = \vec{n} \times (K_1\vec{t}_1 + K_2\vec{t}_2) + J\vec{n} \tag{5.4}$$

the vector stress function representation is equivalent to twelve scalar equations. We will list only two of these scalar equations here, namely,

$$P_1 = K_2 + \frac{F,_2}{\alpha_2} - \frac{H_1}{R_{12}} - \frac{H_2}{R_{22}}\ , \qquad P_2 = -K_1 - \frac{F,_1}{\alpha_1} + \frac{H_1}{R_{11}} + \frac{H_2}{R_{12}}, \tag{5.5}$$

If $P_1 = P_2 = 0$, the above can be used to express K_1 and K_2 in terms of the other four scalar stress functions.

2. Compatibility Equations

Recall that the four vector strain measures are defined in terms of two displacement vectors $\vec{\phi}$ and $\vec{u}$ by

$$\alpha_j \vec{\kappa}_j = \vec{\phi}_{,j} \qquad \alpha_j \vec{\varepsilon}_j = \vec{u}_{,j} + \vec{r}_{,j} \times \vec{\phi} \qquad (j=1,2)$$

So, they cannot be completely independent of each other in the sense that there are two vector differential relations among them in the form

$$(\alpha_2 \vec{\kappa}_2)_{,1} - (\alpha_1 \vec{\kappa}_1)_{,2} = 0 , \tag{5.6a}$$

$$(\alpha_2 \vec{\varepsilon}_2)_{,1} - (\alpha_1 \vec{\varepsilon}_1)_{,2} + \vec{r}_{,1} \times (\alpha_2 \vec{\kappa}_2) - \vec{r}_{,2} \times (\alpha_1 \vec{\kappa}_1) = 0 \tag{5.6b}$$

They are called the compatibility equations.

Corresponding to these two vector compatibility equations are six scalar compatibility equations for the components of the strain resultant and couple vectors. We will list only one of these six equations here, namely,

$$\frac{(\alpha_2\gamma_2)_{,1} - (\alpha_1\gamma_1)_{,2}}{\alpha_1\alpha_2} + \kappa_{21} - \kappa_{12} - \frac{\varepsilon_{21}}{R_{11}} + \frac{\varepsilon_{12}}{R_{22}} + \frac{\varepsilon_{11} - \varepsilon_{22}}{R_{12}} = 0 . \tag{5.7}$$

If $\gamma_1 = \gamma_2 = 0$ throughout the shell, this sixth compatibility equation becomes an algebraic equation, very similar to what happens to the sixth equilibrium equation when $P_1 = P_2 = 0$. Thus, the restriction of vanishing transverse shear strains does reduce the order of the system of shell equations in some sense (keeping in mind that the compatibility equations never get into the picture if we employ the displacement method of solution).

3. Differential Equations for Stress Functions

Just as in three dimensional elasticity theory, there are several alternatives to the displacement method of solution for shell problems. In section (1), we satisfied all equilibrium equations exactly with two arbitrary vector stress functions $\vec{K}$ and $\vec{H}$. Suppose that we can write the stress-strain relations in the form of scalar equations expressing each individual strain measure in terms of one or more stress measures, e.g.,

$$\vec{\varepsilon}_j = \frac{\partial C}{\partial \vec{N}_j} \quad , \quad \vec{\kappa}_j = \frac{\partial C}{\partial \vec{M}_j} \quad , \qquad (k=1,2) \ .$$

We can certainly do this for the linear isotropic uncoupled relations mentioned in the last chapter as long as $t_M \neq -1/2$ and $D_p \neq 0$.(1) Then these stress-strain relations for each strain measure can be written as strain-stress function relations by way of the stress function representation.

Upon substituting these strain-stress functions relations into the two vector compatibility equations, we will get two differential equations for the stress functions $\vec{K}$ and $\vec{H}$. Together with appropriately prescribed boundary conditions, they determine $\vec{K}$ and $\vec{H}$ completely (or up to some arbitrary functions which give rise to no stress resultants and couples). Having $\vec{K}$ and $\vec{H}$, we can then calculate the stress resultants and couples by the stress-stress function relations and the strain resultants and couples by the stress-strain relations. The displacements can also be calculated from the strain measures afterwards.

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(1) We can still do something similar to this even if $t_M = -1/2$ and $D_p = 0$. A discussion along this line will be postponed until a later section.

4. Boundary Conditions for the Stress Function Method

A few words about the boundary conditions for this so-called stress function method for shell problems are appropriate at this point. If the prescribed edge conditions are stress boundary conditions: $\vec{N}_\nu = \vec{N}^*_\nu$ and $\vec{M}_\nu = \vec{M}^*_\nu$, they can be expressed in terms of the stress functions since we have

$$\vec{N}_\nu = \vec{K},_s/\alpha_s \quad \text{and} \quad \vec{M}_\nu = \vec{H},_s/\alpha_s + \vec{r},_s \times \vec{K}/\alpha_s$$

If the edge conditions are prescribed in terms of the displacements, the situation is a little more complicated. In principle, what we have to do is obtain the most general solution for $\vec{K}$ and $\vec{H}$ of the two vector compatibility equations and to calculate the corresponding strain resultants and couples in a manner described earlier. We then treat the strain displacement relations as differential equations for $\vec{u}$ and $\vec{\phi}$ with $\vec{\varepsilon}_j$ and $\vec{\kappa}_j$ known, and obtain the most general solution of these equations. Finally, the displacement fields will be specialized so that the displacement edge conditions are satisfied. In the process, we have also specialized $\vec{K}$ and $\vec{H}$ to the point that whatever arbitrariness left in them should give rise to no stress resultants and couples.

So, it seems that the stress function method is not particularly efficient for a displacement boundary value problem. What is more serious however, is the rather dubious step of solving the strain-displacement relations for $\vec{u}$ and $\vec{\phi}$. The point is that, with $\vec{\kappa}_j$ and $\vec{\varepsilon}_j$ prescribed, the four vector strain displacement relations in some sense overdetermine $\vec{u}$ and $\vec{\phi}$. To be sure, the strain measures are restricted by the two vector compatibility equations in the sense that the most general solution for the strain measures calculated from the most general solution for the stress functions obtained by the stress function method automatically satisfy these compatibility equations. Nevertheless, the integrability of the strain-displacement relations, subject to the compatibility equations, still needs to be investigated. We do not pursue a discussion along this line here as the analysis is very similar to what is done in the three dimensional theory of elasticity and in the linear plate theory (cf. Part III of these lecture notes).

6. Static - Geometric Duality

1. Duality of Differential Equations

At this point, one can not help but sense a certain symmetry or duality between the displacement function and the stress function method of solution for shell problems. Let us summarize once more the relevant differential equations for the two approaches for the case $\vec{p} = \vec{q} = \vec{0}$ as follows:

Displacement Method

Strain Displacement:

$$\alpha_2\vec{\kappa}_2 = \vec{\phi}_{,2} \qquad \alpha_1\vec{\kappa}_1 = \vec{\phi}_{,1}$$

$$\alpha_2\vec{\varepsilon}_2 = \vec{u}_{,2} + \vec{r}_{,2} \times \vec{\phi}$$

$$\alpha_1\vec{\varepsilon}_1 = \vec{u}_{,1} + \vec{r}_{,1} \times \vec{\phi}$$

Stress-Strain:

$$\vec{N}_j = \frac{\partial S}{\partial \vec{\varepsilon}_j} \qquad \vec{M}_j = \frac{\partial S}{\partial \vec{\kappa}_j}$$

Equilibrium:

$$(\alpha_2\vec{N}_1)_{,1} + (\alpha_1\vec{N}_2)_{,2} = 0$$

$$(\alpha_2\vec{M}_1)_{,1} + (\alpha_1\vec{M}_2)_{,2} + \vec{r}_{,1}\times(\alpha_2\vec{N}_1) + \vec{r}_{,2}\times(\alpha_1\vec{N}_2) = 0$$

Stress Function Method

Stress-Function Representation:

$$\alpha_2\vec{N}_1 = \vec{K}_{,2} \qquad -\alpha_1\vec{N}_2 = \vec{K}_{,1}$$

$$\alpha_2\vec{M}_1 = \vec{H}_{,2} + \vec{r}_{,2} \times \vec{K}$$

$$\alpha_1\vec{M}_2 = \vec{H}_{,1} + \vec{r}_{,1} \times \vec{K}$$

Stress-Strain:

$$\vec{\kappa}_j = \frac{\partial C}{\partial \vec{M}_j} \qquad \vec{\varepsilon}_j = \frac{\partial C}{\partial \vec{N}_j}$$

Compatibility

$$(\alpha_2\vec{\kappa}_2)_{,1} - (\alpha_1\vec{\kappa}_1)_{,2} = 0$$

$$(\alpha_2\vec{\varepsilon}_2)_{,1} - (\alpha_1\vec{\varepsilon}_1)_{,2} + \vec{r}_{,1}\times(\alpha_2\vec{\kappa}_2) - \vec{r}_{,2}\times(\alpha_1\vec{\kappa}_1) = 0$$

Evidently, if we replace all strain measures in the compatibility equations by the dual stress measures according to Table (5.1) we will get the homogeneous equilibrium equations and vice versa. The stress-stress function relations are also the duals of the strain-displacement relations if we observe the additional dual relations $(\vec{K},\vec{H}) \longleftrightarrow (\vec{\phi},\vec{u})$.

Table (5.1)

$\vec{N}_1$	$\vec{N}_2$	$\vec{M}_1$	$\vec{M}_2$	$\vec{K}$	$\vec{H}$
$\vec{\kappa}_2$	$-\vec{\kappa}_1$	$\vec{\varepsilon}_2$	$-\vec{\varepsilon}_1$	$\vec{\phi}$	$\vec{u}$

Since we will have to work with scalar equations eventually, the duality among the scalar quantities are also listed in Table (5.2):

Table (5.2)

N_{11}	N_{22}	N_{12}	N_{21}	Q_1	Q_2	K_1	K_2	J
$-\kappa_{22}$	$-\kappa_{11}$	κ_{21}	κ_{12}	λ_2	$-\lambda_1$	ϕ_1	ϕ_2	ω
M_{11}	M_{22}	M_{12}	M_{21}	P_1	P_2	H_1	H_2	F
ε_{22}	ε_{11}	$-\varepsilon_{21}$	$-\varepsilon_{12}$	γ_2	$-\gamma_1$	u_1	u_2	w

Having these dualities, it is only natural to ask whether there is also a duality between the stress strain relations and the corresponding inverted relations. For this additional duality, we will need a duality between the stress potential and the strain potential in the form $S \longleftrightarrow -C$. It is rather remarkable that there is in fact such a duality if we observe the dual relations among the elastic moduli in Table (5.3):

Table (5.3)

A	A_Q	ν_N	t_N
$-D$	$-D_p$	$-\nu_M$	$-t_M$

This complete duality among the differential equations of shell theory is known as the Static-Geometric Duality (or Analogy). The original version of this duality was first noted by A.L. Goldenveiser of the Soviet Union in his doctoral dissertation. The present version of the same duality, though different substantially from the earlier version(s) both in form and in content, is nothing more than a trivial generalization of Goldenveisser's original duality. The importance of the particular form of our duality lies in its simplicity and completeness which allow us to extend it further, making it more useful both for theoretical development and for the solution of specific problems.

2. Duality of Boundary Conditions

The duality between the governing differential equations for the displacement method on the one hand and those for the stress function method on the other hand makes our shell theory a rather elegant subject, allows us to pass easily from one formulation to the other in our effort to understand various aspects of the theory and reduces the amount of labor involved in our manipulations of the differential equations. If we are satisfied with this more or less theoretical nicety, we would have done this far reaching concept of static-geometric duality a grave injustice.

One may ask what more can we expect of this duality. A rather natural place to look for more of the same would be in the boundary conditions. But here we immediately encounter obstacles. Even if we leave aside the variety of possible sets of physically realizable conditions at any given edge and confine ourselves only to the two fundamental problems, we still have to face the obstacle that the stress boundary conditions are not the duals of the displacement conditions! To overcome this obstacle, we may try to reformulate the stress boundary conditions in terms of the stress functions $\vec{K}$ and $\vec{H}$ which are the duals of $\vec{\phi}$ and $\vec{u}$ respectively. This is in fact possible in view of the development in section (1) of this chapter (see also P.M. Naghdi's article given at the end of my Introduction).

The alternative is to try to reformulate the displacement edge conditions completely in terms of the strain measures which are the duals of $\vec{N}_\nu$ and $\vec{M}_\nu$ respectively. This possibility was left unexplored until relatively recently[2] for one very simple reason. The form of the strain-displacement relations for the classical shell theory made it seemingly not feasible to do so. Just why this is so will be discussed later. We merely point out here that a reformulation of displacement edge conditions in terms of strain measures is almost trivial within the framework of our Gunther-Reissner type theory. Since $\vec{\nu}$, $\vec{t}$ and $\vec{n}$ also form a right hand orthogonal triad of unit vectors, we have

$$\alpha_s \vec{\kappa}_s = \vec{\phi},_s \qquad \text{and} \qquad \alpha_s \vec{\varepsilon}_s = \vec{u},_s + \vec{r},_s \times \vec{\phi}$$

If $\vec{u} = \vec{u}^*(\xi_s)$ and $\vec{\phi} = \vec{\phi}^*(\xi_s)$ at an edge of the shell, then the corresponding prescribed $\vec{\kappa}_s^*$ and $\vec{\varepsilon}_s^*$ can be obtained by differentiating $\vec{\phi}^*$ and $\vec{u}^*$ with respect to ξ_s. The important point is that differentiation in the direction of $\vec{\nu}$ (normal to the edge curve) of the components of $\vec{\phi}^*$ and $\vec{u}^*$ is not involved; this is not at all apparent if we insist upon working with the conventional shell theory. Of course, the fact that shell analysts (as well as elasticians) are so ingrained by the idea that geometrical constraints should be prescribed in terms of displacements also did not encourage an exploration of this reformulation.

Needless to say, if we now replace the displacement boundary conditions $\vec{u} = \vec{u}^*$ and $\vec{\phi} = \vec{\phi}^*$ by $\vec{\kappa}_s = \vec{\kappa}_s^*$ and $\vec{\varepsilon}_s = \vec{\varepsilon}_s^*$, then the reformulated displacement conditions (or strain conditions) become the duals of the stress boundary conditions $\vec{N}_\nu = \vec{N}_\nu^*$ and $\vec{M}_\nu = \vec{M}_\nu^*$. With this duality, we not only enlarge the scope of the very elegant

(2) F.Y.M. Wan, "Two variational theorems for thin shells", J. Math. & Phys. 47, 1968, 429-431.

concept of static-geometric duality, but actually provide a missing bridge between the two fundamental problems of shell theory (see next section) as well as make it computationally useful (as we shall see later).

3. Duality of the Two Fundamental Problems

Unlike the displacement boundary conditions, the corresponding (but not necessarily equivalent) strain boundary conditions are particularly appropriate for the stress function method of solution since we no longer have to solve the strain-displacement relations before we specialize the general solution of the compatibility equations. Integration of the strain displacement relations is necessary only if we want to determine the displacement fields specifically and can be done as an independent calculation after we have determined all measures of stress and strain.

Moreover, using these strain edge conditions, the whole first fundamental problem in the stress function formulation now becomes the static-geometric dual of the second fundamental problem in the displacement function formulation. That is, the differential equations and boundary conditions for one problem are the static-geometric duals of the corresponding differential equations and boundary conditions of the other. Hence, if we have obtained the solution of one of these problems, we have also in effect obtained the solution of the other as we can get the latter simply by a formal change of symbols (according to the rules of the static-geometric duality) in the solution of the first problem! In short, once we have understood the (linear) elastostatics of shells for one fundamental problem, we have also understood the same for the other. So, at least in the linear range, we really have only one fundamental problem in shell theory instead of two as in three dimensional elasticity theory!

4. Formulation With Intrinsic Duality

Having seen a complete duality between two different formulations of the linear elasto-static problem of shell theory (namely, the stress function method and the displacement function method), we may ask whether there are formulations which have an intrinsic static-geometric duality, i.e., a formation whose system of governing differential equations is its own dual. Such a formulation should have half of the governing differential equations being the static-geometric dual of the other half. A number of such formulations will be discussed in this section. We will see the advantage of an intrinsically dual formulation of shell problems later when we deal with the solution of specific problems.

(A) Equilibrium-Compatibility Formulation

If we want the differential equations of equilibrium as a part of our formulation, then the static-geometric duality dictates that the compatibility equations must also be a part if the formulation is to have an intrinsic duality. Together, they are four vector equations for eight unknown vector fields. So we need four more vector equations to complete the set, assuming of course we do not introduce any more unknowns. Evidently, these equations must come from the stress-strain relations. But neither the set (3.4) nor the set (3.6) would do since they do not have an intrinsic duality. On the other hand, we have already seen at least one set of stress-strain relations with an intrinsic duality, namely, (3.8) and (3.9). More generally, we may have stress-strain relations in terms of a mixed potential

$$P = P(N_{11},N_{12},N_{21},N_{22},Q_1,Q_2;\kappa_{11},\kappa_{12},\kappa_{21},\kappa_{22},\lambda_1,\lambda_2) \tag{6.2a}$$

with

$$\vec{\varepsilon}_j = \frac{\partial P}{\partial \vec{N}_j} \quad , \quad M_j = \frac{\partial P}{\partial \vec{\kappa}_j} \quad , \tag{6.2b}$$

as long as

$$P(-\kappa_{22},\kappa_{21},\kappa_{12},-\kappa_{11},\lambda_2,-\lambda_1;-N_{22},N_{21},N_{12},-N_{11},-Q_2,Q_1)$$

$$= P(N_{11},\ldots,Q_2;\kappa_{11},\ldots,\lambda_2) \quad .$$

It is not difficult to see that, for (3.8) and (3.9), we have[3]

$$P = \frac{1}{2}A[(N_{11}+N_{22})^2 - 2(1+\nu_N)(N_{11}N_{22}-N_{12}N_{21}) + (1+t_N)(1+\nu_N)(N_{12}-N_{21})^2]$$

$$+ \frac{1}{2}D[(\kappa_{11}+\kappa_{22})^2 - 2(1-\nu_M)(\kappa_{11}\kappa_{22}-\kappa_{12}\kappa_{21}) + (1-t_M)(1-\nu_M)(\kappa_{12}-\kappa_{21})^2]$$

$$+ \frac{1}{2}D_P(\lambda_1^2+\lambda_2^2) + \frac{1}{2}A_Q(Q_1^2+Q_2^2) \quad . \tag{6.3}$$

---

(3) P is positive definite if A_Q , D_p , D and A are positive, $|\nu_M| \le 1$, $|\nu_N| \le 1$, $t_M \le 1/2$, $t_N \le 1/2$.

In this mixed form, the possibility of t_N and t_M assuming the value $\frac{1}{2}$ is no longer any cause for concern.

The present equilibrium-compatibility formulation may be regarded as a system of twelve scalar first order equations for the six scalar stress resultants and six scalar strain couples. Once the solution is obtained, the stress couples and the strain resultants are then obtained from the (algebraic) stress-strain relations. The displacement fields are determined by solving the strain-displacement relations (as a system of first order partial differential equations) only after we have obtained all the resultants and couples and only if we need them.

(B) Stress-Stress Function and Strain-Displacement Formulation

Another possible formulation of shell problems with an intrinsic static-geometric duality consists of the stress function representation and strain-displacement relations as a part of the governing differential equations. Together, they are eight vector equations for twelve unknown vector fields. Since equilibrium and compatibility are automatically satisfied, the remaining four equations are again to come from the stress-strain relations. For an intrinsic duality, these four relations will have to be taken in mixed form such as (3.13) with P given by (3.14). This formulation is essentially a system of twelve scalar first order equations for the six scalar stress functions and six scalar displacement components.

Once the solution is obtained for a specific problem, all resultants and couples are obtained by simple differentiation. Unlike the equilibrium-compatibility formulation in the last section, there is no additional partial differential equations to be solved.

(C) A Mixed Formulation

There is another formulation of boundary value problems in shell theory with an intrinsic duality which has formed the basis of the theory and applications of several special classes of problems (including axisymmetric deformations of shells of revolution, shallow shell theory, etc.)[4]. This formulation consists of the stress function representations (2.1a,b), their dual strain-displacement relations

$$\alpha_i \vec{\kappa}_i = \vec{\phi}_{,i} \quad , \qquad (i = 1,2) \quad , \tag{6.4}$$

the vector moment equilibrium equations (with $\vec{q} = \vec{0}$ for simplicity)

$$(\alpha_2 \vec{M}_1)_{,1} + (\alpha_1 \vec{M}_2)_{,2} + \vec{r}_{,1} \times (\alpha_2 \vec{N}_1) + \vec{r}_{,2} \times (\alpha_1 \vec{N}_2) = \vec{0} \quad , \tag{6.5}$$

its dual vector compatibility equation, and the stress-strain relations (6.2b). It is clear that this system of equations has an intrinsic duality and it, along with appropriate boundary conditions, completely determines $\vec{\phi}$, $\vec{K}$ and all stress and strain measures.

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[4] These special theories were developed without any conscious effort to include a built-in static-geometric duality in the formulation.

If we wish, the displacement vector $\vec{u}$ (and its dual stress function $\vec{H}$) may then be calculated from

$$\alpha_j \vec{\varepsilon}_j = \vec{u}_{,j} + \vec{r}_{,j} \times \vec{\phi} \qquad (j = 1,2) \qquad (6.6)$$

(and their dual stress-stress function relations).

In working with this mixed formulation, it is customary to use (5.16), (6.4) and (6.26) to transform (6.5) and its dual into two vector equations for $\vec{\phi}$ and $\vec{K}$. The corresponding six scalar differential equations are all second order equations. These equations further simplify or uncouple for the special cases of shallow shells, axisymmetric stress and strain in shells of revolution, etc.

7. Variational Principles

1. A Variational Principle for the Displacement Function Formulation

In three dimensional linear elasticity theory, a number of minimum and stationary variational principles follow from the virtual work theorem. We expect that similar results are also available for our linear shell theory as it has the virtual work principle as one of its postulates. Detailed derivations of the several variational principles for shells of interest would not be given here; they are similar to those for the corresponding principles in three dimensional elasticity. In this section, we simply state a variational principle which summarizes the displacement method for linear shell theory. This and other variational principles discussed in this chapter are of course useful for obtaining approximate solutions of boundary value problems; they are also useful for deducing the appropriate (contracted or reduced) boundary conditions for shells with special material properties such as transverse rigidity, etc.

Let

$$I_D = \iint \left\{ \left(\vec{\kappa}_i - \frac{\vec{\phi}_{,i}}{\alpha_i}\right)\cdot\vec{M}_i + \left(\vec{\varepsilon}_i - \frac{\vec{u}_{,i} + \vec{r}_{,i}\times\vec{\phi}}{\alpha_i}\right)\cdot\vec{N}_i - S \right\} \alpha_1\alpha_2 d\xi_1 d\xi_2 \tag{7.1}$$

$$+ \int_{\Gamma_\sigma} (\vec{N}^*_\nu\cdot\vec{u} + \vec{M}^*_\nu\cdot\vec{\phi})ds + \int_{\Gamma_d} (\vec{N}_\nu\cdot\Delta\vec{u} + \vec{M}_\nu\cdot\Delta\vec{\phi})ds \quad ,$$

where $S = S(\vec{\kappa}_j, \vec{\varepsilon}_j)$ is the stress potential or (surface) strain energy density function for the shell. With $\delta\vec{\kappa}_j$, $\delta\vec{\varepsilon}_j$, $\delta\vec{M}_j$, $\delta\vec{N}_j$, $\delta\vec{\phi}$ and $\delta\vec{u}$ varying independently, the variational equation $\delta I_D = 0$ has as its Euler differential equations the strain-displacement relations, the stress-strain relations (3.4) and the homogeneous differential equations of equilibrium; it also has as its Euler boundary conditions the stress boundary conditions

$$\Delta\vec{N}_\nu = \Delta\vec{M}_\nu = \vec{0} \qquad (7.2)$$

on Γ_σ and the displacement boundary conditions

$$\Delta\vec{u} = \Delta\vec{\phi} = \vec{0}$$

on Γ_d where $\Delta\vec{f} \equiv \vec{f} - \vec{f}^*$ with f^* being a prescribed function[(5)].

External load terms $\vec{p}$ and $\vec{q}$ can be included by way of a load potential $-L(\vec{u}, \vec{\phi})$ term in the surface integral with

$$\vec{p} = -\frac{\partial L}{\partial \vec{u}} \quad , \quad \vec{q} = -\frac{\partial L}{\partial \vec{\phi}} \quad . \qquad (7.3)$$

At the same time, the boundary integral can be made more general by replacing all line integrals by a single line integral over the entire edge of $\ell(\vec{N}_\nu, \vec{M}_\nu, \vec{u}, \vec{\phi})$. For the special case of stress boundary

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(5) See E. Reissner, "Variational considerations for elastic beams and shells", Proc. ASCE, J. Eng. Mech. 88, 1962, 23-57, and "Variational methods and boundary conditions in shell theory", Studies in Optimization, Vol. 1, S.I.A.M. Publications, Philadelphia, 1970, 78-94.

conditions on Γ_σ portion of Γ and displacement boundary conditions on the remaining portion Γ_d of Γ, we have

$$\ell = \begin{cases} \vec{N}^*_\nu \cdot \vec{u} + \vec{M}^*_\nu \cdot \vec{\phi} & (\text{on } \Gamma_\sigma) \\ \\ \vec{N}_\nu \cdot \Delta\vec{u} + \vec{M}_\nu \cdot \Delta\phi & (\text{on } \Gamma_d) \end{cases} .$$

2. A Variational Principle for the Stress Function Formulation

By the static-geometric duality, we have immediately a dual of the variational principle given in the last section. Denote the dual of $-I_D$ by I_S . We have then the following expression for I_S :

$$I_S = \iint \{ (\vec{N}_1 - \frac{\vec{K}_{,2}}{\alpha_2}) \cdot \vec{\varepsilon}_1 + (\vec{N}_2 + \frac{\vec{K}_{,1}}{\alpha_1}) \cdot \vec{\varepsilon}_2$$

$$+ (\vec{M}_1 - \frac{\vec{H}_{,2} + \vec{r}_{,2} \times \vec{K}}{\alpha_2}) \cdot \vec{\kappa}_1 + (\vec{M}_2 + \frac{\vec{H}_{,1} + \vec{r}_{,1} \times \vec{K}}{\alpha_1}) \cdot \vec{\kappa}_2$$

$$- C(\vec{N}_i, \vec{M}_i) \} \alpha_1 \alpha_2 d\xi_1 d\xi_2 \qquad (7.4)$$

$$- \int_{\Gamma_d} (\vec{\kappa}_s^* \cdot \vec{H} + \vec{\varepsilon}_s^* \cdot \vec{K}) ds - \int_{\Gamma_\sigma} (\vec{\kappa}_s \cdot \Delta\vec{H} + \vec{\varepsilon}_s \cdot \Delta\vec{K}) ds$$

where $C(\vec{N}_i, \vec{M}_i)$ is the strain potential or complementary (surface) strain energy density function which is the dual of $-S(\vec{\kappa}_i, \vec{\varepsilon}_i)$. With $\delta\vec{\kappa}_j$, $\delta\vec{\varepsilon}_j$, $\delta\vec{N}_i$, $\delta\vec{M}_i$, $\delta\vec{K}$ and $\delta\vec{H}$ varying independently, the variational equation $\delta I_S = 0$ has as its Euler differential equations, the stress function representations, the stress-strain relations (3.7) and the compatibility equations; it has as its Euler boundary conditions the displacement boundary conditions expressed in terms of strain measure,

$$\Delta\vec{\kappa}_s = \Delta\vec{\varepsilon}_s = \vec{0} \tag{7.5a}$$

on Γ_d, and the stress boundary conditions in terms of the stress functions $\vec{K}$ and $\vec{H}$ on Γ_σ :

$$\Delta\vec{K} = \Delta\vec{H} = \vec{0} \quad . \tag{7.5b}$$

The physical meaning of the conditions on the stress functions is clear from

$$\int_{\Gamma_\sigma} \vec{N}_\nu ds = \int_{s_1}^{s_2} \vec{K}_{,s} ds = [\vec{K}]_{s_1}^{s_2}$$

$$\int_{\Gamma_\sigma} (\vec{M}_\nu + \vec{r} \times \vec{N}_\nu) ds = \int_{s_1}^{s_2} (\vec{H}_{,s} + \vec{r}_{,s} \times \vec{K} + \vec{r} \times \vec{K}_{,s}) ds$$

$$= [\vec{H} + \vec{r} \times \vec{K}]_{s_1}^{s_2} \quad .$$

3. Mixed Variational Principles for $\vec{\phi}$ and $\vec{K}$

In view of the frequent uses of the mixed formulation of shell theory in terms of $\vec{\phi}$ and $\vec{K}$ described in section (4) of chapter (6), it is of interest to give here a variational principle summarizing this mixed method of solution[6]. In this formulation, the strain-displacement relations

$$\alpha_i \vec{\kappa}_i = \vec{\phi}_{,i} \qquad (i = 1,2) \tag{7.6a}$$

and the stress function representation

$$\alpha_2 \vec{N}_1 = \vec{K}_{,2} \quad , \quad \alpha_1 \vec{N}_2 = -\vec{K}_{,1} \tag{7.6b}$$

are taken as equations of definition. For stretching-bending uncoupled stress strain relations, we may write the mixed potential $P(\vec{N}_j, \vec{\kappa}_j)$ as

$$P(\vec{N}_j, \vec{\kappa}_j) = C_N(\vec{N}_j) - S_\kappa(\vec{\kappa}_j) = \vec{N}_j \cdot \vec{\varepsilon}_j - S(\vec{\varepsilon}_j, \vec{\kappa}) \tag{7.7}$$

where $S(\vec{\varepsilon}_j, \vec{\kappa}_j)$ is the stress potential (or strain energy density function) of the shell. With (7.6) and (7.7), we may transform the expression for the potential energy of the shell, for the case of prescribed displacement $\vec{u}^*$ (with $\Delta\vec{u} \equiv \vec{u} - \vec{u}^*$) and stress

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(6) To the knowledge of this writer, the variational principles discussed in this section have not been given elsewhere in the literature except in his own lecture notes for a graduate course in Elasticity at MIT in 1969/70.

couple $\vec{M}^*_\nu$ along the edge curve Γ, into

$$I_{PE} = \iint \{ [S_\kappa - C_N]\alpha_1\alpha_2 + \vec{\phi}\cdot(\vec{r}_{,2} \times \vec{K}_{,1} - \vec{r}_{,1} \times \vec{K}_{,2})\} d\xi_1 d\xi_2$$

$$- \oint_\Gamma (\vec{M}^*_\nu \cdot \vec{\phi} + \vec{e}^*_s \cdot \vec{K}) ds \tag{7.8b}$$

where

$$\vec{e}^*_s \equiv \vec{u}^*_{,s} \quad . \tag{7.8c}$$

With $\delta\vec{\phi}$ and $\delta\vec{K}$ varying independently, the variational equation $\delta I_{PE} = 0$ has as its Euler differential equations, the (homogeneous) moment equilibrium equation in the form

$$(\alpha_2\vec{M}_1)_{,1} + (\alpha_1\vec{M}_2)_{,2} + \vec{r}_{,1} \times \vec{K}_{,2} - \vec{r}_{,2} \times \vec{K}_{,1} = 0 \tag{7.9a}$$

and the compatibility equation in the form

$$(\alpha_2\vec{\varepsilon}_2)_{,1} - (\alpha_1\vec{\varepsilon}_1)_{,2} + \vec{r}_{,1} \times \vec{\phi}_{,2} - \vec{r}_{,2} \times \vec{\phi}_{,1} = 0 \tag{7.9b}$$

where

$$\vec{\varepsilon}_1 = \frac{\partial P}{\partial \vec{N}_1} = \frac{\partial C_N}{\partial \vec{N}_1} = \alpha_2 \frac{\partial C_N}{\partial \vec{K}_{,2}} \; , \quad \vec{\varepsilon}_2 = \frac{\partial P}{\partial \vec{N}_2} = -\alpha_1 \frac{\partial C_N}{\partial \vec{K}_{,1}} \tag{7.10a}$$

$$\vec{M}_1 = \frac{\partial P}{\partial \vec{\kappa}_1} = \frac{\partial S_\kappa}{\partial \vec{\kappa}_1} = \alpha_1 \frac{\partial S_\kappa}{\partial \vec{\phi}_{,1}} \; , \quad \vec{M}_2 = \dots = \alpha_2 \frac{\partial P}{\partial \vec{\phi}_{,2}} \quad . \tag{7.10b}$$

The associated Euler boundary conditions are

$$\Delta\vec{M}_{\nu} = \Delta\vec{e}_s = \vec{0} \quad , \tag{7.11a}$$

where

$$\Delta\vec{e}_s \equiv \vec{e}_s - \vec{e}_s^{\,*} \quad , \quad \vec{e}_s \equiv \vec{\varepsilon}_s - \vec{r}_{,s} \times \vec{\phi} \quad . \tag{7.11b}$$

Other types of Euler boundary conditions can be obtained by modifying I_{PE} suitably with the help of Lagrange multipliers. For example, if the displacement boundary condition $\Delta\vec{u} = \vec{0}$ is replaced by prescribed edge stress resultant vector (which, in this mixed formulation, should be taken in the form of $\Delta\vec{K} = \vec{0}$), then, the last term in the boundary integral, $\vec{e}_s^{\,*} \cdot \vec{K}$, should be replaced by $\vec{e}_s \cdot \Delta\vec{K}$.

As it stands, the expression for I_{PE} in (7.8b) is not symmetric in $\vec{K}$ and $\vec{\phi}$. The surface integral can be made more symmetric by using the identity

$$\begin{aligned}\vec{\phi}\cdot(\vec{r}_{,2} \times \vec{K}_{,1} - \vec{r}_{,1} \times \vec{K}_{,2}) = {} & \tfrac{1}{2}\vec{\phi}\cdot(\vec{r}_{,2} \times \vec{K}_{,1} - \vec{r}_{,1} \times \vec{K}_{,2}) \\ & + \tfrac{1}{2}\{[\vec{\phi}\cdot(\vec{r}_{,2} \times \vec{K})]_{,1} - [\vec{\phi}\cdot(\vec{r}_{,1} \times \vec{K})]_{,2}\} \\ & + \tfrac{1}{2}\vec{K}\cdot(\vec{r}_{,2} \times \vec{\phi}_{,1} - \vec{r}_{,1} \times \vec{\phi}_{,2})\end{aligned}$$

to transform I_{PE} for the <u>stress</u> boundary value problem into

$$I_{PE} = \iint \{ [S_\kappa - C_N]\alpha_1\alpha_2 + \frac{1}{2}\vec{\phi}\cdot(\vec{r}_{,2} \times \vec{K}_{,1} - \vec{r}_{,1} \times \vec{K}_{,2})$$

$$+ \frac{1}{2}\vec{K}\cdot(\vec{r}_{,2} \times \vec{\phi}_{,1} - \vec{r}_{,1} \times \vec{\phi}_{,2}) \} d\xi_1 d\xi_2 \tag{7.12}$$

$$- \oint_\Gamma (\vec{M}^*_\nu \cdot \vec{\phi} + \vec{e}_s \cdot \Delta\vec{K}) ds - \frac{1}{2}\oint_\Gamma (\vec{r}_{,s} \times \vec{\phi})\cdot\vec{K} ds \quad .$$

The Euler differential equations and boundary conditions are not affected by this transformation. Now the static geometric dual of I_{PE} , denoted by I^*_{PE} , is

$$I^*_{PE} = \iint \{ \ldots\ldots \} d\xi_1 d\xi_2$$

$$- \oint_\Gamma (\vec{\varepsilon}^*_s \cdot \vec{K} + \vec{\mu}_\nu \cdot \Delta\vec{\phi}) ds - \frac{1}{2}\oint_\Gamma (\vec{r}_{,s} \times \vec{K})\cdot\vec{\phi} ds \tag{7.13}$$

where the surface integral is identical to the surface integral of (7.11) and where

$$\alpha_s \vec{\mu}_\nu \equiv \vec{M}_\nu - \vec{r}_{,s} \times \vec{K} \equiv (\nu_1 \frac{\partial S_\kappa}{\partial \vec{\kappa}_1} + \nu_2 \frac{\partial S_\kappa}{\partial \vec{\kappa}_2}) - \vec{r}_{,s} \times \vec{K} \quad . \tag{7.14}$$

The Euler differential equations corresponding to $\delta I^*_{PE} = 0$ (with $\delta\vec{K}$ and $\delta\vec{\phi}$ varying independently) are again (7.9a) and (7.9b). The Euler boundary conditions are now

$$\Delta\vec{\varepsilon}_s = \Delta\vec{\phi} = \vec{0} \tag{7.15}$$

which are the static-geometric duals of the stress boundary conditions $\Delta\vec{M}_\nu = \Delta\vec{K} = \vec{0}$ associated with I_{PE} (with $\vec{e}^*_s \cdot\vec{K}$ replaced by $\vec{e}_s\cdot\Delta\vec{K}$). We know by now that (7.15) may be used when edge displacements are prescribed.

8. Lower Order Shell Theories

1. A Theory without Moment-Stress Couples

For conventional structures of elastic media, the effect of moment-stresses is negligible. The conventional elasticity theory for such structures assumes that these moment stresses are absent. Formally, the material is assumed to be incapable of carrying (or developing) moment-stresses. In terms of our shell theory, this means $D_P = 0$ in our linear stress-strain relations (3.9) or more generally, a stress potential S which is independent of λ_1 and λ_2.

We mentioned earlier that $P_1 \equiv P_2 \equiv 0$ will lead to a reduction of the order of our general system of shell equations as, among other things, the sixth equilibrium equations is now algebraic. Such a reduction of order gives rise to no difficulty when stress conditions are prescribed at the edge(s) of the shell. We simply omit the condition on P_ν since we must have $P_\nu^* \equiv 0$ to be consistent. This in turn suggests that the system of governing differential equations is a tenth order system as there remain five stress boundary conditions to be satisfied at each edge.

For prescribed displacement conditions, we may expect intuitively that we can no longer prescribe ω, the normal component of $\vec{\phi}$, as we have lost the freedom of a moment (stress couple) turning about the midsurface normal. It turns out that the algebraic sixth equilibrium equations and some of the stress-strain relations and strain-displacement relations actually allow us to eliminate ω from all the equations in this theory. To see how we can deduce the omission of the condition on ω

mathematically we recall the functional I_D of the variational principle for the displacement function formulation of our shell theory:

$$I_D = \iint \{\cdots\} dS + \int_{\Gamma_\sigma} [\vec{N}^*_\nu \cdot \vec{u} + \vec{M}^*_\nu \cdot \vec{\phi}] ds$$
$$+ \int_{\Gamma_d} [\vec{N}_\nu \cdot \Delta\vec{u} + \vec{M}_\nu \cdot \Delta\vec{\phi}] ds \; . \qquad (7.1)$$

With $P_\nu \equiv 0$, we have along Γ_d

$$\vec{M}_\nu \cdot \Delta\vec{\phi} = M_{\nu\nu} \cdot \Delta\phi_\nu + M_{\nu s} \Delta\phi_s$$

so that $\delta I_D = 0$ requires only

$$\Delta\vec{u} = \vec{0} \; , \qquad \Delta\phi_\nu = \Delta\phi_s = 0 \qquad (8.1)$$

Nothing is said (or can be said) about the edge value of ω even if the edge conditions are prescribed in terms of displacements.

In a stress function formulation, displacement boundary conditions are more appropriately formulated in terms of strain measures. In the variational principle for the stress function formulation, we have the functional

$$I_S = \iint \{\cdots\} dS - \oint_{\Gamma_d} (\vec{\kappa}_s \cdot \vec{H} + \vec{\varepsilon}_s \cdot \vec{K}) dS \qquad (7.4)$$

where we have taken Γ_d to be the entire edge curve. *With*

$\delta F_{,s}$ — δH_ν — δH_s …

the quantities δK_s and δK_ν do not vary independently. Instead we have

$$\delta I_S = \iint\{\cdots\}dS + \oint_{\Gamma_d}\{(\Delta\lambda_s - \Delta\varepsilon_{s\nu,s})\delta F + (\Delta\varepsilon_{ss})(\frac{1}{\alpha_\nu}\delta F_{,\nu}) + (\Delta\gamma_s)\delta J - (\Delta K_{ss} + \frac{\Delta\varepsilon_{ss}}{R_{\nu\nu}} + \frac{\Delta\varepsilon_{s\nu}}{R_{\nu s}})\Delta H_\nu + (\Delta K_{s\nu} - \frac{\Delta\varepsilon_{ss}}{R_{\nu s}} - \frac{\Delta\varepsilon_{s\nu}}{R_{ss}})\Delta H_s\}ds \; .$$

Independent variation of δF, δF_ν , δJ, δH_ν and δH_s give five contracted Euler boundary conditions

$$\Delta\varepsilon_{ss} = \Delta\gamma_s = \Delta\lambda_s^e = \Delta\kappa_{ss}^e = \Delta\kappa_{s\nu}^e = 0 \qquad (8.3)$$

where

$$\kappa_{s\nu}^e \equiv \kappa_{s\nu} - \frac{\varepsilon_{\zeta\nu}}{R_{ss}}, \quad \kappa_{ss}^e \equiv \kappa_{ss} + \frac{\varepsilon_{s\nu}}{R_{\nu s}}, \quad \lambda_s^e \equiv \lambda_s - \frac{\partial\varepsilon_{s\nu}}{\partial s} \qquad (8.4)$$

We have from the strain-displacement relations

$$\varepsilon_{ss} = u_{s,s} + \frac{w}{R_{ss}}, \quad \gamma_s = \phi_s + w_{,s} - \frac{u_s}{R_{ss}} - \frac{u_\nu}{R_{\nu s}}$$

$$\kappa_{s\nu} - \frac{\varepsilon_{s\nu}}{R_{ss}} = \phi_{\nu,s} - \frac{1}{R_{ss}}(u_{\nu,s} + \frac{w}{R_{s\nu}}) \qquad (8.5)$$

$$\kappa_{ss} + \frac{\varepsilon_{s\nu}}{R_{\nu s}} = \phi_{s,s} + \frac{1}{R_{\nu s}}(u_{\nu,s} + \frac{w}{R_{s\nu}})$$

They show that the five contracted strain boundary conditions also do not involve ω .

Since λ_1 and λ_2 do not appear in the stress-strain relations in our tenth order theory and since they appear in undifferentiated form in two of the compatibility equations (the fourth and fifth),

$$\frac{(\alpha_2\varepsilon_{22})_{,1} - (\alpha_1\varepsilon_{21})_{,2} - \alpha_{1,2}\varepsilon_{21} - \alpha_{2,1}\varepsilon_{11}}{\alpha_1\alpha_2} - \lambda_2 + \frac{\gamma_2}{R_{12}} - \frac{\gamma_1}{R_{22}} = 0$$

and

$$-\frac{(\alpha_2\varepsilon_{21})_{,1} - (\alpha_1\varepsilon_{11})_{,2} + \lambda_{2,1}\varepsilon_{12} + \alpha_{1,2}\varepsilon_{22}}{\alpha_1\alpha_2} + \lambda_1 - \frac{\gamma_2}{R_{11}} + \frac{\gamma_1}{R_{12}} = 0 ,$$

we can eliminate them from our theory completely. We are then left with only <u>four</u> compatibility equations and one of them will be a <u>second order</u> partial differential equation while the other three remain as first order equations. These are in fact the compatibility equations in a conventional shell theory with or without transverse shear deformations. Three comments on these four equations are appropriate:

1. Without the explicit appearance of λ_1 and λ_2, it is much more difficult to deduce the compatibility equations from the strain-displacement relations. The derivation of one of these equations for the special case of axisymmetric deformation of spherical shells (by H. Reissner) was considered a stroke of genius!
2. We no longer have a static-geometric duality unless we set $\gamma_i \equiv 0$ (because we have $P_i \equiv 0$) <u>and</u> take as equilibrium equations the first three and the last equilibrium equation of our original formulation with Q_j eliminated by way of the two remaining equilibrium equations. This particular form of equilibrium equations is highly artificial since Q_1 and Q_2 appear naturally in the conditions of force and moment equilibrium.

3. A mathematical theory which involves some first order and some second order differential equations lacks a certain structural elegance and consistency compared to one which involves equations of the same order and should therefore be avoided if at all possible.

Our proposal here is to leave the first order equations alone for the various special cases of the general twelfth order shell theory. In the long run, more progress can be made by this practice.

2. A Theory without Transverse Shear Deformation

If the strain potential C is independent of Q_1 and Q_2, e.g., if $A_Q = 0$ in our linear stress strain relations, we have $\gamma_1 = \gamma_2 = 0$ throughout the shell. Evidently, this is just the dual of the theory without moment-stress couples and the reduced system of shell equations is analogously tenth order in this case.

We have already mentioned the need to analyze the corresponding reduction of boundary conditions. Having reformulated the displacement edge conditions in terms of strain measures, the difficulties associated with reduced displacement edge conditions are automatically removed. We need $\gamma_s^* = 0$ along the edge for consistency. Even if we insist on using the actual displacement edge conditions, we observe that, with reference to $\vec{\nu}$, $\vec{t}$ and $\vec{n}$, the conditions $\gamma_\nu = \gamma_s = 0$ implies

$$\phi_\nu = -\frac{w,_\nu}{\alpha_\nu} + \frac{u_\nu}{R_{\nu\nu}} + \frac{u_s}{R_{\nu s}} , \quad \phi_s = -\frac{w,_s}{\alpha_s} + \frac{u_s}{R_{ss}} + \frac{u}{R_{\nu s}}$$

so that we can no longer prescribe ϕ_s^* independently once w^*, u_ν^*, and u_s^* are given at the edge. On the other hand, we may continue to prescribe ϕ_ν^* independently since $w^*,_\nu$ is not determined by w^*, u_ν^* and u_s^*. The omission of the condition on ϕ_s leaves us with the correct number of edge conditions for the reduced system of equations.

If stress conditions are prescribed, the situation is much more complicated. However, we infer from the static-geometric duality that the five contracted stress boundary conditions should be the duals of (8.3), namely,

$$\Delta M_{\nu\nu} = \Delta P_{\nu} + \Delta Q^{e}_{\nu} = \Delta N^{e}_{\nu\nu} = \Delta N^{e}_{\nu s} = 0 \tag{8.6}$$

where

$$Q^{e}_{\nu} \equiv Q_{\nu} + \frac{\partial M_{\nu s}}{\partial s}, \quad N^{e}_{\nu\nu} \equiv N_{\nu\nu} + \frac{M_{\nu s}}{R_{\nu s}}, \quad N^{e}_{\nu s} \equiv N_{\nu s} + \frac{M_{\nu s}}{R_{ss}} \tag{8.7}$$

The appropriateness of the five conditions in (8.6) may be confirmed by way of the variational principle for the displacement function formulation in section (1) of Chapter (7) similar to the development in the last section. No further comment is necessary.

It should also be noted that the form of the five appropriate stress boundary conditions for shells without transverse shear deformation is considerably simpler if they are to be given in terms of the stress functions. We have in the boundary integral along Γ_{σ} of (7.4)

$$\vec{\kappa}_{s}\cdot\Delta\vec{H} + \vec{\varepsilon}_{s}\cdot\Delta\vec{K} = \vec{\kappa}_{s}\cdot\Delta\vec{H} + \varepsilon_{ss}\Delta K_{\nu} - \varepsilon_{s\nu}\Delta K_{s} .$$

Therefore, nothing is (or can be) said about ΔJ when a shell is not shear deformable.

3. Conventional Shell Theories and a First Approximation Shell Theory

The various shell theories used in engineering literature to analyze specific problems invariably assume the absence of moment stress couples (so that $P_j \equiv 0$, $j = 1,2,$). Until recently, transverse shear deformation is also not allowed in such analyses (so that $\gamma_j \equiv 0$, $j = 1,2$). A shell theory without moment stress couples and transverse shear deformation is known as a conventional shell theory. Many such theories are possible. For hyperelastic shells with a mixed potential $P(N_{ij}, Q_j, \lambda_j \kappa_{ij})$, different conventional theories are obtained by various specialization of P provided of course P is independent of Q_j and λ_j so that $\partial P/\partial Q_j \equiv \partial P/\partial \lambda_j = 0$, $j = 1,2,$. Evidently, a conventional shell theory is an eighth order theory; this can be confirmed by way of the variational principles of Chapter (7). The reduction and contraction of stress or displacement boundary conditions for such a theory are obtained from an appropriate combination of the reductions and contractions required in the last two sections. It is not necessary to discuss further the appropriate boundary conditions for these eighth order theories.

The shell theory with the linear stress-strain relations (3.7) and (3.8) can be specialized to a conventional shell theory by setting $A_Q = D_P = 0$. If we also set $t_N = -t_M = 1/2$, the resulting shell theory is known as a consistent first approximation linear theory for isotropic shells. This theory is now generally accepted as the appropriate theory for the analysis of infinitesimal deformations of linearly elastic isotropic shells. In the subsequent development, it will be used whenever the solution of a specific problem of that type is to be obtained.

4. A Membrane Theory

If in addition to $P_j \equiv \gamma_j \equiv 0$, $j = 1,2$, a shell theory also requires $M_{ij} \equiv 0$, $i,j = 1,2$, we have what is known as a membrane theory. For a hyperelastic shell, we have a membrane theory if the mixed potential $P(N_{ij}, Q_j, \lambda_j, \kappa_{ij})$ is independent of Q_j, λ_j and κ_{ij}, $i,j = 1,2$. For the linear theory of (3.7) and (3.8), we get a membrane theory if $A_Q = D_P = D = 0$. For a membrane theory, the order of the system of differential equations is further reduced to four. This can be seen from the following development.

With $M_{ij} \equiv 0$, the moment equilibrium equations give us

$$Q_1 \equiv Q_2 \equiv 0 \quad \text{and} \quad N_{12} = N_{21}$$

and the three force equilibrium equations become

$$(\alpha_2 N_{11})_{,1} + (\alpha_1 N_{21})_{,2} + \alpha_{1,2} N_{12} - \alpha_{2,1} N_{22} \quad \alpha_1 \alpha_2 p_1 = 0$$

$$(\alpha_2 N_{12})_{,1} + (\alpha_1 N_{22})_{,2} + \alpha_{2,1} N_{21} - \alpha_{1,2} N_{11} + \alpha_1 \alpha_2 p_2 = 0$$

$$\frac{N_{11}}{R_{11}} + \frac{N_{22}}{R_{22}} + \frac{N_{12} + N_{21}}{R_{12}} - \alpha_1 \alpha_2 p_n = 0$$

which form a second order system. These equations are for only three unknowns $(N_{12} = N_{21})$ and the problem becomes statically determinate. That is, we can now determine the only three nonvanishing internal reactions N_{ij} by equilibrium considerations alone.

Having N_{ij}, we can calculate the strain resultants ε_{ij} by the stress-strain relations, and then the displacement components u_1, u_2 and w by solving the strain displacement relations as differential equations for these displacement components[6]. This system of equations is also a second order system:

$$\varepsilon_{11} = \frac{u_{1,1}}{\alpha_1} + \frac{\alpha_{1,2}u_2}{\alpha_1\alpha_2} + \frac{w}{R_{11}}, \qquad \varepsilon_{22} = \frac{u_{2,2}}{\alpha_1} + \frac{\alpha_{2,1}u_1}{\alpha_1\alpha_2} + \frac{w}{R_{22}}$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left[\frac{u_{2,1}}{\alpha_1} + \frac{u_{1,2}}{\alpha_2} - \frac{\alpha_{1,2}u_1}{\alpha_1\alpha_2} - \frac{\alpha_{2,1}u_2}{\alpha_1\alpha_2} + \frac{2w}{R_{12}}\right]$$

Finally, κ_{ij} are obtained from their expressions in terms of the displacement components keeping in mind that ϕ_1 and ϕ_2 are now given in terms of u_1, u_2 and w by the conditions of vanishing transverse shearing strains.

Unlike the case of moment stress couple, shell structures do exhibit significant bending action under some load conditions. This so-called membrane theory of shells is an idealized (or limiting) form of an approximate shell theory in which the bending action (characterized by the stress and strain couples) is not absent but expected to be insignificant compared to the stretching action (characterized by the stress and strain resultants) of the shell. (In the context of the three dimensional theory of elasticity, the bending stresses

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[6] ω is immediately expressed in terms of u_1, u_2 and w given that $\varepsilon_{12} = \varepsilon_{21}$

are negligibly small compared to the direct (or membrane) stresses in this case.) That this cannot be true for the entire shell is indicated by the fact that we cannot satisfy all the expected physical edge conditions by a membrane theory alone. Bending stresses are therefore expected to be of significant magnitude near an edge of the shell. In the language of singular perturbation theory, a membrane solution may be considered as the leading term outer solution of a matched asymptotic expansion solution for the shell problem. This will become evident when we turn to methods of solution for specific problems later.

The simplicity of the membrane theory lies in the lower order system of equations and the uncoupling of statics and deformation. It is extremely attractive for the solution of specific shell problems and was therefore very popular among the practicing engineers before high speed computing became available and before asymptotic analysis became a part of an advanced engineering curriculum. It is important to note the following observations on this simple theory:

1. There is of course a reduction to only <u>two</u> boundary conditions to be described at each edge.
2. There is no difficulty with the reduction of <u>stress</u> boundary conditions since we must have $M^*_{\nu\nu}=M^*_{\nu s}=P^*_{\nu}=Q^*_{\nu}=0$ to be consistent leaving us with $\Delta N_{\nu\nu} = \Delta N_{\nu t} = 0$ along Γ_s . The appropriate reduction of displacement edge conditions will be discussed later.

3. What is more surprising from a shell-theoretical point of view is that not all of the allowed edge conditions can be prescribed in terms of $N_{\nu\nu}$ and $N_{\nu s}$. Some of the nontrivial edge conditions must be prescribed in terms of displacement components or strain measures. The reason is that the N_{ij}'s are completely determined by a second order system of equations.

Just what physical laws prevent us from having an appropriately reduced stress boundary value problem (with stress conditions prescribed at all edges) for a membrane theory (even when all external loads are self-equilibrating[7] is not known to this writer. However, his own calculations using the conventional theory showed that the dominant stresses in the "interior" of a shallow spherical shell are not those associated with N_{ij}'s if only stress measures are prescribed at the edge(s)[8]. For these problems the bending stresses associated with M_{ij} dominate or are equally significant as the direct (or membrane) stresses associated with N_{ij}. Therefore, the use of a membrane theory is not appropriate for these problems.

To obtain the proper displacement boundary conditions at an edge Γ_d of the shell for a membrane theory, we only have to observe that, in the boundary integral along Γ_d in (7.1), we have now $\vec{M} \equiv \vec{0}$ and $Q_\nu \equiv 0$ so that

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(7) The loads give rise to no net resultant force and moment.

(8) F.Y.M. Wan, "Membrane and bending stresses in shallow spherical shells", Intern'l. J. Solids & Structures 3, 1967, pp. 353-366.

$$\int_{\Gamma_d} (\vec{N}_\nu \cdot \Delta\vec{u} + \vec{\mu}_\nu \cdot \Delta\vec{\phi})ds = \int_{\Gamma_d} (N_{\nu\nu}\Delta u_\nu + N_{\nu t}\Delta u_t)ds.$$

The only independent variations $\delta N_{\nu\nu}$ and $\delta N_{\nu t}$ give

$$\Delta u_\nu = \Delta u_t = 0$$

as the two appropriate displacement boundary conditions along an edge of the shell. Thus, for a membrane theory, only the displacement components tangent to the middle surface can be prescribed.

5. An Inextensional Bending Theory

If bending (instead of membrane) shell action is dominant and the shell deforms effectively without stretching or compression, an approximate shell theory which neglects midsurface strains is appropriate. In its limiting form, this leads to a shell theory which assumes the shell is inextensible with $\varepsilon_{ij} = 0$ (while N_{ij} stay finite). A hyper-elastic shell is inextensible if the mixed potential $P(N_{ij}, \kappa_{ij}, \ldots)$ is independent of N_{ij}, $i,j = 1,2$. A conventional shell theory which also stipulates midsurface inextensibility is called an <u>inextensional bending shell theory</u>. The linear shell theory with (3.7) and (3.8) as its stress-strain relations is an inextensional bending theory if $A_Q = D_P = A = 0$.

With $\varepsilon_{ij} \equiv 0$, three of the scaler compatibility equations in (5.6) gives

$$\lambda_1 = \lambda_2 = 0 \quad \text{and} \quad \kappa_{12} = \kappa_{21}$$

The remaining three compatibility equations become

$$-(\alpha_2\kappa_{22})_{,1} + (\alpha_1\kappa_{12})_{,2} + \alpha_{1,2}\kappa_{21} + \alpha_{2,1}\kappa_{11} = 0$$

$$(\alpha_2\kappa_{21})_{,1} - (\alpha_1\kappa_{11})_{,2} + \alpha_{2,1}\kappa_{12} + \alpha_{1,2}\kappa_{22} = 0$$

$$-\frac{\kappa_{22}}{R_{11}} - \frac{11}{R_{22}} + \frac{\kappa_{21} + \kappa_{12}}{R_{12}} = 0$$

and form a <u>second order</u> system for the three unknowns κ_{11}, κ_{22} and $\kappa_{12} = \kappa_{21}$. The problem is therefore "geometrically determinate". Having κ_{ij}, we can calculate M_{ij} by the stress-strain relations and then the transverse shear resultants, Q_1 and Q_2, by the first two moment equilibrium equations. Finally, the remaining four equilibrium equations form a second order system for the determination of the four stress resultants N_{ij} (not necessarily symmetric in subscripts).

It should be evident by now that the inextensional bending theory of shells is the static-geometric dual of the membrane theory. What was said about the stress boundary conditions for the latter applies to the strain (and therefore displacement) boundary conditions here. Also, the appropriate stress boundary conditions for the inextensional bending theory may be taken in the form $\Delta H_\nu \equiv \Delta H_t = 0$ along Γ_s which is just the static-geometric analogues of the displacement conditions for the membrane theory.

Other remarks on the boundary conditions for the membrane theory also apply to the dual quantities in the inextensional bending theory. In particular, half of the boundary conditions must be prescribed in terms of the stress measures. The writer's own calculations for the displacement BVP for shallow spherical shells show that membrane action dominates in the shell interior if all edge data are prescribed in terms of displacements.[8,9]

The inextensional bending theory of shell considered here is an idealized (or limiting) form of an approximate shell theory in which the stretching action of the shell is insignificant compared to the bending action. This is essentially the situation in most free vibration problems. But for static problems, such a theory is rarely employed by

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(9) See F.Y.M. Wan, "On the displacement boundary value problem of shallow spherical shells", Int'l J. Solids & Structures (IJSS) 4, 1968, 661-666 (and other references cited there).

practicing engineers although it should be used for some problems[6]

If the stress boundary conditions are prescribed in terms of the stress resultants and couples (instead of stress functions) and the two appropriate stress boundary conditions at an edge for the inextensional bending theory are to be given in terms of the same quantities, the five conditions $\Delta Q_{\nu} = \Delta M_{\nu\nu} = \Delta M_{\nu t} = \Delta N_{\nu\nu} = \Delta N_{\nu t} = 0$ will have to be contracted to two, analogous to the Kirchhoff-Bassett conditions for a theory with no transverse shear deformation. These two appropriate contracted stress boundary conditions have been obtained by E. Reissner[5] and will not be given here.

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III THE FOUNDATIONS OF LINEAR SHELL THEORY

9. The Kirchhoff-Love Hypotheses

1. Linear Elasticity Theory and Lines of Curvature Shell Coordinates

Let $\vec{r}(\xi_1,\xi_2)$ be the position vector of a point on the middle surface of a thin shell. For simplicity, we take the middle surface coordinate system, ξ_1 and ξ_2 , to coincide with the orthogonal lines of principal curvature of the middle surface. We call such surface coordinates lines of curvature (l.o.c.) coordinates. A point in the shell body may then be described by the position vector

$$\vec{x}(\xi_1,\xi_2,\zeta) = \vec{r}(\xi_1,\xi_2) + \zeta\vec{n}(\xi_1,\xi_2)$$

where $\vec{n}$ is the unit normal at the point $\vec{r}(\xi_1,\xi_2)$ of the middle surface $(\zeta = 0)$. The unit tangent vectors $\vec{t_1}$ and $\vec{t_2}$ along the surface coordinate lines ξ_2 = constant and ξ_1 = constant are given by

$$\vec{t_k} = \frac{1}{\hat{\alpha}_k}\frac{\partial\vec{x}}{\partial\xi_k} \quad ,$$

with

$$\hat{\alpha}_k = \alpha_k(1 + \frac{\zeta}{R_k}) \ , \quad \alpha_k = |\vec{r}_{,k}|$$

where R_1 and R_2 are the two principal radii of curvature of the surface. The positive direction of the unit normal is defined by the convention $\vec{n} = \vec{t_1} \times \vec{t_2}$.

For the orthogonal curvilinear space coordinates ξ_1, ξ_2 and $\xi_3 \equiv \zeta$, the linear strain components e_{ij} , $i , j = 1,2,3,$ of the three dimensional linear elasticity theory are given in terms of the displacement components U_k , $k = 1,2,3,$ by the six linear strain-displacement relations:

$$\begin{aligned}
&\left(1+\frac{\zeta}{R_1}\right)e_{11} = \frac{1}{\alpha_1} U_{1,1} + \frac{\alpha_{1,2}}{\alpha_1\alpha_2} U_2 + \frac{1}{R_1} U_3 \ , \ \left(1+\frac{\zeta}{R_2}\right)e_{22} = \ldots . \\
&e_{12} = e_{21} = \frac{1}{1+\frac{\zeta}{R_1}} \left(\frac{1}{\alpha_1} U_{2,1} - \frac{\alpha_{1,2}}{\alpha_1\alpha_2} U_1\right) + \frac{1}{1+\frac{\zeta}{R_2}} \left(\frac{1}{\alpha_2} U_{1,2} - \frac{\alpha_{2,1}}{\alpha_1\alpha_2} U_2\right) \\
&e_{13} = e_{31} = \frac{1}{1+\frac{\zeta}{R_1}} \left(\frac{1}{\alpha_1} U_{3,1} - \frac{1}{R_1} U_1\right) + U_{1,3} \ , \ e_{23} = e_{32} = \ldots . \\
&e_{33} = U_{3,3}
\end{aligned} \tag{9.1}$$

where $(\ )_{,k} \equiv \partial(\ )/\partial\xi_k$. The three linear differential equations of force equilibrium for the stress components σ_{ij} , $i,j = 1,2,3,$ take the form

$$\begin{aligned}
&(\hat{\alpha}_2\sigma_{11})_{,1} + (\hat{\alpha}_1\sigma_{21})_{,2} + \left(1+\frac{\zeta}{R_2}\right) \alpha_{1,2}\sigma_{12} - \left(1+\frac{\zeta}{R_1}\right) \alpha_{2,1}\sigma_{22} \\
&\qquad + \frac{\alpha_1\hat{\alpha}_2}{R_1} \sigma_{13} + (\hat{\alpha}_1\hat{\alpha}_2\sigma_{31})_{,3} + \hat{\alpha}_1\hat{\alpha}_2 f_1 = 0 \\
&(\hat{\alpha}_2\sigma_{12})_{,1} + (\hat{\alpha}_1\sigma_{22})_{,2} + \left(1+\frac{\zeta}{R_1}\right)\alpha_{2,1}\sigma_{21} - \left(1+\frac{\zeta}{R_2}\right) \alpha_{1,2}\sigma_{11} \\
&\qquad + \frac{\hat{\alpha}_1\alpha_2}{R_2} \sigma_{23} + (\hat{\alpha}_1\hat{\alpha}_2\sigma_{32})_{,3} + \hat{\alpha}_1\hat{\alpha}_2 f_2 = 0 \\
&(\hat{\alpha}_2\sigma_{13})_{,1} + (\hat{\alpha}_1\sigma_{23})_{,2} - \frac{\alpha_1\hat{\alpha}_2}{R_1} \sigma_{11} - \frac{\hat{\alpha}_1\alpha_2}{R_2} \sigma_{22} + (\hat{\alpha}_1\hat{\alpha}_2\sigma_{33})_{,3} + \hat{\alpha}_1\hat{\alpha}_2 f_3 = 0
\end{aligned} \tag{9.2}$$

where f_k, $k = 1,2,3,$ are the components of the distributed body force intensity vector per unit volume. We also note the symmetry of σ_{ij} in its subscripts, taken in form

$$\sigma_{ij} - \sigma_{ji} = 0 \quad (i \neq j,\ i,j = 1,2,3) \ .$$

For $i \neq j$, it is a consequence of the condition of moment equilibrium of a differential volume of the solid body (assuming the absence of couple stresses and distributed body moment intensity vector per unit volume).

2. Stress Resultants and Couples

For a shell of thickness h, we define

$$\{N_{1j},\ M_{1j}\} = \int_{-h/2}^{h/2} \sigma_{1j}\left(1+\frac{\zeta}{R_2}\right)\{1,\zeta\}\ d\zeta \ , \qquad (j = 1,2)$$

$$\{N_{2j},\ M_{2j}\} = \int_{-h/2}^{h/2} \sigma_{2j}\left(1+\frac{\zeta}{R_1}\right)\{1,\zeta\}\ d\zeta \ , \tag{9.3}$$

$$Q_1, = \int_{-h/2}^{h/2} \sigma_{13}\left(1+\frac{\zeta}{R_2}\right)d\zeta \ , \quad Q_2 = \int_{-h/2}^{h/2} \sigma_{23}\left(1+\frac{\zeta}{R_1}\right)d\zeta \ .$$

It is important to keep in mind that N_{12} is in general not equal to N_{21} and M_{12} is in general not equal to M_{21}, although we have $\sigma_{12} = \sigma_{21}$ in three dimensional elasticity theory.

Upon integrating the three differential equations of force equilibrium of three dimensional linear elasticity theory, we get

$$\boxed{\begin{aligned} &(\alpha_2 N_{11})_{,1} + (\alpha_1 N_{21})_{,2} + \alpha_{1,2}N_{12} - \alpha_{2,1}N_{22} + \frac{\alpha_1\alpha_2}{R_1}Q_1 + \alpha_1\alpha_2 p_1 = 0 \\ &(\alpha_2 N_{12})_{,1} + (\alpha_1 N_{22})_{,2} + \alpha_{2,1}N_{21} - \alpha_{1,2}N_{11} + \frac{\alpha_1\alpha_2}{R_2}Q_2 + \alpha_1\alpha_2 p_2 = 0 \\ &(\alpha_2 Q_1)_{,1} + (\alpha_1 Q_2)_{,2} - \frac{\alpha_1\alpha_2}{R_1}N_{11} - \frac{\alpha_1\alpha_2}{R_2}N_{22} + \alpha_1\alpha_2 p_3 = 0 \end{aligned}} \tag{9.4}$$

where

$$p_k = [(1+\frac{\zeta}{R_1})(1+\frac{\zeta}{R_2})\sigma_{3k}]_{-h/2}^{h/2} + \int_{-h/2}^{h/2}(1+\frac{\zeta}{R_1})(1+\frac{\zeta}{R_2})f_k(\xi_1,\xi_2,\zeta)d\zeta$$

Next, we multiply the two in-(tangent) plane force equilibrium equations through by ζ and integrate across the shell thickness. In this way, we get

$$\boxed{\begin{aligned}&(\alpha_2 M_{11})_{,1} + (\alpha_1 M_{21})_{,2} + \alpha_{1,2}M_{12} - \alpha_{2,1}M_{22} - \alpha_1\alpha_2 Q_1 + \alpha_1\alpha_2 q_1 = 0\\ &(\alpha_2 M_{12})_{,1} + (\alpha_1 M_{22})_{,2} + \alpha_{2,1}M_{21} - \alpha_{1,2}M_{11} - \alpha_1\alpha_2 Q_2 + \alpha_1\alpha_2 q_2 = 0\end{aligned}} \tag{9.5}$$

where

$$q_k = [\zeta(1+\frac{\zeta}{R_1})(1+\frac{\zeta}{R_2})\sigma_{3k}]_{-h/2}^{h/2} + \int_{-h/2}^{h/2}\zeta(1+\frac{\zeta}{R_1})(1+\frac{\zeta}{R_2})f_k(\xi_1,\xi_2,\zeta)d\zeta$$

Finally, we form from $\sigma_{12} - \sigma_{21} = 0$

$$\int_{-h/2}^{h/2}(\sigma_{12} - \sigma_{21})(1+\frac{\zeta}{R_1})(1+\frac{\zeta}{R_2})d\zeta = 0$$

to get

$$\boxed{N_{12} - N_{21} + \frac{M_{12}}{R_1} - \frac{M_{21}}{R_2} = 0} \tag{9.6}$$

The six integrated equilibrium equations are <u>exact</u> consequences of the three dimensional equilibrium equations. They are identical to the three force and three moment equilibrium equations for a thin shell obtained by treating the shell as a surface endowed with certain mechanical and material properties and specialized to the case of l.o.c. (surface) coordinates and $P_j = q_n = 0$, $j = 1,2$.

3. The First Approximation Shell Theory

As we know by now, the Kirchhoff plate theory so important to theoretical and applied (solid) mechanics, is the leading term of the outer (asymptotic expansion of the exact) solution of three dimensional elasto-static boundary value problem for flat solid bodies which are thin in one direction. A corresponding leading term outer solution for thin shells is similarly important to engineering mechanics. The main body of this leading term outer solution, designated as a first approximation shell theory in the literature, was developed by A.E.H. Love(1) in 1888. Similar to the Kirchhoff theory, Love's development of his first approximation shell theory was based on a set of assumptions (instead of an asymptotic analysis). For the linear theory, these assumptions may be taken in the form of three postulates:

1. Normals to the undeformed middle surface remain normal to the deformed middle surface without change in length.
2. Transverse normal stress σ_{33} may be neglected in the in-plane stress strain relations.
3. The shell is thin so that $1 + \zeta/R_k$ may be approximated by 1 .

First approximation theories for isotropic elastic shells derived by Love and other more recent investigators based on these postulates are usually deficient in some way, in the sense that there are some inconsistencies

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(1) See reference [1] at the end of Part III.

among the governing differential equations of these theories. These deficiencies were removed independently by J.L. Sanders, Jr.[2] and W.T. Koiter [2] without using the method of matched asymptotic expansions in 1959 to arrive at what is now generally accepted as a consistent first approximation shell theory. Ten years earlier, F.B. Hildebrand, E. Reissner and G.B. Thomas [2] also succeeded in eliminating the inconsistencies by assigning a certain limiting orthotropicity to the shell structure. Of the three approaches which do not involve an asymptotic analysis, the last seems more straightforward and will be described below.

The first assumption of Love, generally adopted by other investigators and called the Love-Kirchhoff hypothesis, is effectively equivalent to the assumption that the deformed shell is essentially in a plane strain state so that $e_{j3} \equiv 0$, $j = 1,2,3$. It follows immediately that

$$\begin{aligned} U_3(\xi_1,\xi_2,\zeta) &= W(\xi_1,\xi_2) \\ U_k(\xi_1,\xi_2,\zeta) &= \overline{U}_k(\xi_1,\xi_2) + \zeta\Phi_k(\xi_1,\xi_2) \quad (k = 1,2) \end{aligned} \tag{9.7}$$

and remaining three nonvanishing strain components become

$$\begin{aligned} (1+\frac{\zeta}{R_k})e_{kk} &= \tilde{\varepsilon}_{kk} + \zeta\tilde{\kappa}_{kk} , \qquad (k=1,2) \\ e_{12} = e_{21} &= \frac{\tilde{\varepsilon}_{12}+\zeta\tilde{\kappa}_{12}}{1+\zeta/R_1} + \frac{\tilde{\varepsilon}_{21}+\zeta\tilde{\kappa}_{21}}{1+\zeta/R_2} , \end{aligned} \tag{9.8}$$

where

$$\tilde{\varepsilon}_{11} = \frac{1}{\alpha_1}\overline{U}_{1,1} + \frac{\alpha_{1,2}}{\alpha_1\alpha_2}\overline{U}_2 + \frac{1}{R_1}W , \quad \tilde{\varepsilon}_{22} = \dots .$$

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[2] See reference [2-4] at the end of Part III.

$$\tilde{\varepsilon}_{12} = \frac{1}{\alpha_1}\bar{U}_{2,1} - \frac{\alpha_{1,2}}{\alpha_1\alpha_2}\bar{U}_1 , \qquad \tilde{\varepsilon}_{21} = \cdots$$

$$\tilde{\kappa}_{11} = \frac{1}{\alpha_1}\Phi_{1,1} + \frac{\alpha_{1,2}}{\alpha_1\alpha_2}\Phi_2 , \qquad \tilde{\kappa}_{22} = \cdots \tag{9.9}$$

$$\tilde{\kappa}_{12} = \frac{1}{\alpha_1}\Phi_{2,1} - \frac{\alpha_{1,2}}{\alpha_1\alpha_2}\Phi_1 , \qquad \tilde{\kappa}_{21} = \cdots$$

where

$$\Phi_1 = \frac{\bar{U}_1}{R_1} - \frac{W_{,1}}{\alpha_1} \quad , \quad \Phi_2 = \frac{\bar{U}_2}{R_2} - \frac{W_{,2}}{\alpha_2} \quad . \tag{9.10}$$

For an isotropic shell, the conditions of vanishing transverse shearing strains, $e_{13} = e_{23} = 0$, require $\sigma_{13} = \sigma_{23} = 0$. But for many problems involving out-of-plane bending shell action, we need nonvanishing transverse shear resultant(s), Q_1 and/or Q_2 , to maintain equilibrium. (The bending of a flat plate, the limiting case of a shell with no curvature, is a familiar example). To remove the inconsistency of $e_{13} = e_{23} = 0$ on the one hand and $Q_j \neq 0$ $(j=1,2)$ on the other hand (without embarking on an asymptotic analysis), we recall that for a <u>transversely isotropic</u> material, the (linear) stress-strain relations take the form

$$e_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22} - \nu_3\sigma_{33}) , \quad e_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11} - \nu_3\sigma_{33})$$

$$e_{12} = \frac{1}{G}\sigma_{12} , \quad e_{j3} = \frac{1}{G_3}\sigma_{j3} \quad (j = 1,2) , \tag{9.11}$$

$$e_{33} = \frac{1}{E_3}\sigma_{33} - \frac{\nu_3}{E}(\sigma_{11} + \sigma_{22})$$

Love's first assumption is formally equivalently to the shell being transversely rigid so that $1/G_3 = 0$ (as well as $1/E_3 = \nu_3 = 0$ which will be needed below). In this way, e_{13} and e_{23} vanish identically with no implication on the magnitude of σ_{13} and σ_{23}.

For an isotropic shell, $e_{33} = 0$ implies $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$. As long as $\nu = 0(1)$, this consequence is inconsistent with Love's second assumption as $\nu(\sigma_{11} + \sigma_{22})$ is not negligibly small in the expression for the two remaining normal strain components e_{11} and e_{22}. For a transversely rigid shell, we have $1/E_3 = \nu_3 = 0$ so that we have $e_{33} \equiv 0$ as well as

$$e_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22}) \;, \quad e_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11}) \qquad (9.12)$$

without any restriction on the magnitude of σ_{11}, σ_{22} and σ_{33}.

10. The Flügge-Lurje-Byrne Stress-Strain Relations

1. Transversely Rigid Shells

The three nontrivial stress-strain relations for our transversely rigid shell may be written as

$$\sigma_{11} = \frac{E}{1-\nu^2}\,(e_{11} + \nu e_{22}) \;, \quad \sigma_{22} = \frac{E}{1-\nu^2}\,(e_{22} + \nu e_{11}) \qquad (10.1)$$

$$\sigma_{12} = \sigma_{21} = G\, e_{12}$$

From the first expression, we get

$$N_{11} = \int_{-h/2}^{h/2} \sigma_{11}\left(1 + \frac{\zeta}{R_2}\right) d\zeta \equiv C_{11}\tilde{\varepsilon}_{11} + C_{12}\tilde{\varepsilon}_{22} + B_{11}\tilde{\kappa}_{11} + B_{12}\tilde{\kappa}_{22} \qquad (10.2a)$$

where

$$C_{11} = \int_{-h/2}^{h/2} \frac{1+\zeta/R_2}{1+\zeta/R_1}\,\frac{E}{1-\nu^2}\, d\zeta, \quad C_{12} = \int_{-h/2}^{h/2} \frac{\nu E}{1-\nu^2}\, d\zeta$$

$$B_{11} = \int_{-h/2}^{h/2} \frac{1+\zeta/R_2}{1+\zeta/R_1}\,\frac{E\zeta}{1-\nu^2}\, d\zeta \;, \quad B_{12} = \int_{-h/2}^{h/2} \frac{\nu E\zeta}{1-\nu^2}\, d\zeta \qquad (10.2b)$$

Similarly, we get the following expressions for stress resultants and couples:

$$N_{22} = C_{22}\tilde{\varepsilon}_{22} + C_{21}\tilde{\varepsilon}_{11} + B_{22}\tilde{\kappa}_{22} + B_{21}\tilde{\kappa}_{11} \tag{10.2c}$$

$$M_{11} = B_{11}\tilde{\varepsilon}_{11} + B_{12}\tilde{\varepsilon}_{22} + D_{11}\tilde{\kappa}_{11} + D_{12}\tilde{\kappa}_{22} \tag{10.2d}$$

$$M_{22} = B_{22}\tilde{\varepsilon}_{22} + B_{21}\tilde{\varepsilon}_{11} + D_{22}\tilde{\kappa}_{22} + D_{21}\tilde{\kappa}_{11} \tag{10.2e}$$

with

$$\{C_{kk}, B_{kk}, D_{kk}\} = \int_{-h/2}^{h/2} \frac{1+\zeta/R_j}{1+\zeta/R_k} \frac{E}{1-\nu^2} \{1,\zeta,\zeta^2\}d\zeta \tag{10.2f}$$

$(j \neq k)$

$$\{C_{jk}, B_{jk}, D_{jk}\} = \int_{-h/2}^{h/2} \frac{\nu E}{1-\nu^2} \{1,\zeta,\zeta^2\}d\zeta \tag{10.2g}$$

and

$$N_{12} = C_{s1}\tilde{\varepsilon}_{12} + C_{ss}\tilde{\varepsilon}_{21} + B_{s1}\tilde{\kappa}_{12} + B_{ss}\tilde{\kappa}_{21} \tag{10.3a}$$

$$N_{21} = C_{s2}\tilde{\varepsilon}_{21} + C_{ss}\tilde{\varepsilon}_{12} + B_{s2}\tilde{\kappa}_{21} + B_{ss}\tilde{\kappa}_{12} \tag{10.3b}$$

$$M_{12} = B_{s1}\tilde{\varepsilon}_{12} + B_{ss}\tilde{\varepsilon}_{21} + D_{s1}\tilde{\kappa}_{12} + D_{ss}\tilde{\kappa}_{21} \tag{10.3c}$$

$$M_{21} = B_{s2}\tilde{\varepsilon}_{21} + B_{ss}\tilde{\varepsilon}_{12} + D_{s2}\tilde{\kappa}_{21} + D_{ss}\tilde{\kappa}_{12} \tag{10.3d}$$

with

$$\{C_{sk}, B_{sk}, D_{sk}\} = \int_{-h/2}^{h/2} \frac{1+\zeta/R_j}{1+\zeta/R_k} G\{1,\zeta,\zeta^2\}d\zeta \qquad (j \neq k) \tag{10.3e}$$

$$\{C_{ss}, B_{ss}, D_{ss}\} = \int_{-h/2}^{h/2} G\{1,\zeta,\zeta^2\}d\zeta \tag{10.3f}$$

For a transversely homogeneous shell medium so that the elastic moduli E and ν are independent of ζ, we have

$$B_{12} = B_{21} = B_{ss} = 0 \quad ,$$

$$C_{12} = C_{21} = \frac{\nu Eh}{1-\nu^2} , \quad C_{ss} = Gh \quad , \tag{10.4}$$

$$D_{12} = D_{21} = \frac{\nu Eh^3}{12(1-\nu^2)} , \quad D_{ss} = \frac{Gh^3}{12} .$$

2. <u>The Flügge-Lurje-Byrne Approximate Relations:</u>

With $|\zeta/R_k| << 1$, we may write

$$\frac{1+\zeta/R_1}{1+\zeta/R_2} = 1 - \rho\zeta + \rho\frac{\zeta^2}{R_2} + \ldots$$
$$\frac{1+\zeta/R_2}{1+\zeta/R_1} = 1 + \rho\zeta - \rho\frac{\zeta^2}{R_1} + \ldots \tag{10.5}$$

where $\rho \equiv (1/R_2) - (1/R_1)$. Upon omitting terms of order h^3/R^3 compared to unity, we get for the nonvanishing C_{ij}, B_{ij}, D_{ij}, etc.,

$$C_{11} \simeq C\left[1 - \frac{\rho h^2}{12R_1}\right], \quad C_{22} \simeq C\left[1 + \frac{\rho h^2}{12R_2}\right],$$
$$C_{s1} \simeq Gh\left[1 - \frac{\rho h^2}{12R_1}\right], \quad C_{s2} \simeq Gh\left[1 + \frac{\rho h^2}{12R_2}\right],$$
$$B_{11} \simeq \rho D, \qquad B_{22} \simeq -\rho D \tag{10.6}$$
$$B_{s1} \simeq \frac{1}{2}\rho D(1-\nu) \qquad B_{s2} \simeq -\frac{1}{2}\rho D(1-\nu)$$
$$D_{11} \simeq D_{22} \simeq D, \qquad D_{s1} \simeq D_{s2} \simeq \frac{1}{2}D(1-\nu)$$

with

$$C = \frac{Eh}{1-\nu^2}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}. \tag{10.7}$$

The corresponding system of <u>approximate</u> linear stress-strain relations,

$$N_{11} = C(\tilde{\varepsilon}_{11} + \nu\tilde{\varepsilon}_{22}) + \rho D(\tilde{\kappa}_{11} - \frac{\tilde{\varepsilon}_{11}}{R_1}) \tag{10.8a}$$

$$N_{22} = C(\tilde{\varepsilon}_{22} + \nu\tilde{\varepsilon}_{11}) - \rho D(\tilde{\kappa}_{22} - \frac{\tilde{\varepsilon}_{22}}{R_2}) \tag{10.8b}$$

$$N_{12} = Gh(\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21}) + \frac{1}{2}\rho D(1-\nu)(\tilde{\kappa}_{12} - \frac{\tilde{\varepsilon}_{12}}{R_1}) \tag{10.8c}$$

$$N_{21} = Gh(\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21}) - \frac{1}{2}\rho D(1-\nu)(\tilde{\kappa}_{21} - \frac{\tilde{\varepsilon}_{21}}{R_2}) \tag{10.8d}$$

$$M_{11} = D(\tilde{\kappa}_{11} + \nu\tilde{\kappa}_{22} + \rho\tilde{\varepsilon}_{11}) \tag{10.8e}$$

$$M_{22} = D(\tilde{\kappa}_{22} + \nu\tilde{\kappa}_{11} - \rho\tilde{\varepsilon}_{22}) \tag{10.8f}$$

$$M_{12} = \frac{1}{2} D(1-\nu)(\tilde{\kappa}_{12} + \tilde{\kappa}_{21} + \rho\tilde{\varepsilon}_{12}) \tag{10.8g}$$

$$M_{21} = \frac{1}{2} D(1-\nu)(\tilde{\kappa}_{12} + \tilde{\kappa}_{21} - \rho\tilde{\varepsilon}_{21}) \ , \tag{10.8h}$$

is known as the Flügge-Lurie-Byrne (F-L-B) stress-strain relations for thin shells.

With $\tilde{\varepsilon}_{ij}$ and $\tilde{\kappa}_{ij}$ given in terms of $U_1(\xi_1,\xi_2)$, $U_2(\xi_1,\xi_2)$ and $W(\xi_1,\xi_2)$, the stress resultants and couples N_{ij} and M_{ij} are also given by the same three quantities by way of the F-L-B stress-strain relations. The first two moment equilibrium equations for thin shells may be used to define Q_1 and Q_2 also in terms of U_1 , U_2 and W . In that case, the three force equilibrium equations for thin shells become three partial differential equations the three middle surface displacement components. It would seem that the remaining equilibrium equation,

$$N_{12} - N_{21} + \frac{M_{12}}{R_1} - \frac{M_{21}}{R_2} = 0 \ , \tag{10.9}$$

would impose an additional relation among U_1, U_2 and W , (possibly) leading to a situation where the middle surface displacement components are overdetermined. Fortunately, this is not the case. Instead, the so-called sixth equilibrium equation is identically satisfied by the shear resultants and twisting couples if we use the F-L-B stress-strain relations to express them in terms of U_1, U_2 and W. (Readers are urged to check out this remarkable fact themselves.)

3. Stress and Strain Potentials

For the purpose of formulating variational principles, we need to express the F-L-B stress strain relations (10.8) in terms of a stress potential $S(\varepsilon_{ij}, \kappa_{ij}, \ldots)$ where ε_{ij} and κ_{ij} are the strain resultants and strain couples of the two-dimensional development in chapter (2). From (9.9) and (4.10) (specialized to lines of curvature coordinates), we see that $\tilde{\varepsilon}_{ij}$ and $\tilde{\kappa}_{ij}$ differ from ε_{ij} and κ_{ij}, respectively, only in terms involving ω, once we identify Φ_j, U_j and W with ϕ_j, u_j and ω and recall $\gamma_j = 0$ for a transversely rigid shells. The differences are such that $\tilde{\varepsilon}_{ij}$ and $\tilde{\kappa}_{ij}$ may be replaced by ε_{ij} and κ_{ij} in the F-L-B relations (10.8) as the ω terms introduced by ε_{ij} and κ_{ij} cancel out[3].

Having the F-L-B relations in terms of strain resultants and couples (and denoting them by (10.8a')-(10.8h')) it is not difficult to see that they can be written as[3]

$$N_{ij} = \frac{\partial S}{\partial \varepsilon_{ij}}, \quad M_{ij} = \frac{\partial S}{\partial \kappa_{ij}}, \quad P_j = \frac{\partial S}{\partial \lambda_j}, \quad \gamma_j = 0 \qquad (10.10)$$

for $i,j = 1,2$, with

$$\begin{aligned} S = {} & \frac{1}{2} C[\varepsilon_{11}^2(1-\frac{\rho D}{R_1 C}) + \varepsilon_{22}^2(1+\frac{\rho D}{R_2 C}) + 2\nu\varepsilon_{11}\varepsilon_{22}] \\ & + \rho D[\kappa_{11}\varepsilon_{11} - \kappa_{22}\varepsilon_{22}] + \frac{1}{2}D[\kappa_{11}^2 + \kappa_{22}^2 + 2\nu\kappa_{11}\kappa_{22}] \qquad (10.11) \\ & + \frac{1}{2}Gh[\varepsilon_{12} + \varepsilon_{21}]^2 + \frac{1}{2}\rho D(1-\nu)[(\varepsilon_{12}\kappa_{12} - \varepsilon_{21}\kappa_{21}) - \frac{1}{2}(\frac{\varepsilon_{12}^2}{R_1} - \frac{\varepsilon_{21}^2}{R_2})] \end{aligned}$$

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(3) See reference [5] at the end of Part III.

With the help of known transformation formulas for ε_{ij}, κ_{ij} and R_{ij} $(R_{ii} \equiv R_i)$, it is not difficult to transform the stress potential for lines of curvature coordinates in (10.11) into the corresponding potential for general orthogonal surface coordinates. From this new potential we obtain the F-L-B type of stress-strain relation for general orthogonal coordinates by (10.10). These more general stress-strain relations will not be recorded here but can be found in reference [5] at the end of Part III of these notes.

To formulate variational principles, we also have to be able to express the inverted form of the F-L-B relations (10.8) in terms of strain potential $C(N_{ij}, M_{ij}, \ldots)$ in the form

$$\varepsilon_{ij} = \frac{\partial C}{\partial N_{ij}}, \quad \kappa_{ij} = \frac{\partial C}{\partial M_{ij}}, \quad \gamma_j = \frac{\partial C}{\partial Q_j}, \quad P_j = 0 \tag{10.12}$$

for $i,j = 1,2$. It is not difficult to solve the four relations for N_{11}, N_{22}, M_{11} and M_{22} for ε_{11}, ε_{22}, κ_{11} and κ_{22} to get

$$\begin{aligned} \varepsilon_{11} = \frac{A}{\Delta_1}\Big\{ & N_{11}\Big[1- \frac{(\rho h)^2}{12(1-\nu^2)} + \frac{\rho h^2}{12R_2}\Big] - \nu N_{22}\Big[1- \frac{(\rho h)^2}{12(1-\nu^2)}\Big] \\ & - \rho M_{11}\Big[1+ \frac{\rho h^2}{12(1-\nu^2)R_1}\Big] + \nu\rho M_{22}\Big[\frac{\rho h^2}{12(1-\nu^2)R_2}\Big]\Big\} \end{aligned} \tag{10.13a}$$

$$\begin{aligned} \kappa_{11} = & \frac{1}{D(1-\nu^2)\Delta_1}\Big\{M_{11}\Big[1- \frac{(\rho h)^2 h^2}{144(1-\nu^2)R_1^2}\Big] - \nu M_{22}\Big[1- \frac{(\rho h)^2 h^2}{144(1-\nu^2)R_1R_2}\Big]\Big\} \\ & - \frac{\rho A}{\Delta_1}\Big\{N_{11}\Big[1+ \frac{\rho h^2}{12(1-\nu^2)R_1}\Big] - \nu N_{22}\Big[\frac{\rho h^2}{12(1-\nu^2)R_1}\Big]\Big\} \end{aligned} \tag{10.13b}$$

where

$$A = \frac{1}{Eh}, \quad \Delta_1 = 1- \frac{(\rho h)^2}{12(1-\nu^2)} - \frac{(\rho h)^2 h^2}{144(1-\nu^2)R_1R_2} \tag{10.13c}$$

with corresponding expressions for ε_{22} and κ_{22} .

To solve the remaining relations for ε_{12}, κ_{21}, ε_{12} and κ_{21} , we first add (10.8c) and (10.8d) to get

$$N_{12} + N_{21} = 2Gh(\varepsilon_{12}+\varepsilon_{21}) + \frac{1}{2}\rho D(1-\nu)[\kappa_{12}-\kappa_{21}-\frac{\varepsilon_{21}}{R_1}+\frac{\varepsilon_{21}}{R_2}] .$$

We now use the compatibility equation

$$\kappa_{12} - \kappa_{21} + \frac{\varepsilon_{21}}{R_1} - \frac{\varepsilon_{12}}{R_2} = 0 , \tag{10.14}$$

which follows from (5.7) for $\gamma_1 = \gamma_2 = 0$ and lines of curvature coordinates, to simplify the above expression to get

$$N_{12} + N_{21} = 2Gh(\varepsilon_{12}+\varepsilon_{21}) + \frac{1}{2}\rho^2 D(1-\nu)(\varepsilon_{12}+\varepsilon_{21})$$

or

$$\varepsilon_{12} + \varepsilon_{21} = \frac{1}{\Delta_2} A(1+\nu)(N_{12}+N_{21}) \tag{10.15}$$

where $\Delta_2 = 1 + (\rho h)^2/24$. Next, we use (10.14) to write (10.8g') and (10.8h') as

$$M_{12} = D(1-\nu)[\kappa_{12} + \frac{1}{2R_1}(\varepsilon_{12}-\varepsilon_{21})] \tag{10.16a}$$

$$M_{21} = D(1-\nu)[\kappa_{21} + \frac{1}{2R_2}(\varepsilon_{12} -\varepsilon_{21}] \tag{10.16b}$$

To the extent that the quantity ω does not induce any stress resultant and couple within the present approximation of the exact theory of elasticity and cannot be determined in this theory, we may choose its value so that ε_{12} is the inplane shear strain which is symmetric in its subscript. In other words, we specify the additional condition for determining ω to be the symmetry condition $\varepsilon_{12} = \varepsilon_{21}$. In that case, we get from (10.15) the remarkably simple relations

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2\Delta_2} A(1+\nu)(N_{12} + N_{21}) \qquad (10.15')$$

and from (10.16) the even simpler relations

$$M_{12} = D(1-\nu)\kappa_{12}, \quad M_{21} = D(1-\nu)\kappa_{21} \qquad (10.16a',b')$$

Note that M_{ij}, N_{ij} and x_{ij} are generally not symmetric with respect to their subscripts even though ε_{ij} is.

With (10.13), (10.15') and (10.16') we can now write down an expression for the strain potential C. Before we do so, these inverted relations may first be simplified. This is done by making use of the consistent approximations

$$\frac{1}{\Delta_1} \cong 1 + \frac{(\rho h)^2}{12(1-\nu^2)}, \quad \frac{1}{\Delta_2} \cong 1 - \frac{(\rho h)^2}{24} \qquad (10.17)$$

and by the observation that the simplified inverted relations should give us back (10.8a')-(10.8h') correctly in h and h^3 terms only but not terms involving higher powers of h. In this way, we get

$$\varepsilon_{11} = A\{N_{11}[1+ \frac{\rho h^2}{12R_2}] - \nu N_{22} - \rho M_{11}\} \qquad (10.18a)$$

$$\kappa_{11} = \frac{1}{D(1-\nu^2)}\{M_{11}-\nu M_{22}\} - \rho A N_{11} \qquad (10.18b)$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}(1+\nu)A[1- \frac{(\rho h)^2}{24}](N_{12}+N_{21}) \qquad (10.8c)$$

$$\kappa_{12} = \frac{M_{21}}{D(1-\nu)}, \qquad \kappa_{21} = \frac{M_{21}}{D(1-\nu)} \qquad (10.18d,e)$$

with a corresponding formula for ε_{22} and κ_{22}. The strain potential corresponding to (10.18) is

$$C = \frac{A}{2}\{(1+\frac{\rho h^2}{12R_2})N_{11}^2 + (1-\frac{\rho h^2}{12R_1})N_{22}^2 - 2\nu N_{11}N_{22}\}$$

$$+ \frac{1}{2}(1+\nu)[1-\frac{(\rho h)^2}{24}](N_{12}+N_{21}) + \rho A\{N_{22}M_{22} - N_{11}M_{11}\} \qquad (10.19)$$

$$+ \frac{1}{2D(1-\nu^2)}\{M_{11}^2 + M_{22}^2 - 2\nu M_{11}M_{22} + (1+\nu)(M_{12}^2+M_{21}^2)\}$$

with $\varepsilon_{ij} = \partial C/\partial N_{ij}$, etc. and $P_j = 0$, $i,j = 1,2$.

A more thorough discussion of the material in this section can be found in reference [5] at the end of Part III of these notes.

11. First Approximation Linear Stress-Strain Relations

1. Simplifications of the F-L-B Relations

The F-L-B stress-strain relations seem to contain many terms which are negligible for thin shells. For example, the last term in N_{11} is $0(h^2/R^2)$ compared to the first term and seems negligible. However, there is a distinct possibility that the solution of a specific problem is such that $C(\tilde{\varepsilon}_{11} + \nu\tilde{\varepsilon}_{22})$ is $0(\rho D\tilde{\varepsilon}_{11}/R_1)$ so that the $\tilde{\varepsilon}_{11}/R_1$ term may in fact be important (if not dominant). What this illustrates is that it is difficult to decide what are the significant terms unless we have a well-defined criterion. Throughout the subsequent development, we adopt a single criterion to decide whether or not something contributes significantly to the final solution of a given problem. The criterion is in terms of the stress state of the shell.

To describe our stress criterion, we recall that the in-plane stress components σ_{11}, σ_{22} and σ_{12} for a homogeneous, transversely rigid shell are linear in ζ . Upon writing them as

$$\sigma_{ij} = \sigma^D_{ij}(\xi_1,\xi_2) + \sigma^B_{ij}(\xi_1,\xi_2)\frac{\zeta}{(h/2)} \qquad (i,j = 1,2) \;, \tag{11.1}$$

we expect σ^D and σ^B to be of comparable magnitude when <u>direct</u> stresses and <u>bending</u> stresses both contribute significantly to the stress state of the shell. In that case, we have

$$\sigma^D = 0(\frac{N}{h}) \;, \quad \sigma^B = 0\left(\frac{6M}{h^2}\right) \tag{11.2}$$

where N and M are the corresponding stress resultant and couple, respectively.

By comparing the magnitude of σ^D and σ^B corresponding to different terms in the F-L-B stress-strain relations, we can simplify the relations for the normal resultants and couples immediately to read

$$\begin{aligned} N_{11} &= C(\tilde{\varepsilon}_{11} + \nu\tilde{\varepsilon}_{22}) , \qquad & N_{22} &= C(\tilde{\varepsilon}_{22} + \nu\tilde{\varepsilon}_{11}) \\ M_{11} &= D(\tilde{\kappa}_{11} + \nu\tilde{\kappa}_{22}) , \qquad & M_{22} &= D(\tilde{\kappa}_{22} + \nu\tilde{\kappa}_{11}) \end{aligned} \tag{11.3a-d}$$

The simplifications can be justified as follows:

i) If the terms neglected in N_{11} are important, then

$$\sigma^D_{11} = 0\left(\frac{N_{11}}{h}\right) = \frac{\rho D}{h}\, 0(\tilde{\kappa}_{11}, \tilde{\varepsilon}_{11}/R_1)$$

$$\sigma^B_{11} = 0\left(\frac{M_{11}}{h^2}\right) = \frac{D}{h^2}\, 0(\tilde{\kappa}_{11}, \rho\tilde{\varepsilon}_{11})$$

and

$$\frac{\sigma^D_{11}}{\sigma^B_{11}} = 0(\frac{h}{R})$$

where R is a representative magnitude of R_1 and R_2. Therefore, the accuracy of N_{11} itself is of no consequence insofar as the accuracy of the stress state of the shell is concerned.

(ii) If the term involving $\tilde{\varepsilon}_{11}$ in M_{11} is important, then

$$\sigma^B_{11} = \frac{D}{h^2}\, 0(\rho\tilde{\varepsilon}_{11}) , \qquad \sigma^D_{11} = \frac{C}{h}\, 0(\tilde{\varepsilon}_{11})$$

and

$$\frac{\sigma^B_{11}}{\sigma^D_{11}} = 0(\frac{D}{ChR}) = 0(\frac{h}{R})$$

Therefore, the accuracy of M_{11} itself is of no consequence.

In contrast to the above, any simplification of the relation for the shear resultants and twisting couples must be done with some care. For example, $\tilde{\varepsilon}_{12}$ and $\tilde{\varepsilon}_{21}$ individually may be large compared to the sum $\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21}$ so that the technique used to simplify N_{11}, M_{11}, etc. cannot be applied directly without some preliminary transformations. In order to simplify the expressions for shear resultants and twisting couples, we begin by writing M_{12} as

$$M_{12} = \frac{1}{2} D(1-\nu)[\tilde{\kappa}_{12} + \tilde{\kappa}_{21} + \frac{1}{2} \rho(\tilde{\varepsilon}_{12} - \tilde{\varepsilon}_{21}) + \frac{1}{2} \rho(\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21})]$$

The last term involving $\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21}$ can now be omitted in view of a similar term in N_{12}, so that

$$\boxed{M_{12} = \frac{1}{2} D(1-\nu)[\tilde{\kappa}_{12} + \tilde{\kappa}_{21} + \frac{1}{2} \rho(\tilde{\varepsilon}_{12} - \tilde{\varepsilon}_{21})]} \qquad (11.3e)$$

A similar argument gives the simplified relation

$$\boxed{M_{21} = M_{12}} \qquad (11.3f)$$

For the shear resultants, we write the F-L-B relation for N_{12} as

$$N_{12} = Gh(\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21}) + \frac{1}{4} \rho D(1-\nu)[(\tilde{\kappa}_{12} + \tilde{\kappa}_{21}) + \frac{1}{2} \rho(\tilde{\varepsilon}_{12} - \tilde{\varepsilon}_{21})$$
$$+ (\tilde{\kappa}_{12} - \tilde{\kappa}_{21} + \frac{\tilde{\varepsilon}_{21}}{R_1} - \frac{\tilde{\varepsilon}_{12}}{R_2}) + \frac{1}{2}(\frac{1}{R_2} - \frac{3}{R_1})(\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21})]$$

The last term involving $\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21}$ can be omitted in view of the very first term. From the definition of $\tilde{\kappa}_{ij}$ and $\tilde{\varepsilon}_{ij}$ $(i \neq j)$, it can be verified that

$$\tilde{\kappa}_{12} - \tilde{\kappa}_{21} + \frac{\tilde{\varepsilon}_{21}}{R_1} - \frac{\tilde{\varepsilon}_{12}}{R_2} = 0 \ .$$

What is left can be written as

$$\boxed{N_{12} = Gh(\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21}) + \frac{1}{2}\,\rho\,M_{12}} \qquad (11.3g)$$

A similar argument gives the following simplified relation for N_{21} :

$$\boxed{N_{21} = Gh(\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21}) - \frac{1}{2}\,\rho\,M_{21}} \qquad (11.3h)$$

The eight simplified relations for N_{ij} and M_{ij} derived in this section (and displayed in boxed form) or their equivalence are known as the first approximation linear stress strain relations for thin shells. It has been shown (4) that more refined or higher order relations such as the F-L-B relations for (isotropic) and transversely homogeneous thin shells are <u>inconsistent</u> with the three basic assumptions of shell theory listed in section (2). More specifically, such higher order relations do not improve upon the accuracy of the first approximation shell theory unless we also remove the assumption on the transverse shear strains and the transverse normal stress.

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(4) See reference [3] at the end of Part III.

2. First Approximation Stress Strain Relations in Terms of Strain Resultants and Strain Couples

Comparing the definition of $\tilde{\varepsilon}_{ij}$ and $\tilde{\kappa}_{ij}$ with the definition of strain resultants ε_{ij} and strain couples κ_{ij} (for l.o.c. coordinates) of the two dimensional treatment of shells, we can immediately identify $\tilde{\varepsilon}_{jj} = \varepsilon_{jj}$, $\tilde{\kappa}_{jj} = \kappa_{jj}$, $(j = 1,2)$, and

$$\tilde{\varepsilon}_{12} = \varepsilon_{12} + \omega , \qquad \tilde{\varepsilon}_{21} = \varepsilon_{21} - \omega ,$$

$$\tilde{\kappa}_{12} = \kappa_{12} + \frac{\omega}{21} , \qquad \tilde{\kappa}_{21} = \kappa_{21} - \frac{\omega}{R_2} .$$

In terms of the strain resultants and strain couples, we have for the expressions for the normal resultants and the bending couples

$$\boxed{\begin{array}{ll} N_{11} = C(\varepsilon_{11} + \nu\varepsilon_{22}) , & N_{22} = C(\varepsilon_{22} + \nu\varepsilon_{11}) \\ M_{11} = D(\kappa_{11} + \nu\kappa_{22}) , & M_{22} = D(\kappa_{22} + \nu\kappa_{11}) \end{array}} \qquad (11.3a'\text{-}d')$$

and for shear resultants and twisting couples

$$N_{12} = \frac{1}{2} (1-\nu)C(\varepsilon_{12} + \varepsilon_{21}) + \frac{1}{2}\rho M_{12} \qquad (11.3g')$$

$$N_{21} = \frac{1}{2} (1-\nu)C(\varepsilon_{12} + \varepsilon_{21}) - \frac{1}{2}\rho M_{21} \qquad (11.3h')$$

$$M_{12} = M_{21} = \frac{1}{2} (1-\nu)D[\kappa_{12} + \kappa_{21} + \frac{1}{2}\rho (\varepsilon_{12} - \varepsilon_{21})] \qquad (11.3e')$$

That ω is no longer an independent kinematic field quantity is consistent with the absence of stress couples about the midsurface normal. With ϕ_1, ϕ_2 and ω expressed in terms of u_1, u_2 and w (though the conditions $\gamma_1 = \gamma_2 = \varepsilon_{12} - \varepsilon_{21} = 0$), we may now identify u_k and w with U_k and W respectively.

Now, if we want ε_{ij} to have the meaning of midsurface strain components with ε_{12} being the midsurface shear strain, then we must insist $\varepsilon_{12} = \varepsilon_{21}$. In that case, the $\varepsilon_{12} - \varepsilon_{21}$ term in the expression for M_{12} and M_{21} drops out. In view of the fact $M_{12} = M_{21}$ in our first approximation shell theory, it is desirable to have $\varepsilon_{12} = \varepsilon_{21}$ to maintain static geometric duality.

The relations for N_{ij} can then be inverted to read

$$\boxed{\begin{gathered} \varepsilon_{11} = A(N_{11} - \nu N_{22}) \qquad \varepsilon_{22} = A(N_{22} - \nu N_{11}) \\ \varepsilon_{12} = \varepsilon_{21} = \frac{1}{2} A(1+\nu)(N_{12} + N_{21}) \end{gathered}} \tag{11.4a-c}$$

where $A = 1/Eh$. The relations for M_{ij} are simplified to read

$$\boxed{\begin{gathered} M_{22} = D(\kappa_{22} + \nu\kappa_{11}) \, , \qquad M_{11} = D(\kappa_{11} + \nu\kappa_{22}) \\ M_{12} = M_{21} = \frac{1}{2} D(1-\nu)(\kappa_{12} + \kappa_{21}) \end{gathered}} \tag{11.4d-f}$$

These two sets of relations are in fact the static geometric dual of each other.

Finally, the condition $\varepsilon_{12} = \varepsilon_{21}$ determines ω in terms of u_1 and u_2,

$$2\omega = \left(\frac{u_{2,1}}{\alpha_1} - \frac{\alpha_{1,2}u_1}{\alpha_1\alpha_2}\right) - \left(\frac{u_{1,2}}{\alpha_2} - \frac{\alpha_{2,1}u_2}{\alpha_1\alpha_2}\right) .$$

3. Stress and Strain Potentials

For the purpose of expressing the stress-strain relations (11.3a'-h') in terms of a stress potential, we first write (11.3g') and (11.3h') as

$$N_{12} = \tfrac{1}{2}(1-\nu)C(\varepsilon_{12}+\varepsilon_{21}) + \tfrac{1}{4}\rho D(1-\nu)(\kappa_{12}+\kappa_{21}) \tag{11.3g'}$$

$$N_{21} = \tfrac{1}{2}(1-\nu)C(\varepsilon_{12}+\varepsilon_{21}) - \tfrac{1}{4}\rho D(1-\nu)(\kappa_{12}+\kappa_{21}) \tag{11.3h'}$$

The stress potential for the system (11.3) is easily seen to be

$$\begin{aligned} S = {} & \tfrac{1}{2}C(\varepsilon_{11}^2+\varepsilon_{22}^2+2\nu\varepsilon_{11}\varepsilon_{22}) + \tfrac{1}{4}(1-\nu)C(\varepsilon_{12}+\varepsilon_{21})^2 \\ & + \tfrac{1}{2}D(\kappa_{11}^2+\kappa_{22}^2+2\nu\kappa_{11}\kappa_{22}) + \tfrac{1}{4}(1-\nu)D(\kappa_{12}+\kappa_{21})^2 \\ & + \tfrac{1}{4}(1-\nu)D\rho(\kappa_{12}+\kappa_{21})(\varepsilon_{12}-\varepsilon_{21}) \end{aligned}$$

with $N_{ij} = \partial S/\partial\varepsilon_{ij}$ etc. and with $\varepsilon_{12} = \varepsilon_{21}$ being a consequence of moment equilibrium equation (10.9).

To construct the strain potential, we invert (11.4d) and (11.4e) to get

$$\kappa_{11} = \frac{M_{11}-\nu M_{22}}{D(1-\nu^2)}, \qquad \kappa_{22} = \frac{M_{22}-\nu M_{11}}{D(1-\nu^2)} . \tag{11.6a,b}$$

With $\varepsilon_{12} = \varepsilon_{21}$, (11.3e'), (10.14) and (11.4c) give

$$\kappa_{12} = \frac{M_{12}}{D(1-\nu)} + \frac{1}{4} A(1+\nu)\rho(N_{12}+N_{21}) \tag{11.6c}$$

$$\kappa_{21} = \frac{M_{21}}{D(1-\nu)} - \frac{1}{4} A(1+\nu)\rho(N_{12}+N_{21}) \tag{11.6d}$$

With (11.4a-c) and (11.6a-d) it is not difficult to see that the strain potential is given by

$$
\begin{aligned}
C = {} & \tfrac{1}{2}A[N_{11}^2+N_{22}^2-2\nu N_{11}N_{22}] + \frac{1}{2D(1-\nu^2)}[M_{11}^2+M_{22}^2-2\nu M_{11}M_{22}] \\
& + \tfrac{1}{4}A(1+\nu)(N_{12}+N_{21})^2 + \frac{1}{2(1-\nu)D}(M_{12}^2+M_{21}^2) \\
& + \tfrac{1}{4}A(1+\nu)\rho(N_{12}+N_{21})(M_{12}^2+M_{21}^2)
\end{aligned}
\qquad (11.7)
$$

with $\varepsilon_{ij} = C/\ N_{ij}$, etc., and with $M_{12} = M_{21}$ being a consequence of the compatibility equation (10.14).

The corresponding stress and strain potentials for general orthogonal surface coordinates may be obtained from (11.5) and (11.7), respectively, with the help of standard transformation formulas. A more extensive discussion of the development in this and the last section can be found in reference [6] at the end of Part III of these notes.

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