A Note on Stress Strain Relations of the Linear Theory of Shells[1])

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In a recent paper [1][2]), we have shown, among other things, that the lines-of-curvature stress strain relations of FLÜGGE, LURJE, and BYRNE,

$$N_{11} = C\,(\tilde{\varepsilon}_{11} + \nu\,\tilde{\varepsilon}_{22}) + \varrho\,D\left(\tilde{\varkappa}_{11} - \frac{\tilde{\varepsilon}_{11}}{R_1}\right), \quad M_{11} = D\,(\tilde{\varkappa}_{11} + \nu\,\tilde{\varkappa}_{22} + \varrho\,\tilde{\varepsilon}_{11})\,, \tag{1}$$

$$\left.\begin{aligned} N_{12} &= \frac{1}{2}\,(1-\nu)\,C\,(\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21}) + \frac{1}{2}\,(1-\nu)\,\varrho\,D\left(\tilde{\varkappa}_{12} - \frac{\tilde{\varepsilon}_{12}}{R_1}\right), \\ M_{12} &= \frac{1}{2}\,(1-\nu)\,D\,(\tilde{\varkappa}_{12} + \tilde{\varkappa}_{21} + \varrho\,\tilde{\varepsilon}_{12}) \end{aligned}\right\} \tag{2}$$

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2) Numbers in brackets refer to References, page 681.

etc., where

$$C = \frac{E\,h}{1-\nu^2}, \quad D = \frac{E\,h^3}{12\,(1-\nu^2)}, \quad \varrho = \frac{1}{R_2} - \frac{1}{R_1}$$

could, in inverted form, be written as a system of relations

$$\varepsilon_{jk} = \frac{\partial W}{\partial N_{jk}}, \quad \varkappa_{jk} = \frac{\partial W}{\partial M_{jk}} \tag{3}$$

where the ε_{jk} and $\varkappa_{jk}$, in addition to the usual translational displacement components, involved an angular displacement component ω, such that $\varepsilon_{jj} = \tilde{\varepsilon}_{jj}$, $\varkappa_{jj} = \tilde{\varkappa}_{jj}$, while

$$\varepsilon_{12} = \tilde{\varepsilon}_{12} - \omega, \quad \varepsilon_{21} = \tilde{\varepsilon}_{21} + \omega, \quad \varkappa_{12} = \tilde{\varkappa}_{12} - \frac{\omega}{R_1}, \quad \varkappa_{21} = \tilde{\varkappa}_{21} + \frac{\omega}{R_2}. \tag{4}$$

The inversion of part (1) of the FLB system is straightforward and does not lead to unexpected results. The inversion of part (2) of the system is not straightforward to the same degree and leads to results of a form somewhat simpler than expected, namely

$$\varepsilon_{12} = \varepsilon_{21} = \frac{(N_{12} + N_{21})}{2\,(1-\nu)\,C\,\Delta_2}, \quad \varkappa_{12} = \frac{M_{12}}{(1-\nu)\,D}, \quad \varkappa_{21} = \frac{M_{21}}{(1-\nu)\,D} \tag{5}$$

where

$$\Delta_2 = 1 + \frac{1}{24}\,(\varrho\,h)^2 .$$

In the present note, we show that a corresponding inversion, with somewhat less simple final results, is possible for a system of stress-strain relations which has been proposed as one combining simplicity and adequacy [2]. This system follows upon omitting terms with ϱ in (1) and upon writing in (2)

$$\tilde{\varepsilon}_{12} = \frac{1}{2}\,(\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21}) + \frac{1}{2}\,(\tilde{\varepsilon}_{12} - \tilde{\varepsilon}_{21}) \approx \frac{1}{2}\,(\tilde{\varepsilon}_{12} - \tilde{\varepsilon}_{21}), \tag{6}$$

$$\left.\begin{aligned} \tilde{\varkappa}_{12} = \frac{1}{2}\,(\tilde{\varkappa}_{12} + \tilde{\varkappa}_{21}) + \frac{1}{2}\,(\tilde{\varkappa}_{12} - \tilde{\varkappa}_{21}) &= \frac{1}{2}\,(\tilde{\varkappa}_{12} + \tilde{\varkappa}_{21}) + \frac{1}{2}\left(\frac{\tilde{\varepsilon}_{12}}{R_2} - \frac{\tilde{\varepsilon}_{21}}{R_1}\right) \\ &\approx \frac{1}{2}\,(\tilde{\varkappa}_{12} + \tilde{\varkappa}_{21}) + \frac{1}{4}\left(\frac{1}{R_2} + \frac{1}{R_1}\right)(\tilde{\varepsilon}_{12} - \tilde{\varepsilon}_{21}) \end{aligned}\right\} \tag{7}$$

with corresponding modifications of $\tilde{\varepsilon}_{21}$ and $\tilde{\varkappa}_{21}$. By this Equations (2) are changed to

$$M_{12} = M_{21} = \frac{1}{2}\,(1-\nu)\,D\left[\tilde{\varkappa}_{12} + \tilde{\varkappa}_{21} + \frac{1}{2}\,\varrho\,(\tilde{\varepsilon}_{12} - \tilde{\varepsilon}_{21})\right], \tag{8}$$

$$N_{12} = \frac{1}{2}\,(1-\nu)\,C\,(\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21}) + \frac{1}{4}\,(1-\nu)\,\varrho\,D\left[\tilde{\varkappa}_{12} + \tilde{\varkappa}_{21} + \frac{1}{2}\,\varrho\,(\tilde{\varepsilon}_{12} - \tilde{\varepsilon}_{21})\right], \tag{9}$$

N_{21} being given analogously. Equation (9) and the corresponding equation for N_{21} may alternately be written as

$$(N_{12}, N_{21}) = \frac{1}{2}\,(1-\nu)\,C\,(\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21}) + \frac{1}{2}\,(\varrho\,M_{12}, -\varrho\,M_{21}). \tag{9'}$$

We note particularly, that now $M_{12} = M_{21}$, while N_{12} and N_{21} as given in (9′) are consistent with the moment equilibrium equation

$$N_{12} - N_{21} + \frac{M_{12}}{R_1} - \frac{M_{21}}{R_2} = 0. \tag{10}$$

While it is possible to simply state the final results of our inversion procedure and then verify their correctness, it is of interest here to give the steps leading to these results. We first solve (8) and (9) in the form

$$\begin{pmatrix}\tilde{\varepsilon}_{12}\\ \tilde{\varepsilon}_{21}\end{pmatrix} = \frac{N_{12}+N_{21}}{2\,(1-\nu)\,C} \pm \Omega\,, \quad \begin{pmatrix}\tilde{\varkappa}_{12}\\ \tilde{\varkappa}_{21}\end{pmatrix} = \frac{1}{(1-\nu)\,D}\begin{pmatrix}M_{12}\\ M_{21}\end{pmatrix} - \frac{\varrho\,\Omega}{2} \pm \Lambda \tag{11}$$

where Ω and Λ are arbitrary functions. We then use the compatibility relation

$$\tilde{\varkappa}_{12} - \tilde{\varkappa}_{21} = \frac{\tilde{\varepsilon}_{12}}{R_2} - \frac{\tilde{\varepsilon}_{21}}{R_1} \tag{12}$$

to express Λ in terms of Ω,

$$\Lambda = \left(\frac{1}{R_2} + \frac{1}{R_1}\right)\frac{\Omega}{2} + \frac{\varrho}{2}\,\frac{N_{12}+N_{21}}{2\,(1-\nu)\,C}\,. \tag{13}$$

With this, we write

$$\tilde{\varkappa}_{12} = \frac{M_{12}}{(1-\nu)\,D} + \frac{\varrho}{2}\,\frac{N_{12}+N_{21}}{2\,(1-\nu)\,C} + \frac{\Omega}{R_1}\,, \qquad \tilde{\varkappa}_{21} = \frac{M_{21}}{(1-\nu)\,D} - \frac{\varrho}{2}\,\frac{N_{12}+N_{21}}{2\,(1-\nu)\,C} - \frac{\Omega}{R_2}\,. \tag{14}$$

We now see that it is consistent to identify Ω with the angular displacement component ω in (4) and write, in place of (14) and the first part of (11)

$$\varepsilon_{12} = \varepsilon_{21} = \frac{N_{12}+N_{21}}{2\,(1-\nu)\,C}\,, \tag{15}$$

$$\varkappa_{12} = \frac{M_{12}}{(1-\nu)\,D} + \varrho\,\frac{N_{12}+N_{21}}{4\,(1-\nu)\,C}\,, \qquad \varkappa_{21} = \frac{M_{21}}{(1-\nu)\,D} - \varrho\,\frac{N_{12}+N_{21}}{4\,(1-\nu)\,C}\,. \tag{16}$$

The system (15) and (16) is not yet in such a form that we can write it in terms of a strain energy function W, as in (3). The appropriate modification consists in making use of the symmetry relation $M_{12} = M_{21}$ in order to write ε_{12} and ε_{21} as

$$\varepsilon_{12} = \varepsilon_{21} = \frac{N_{12}+N_{21}}{2\,(1-\nu)\,C} + \varrho\,\frac{M_{12}-M_{21}}{4\,(1-\nu)\,C}\,. \tag{17}$$

In this and in (16) the relation $M_{12} = M_{21}$ is now to be considered not as an identity but as a consequence of the finite compatibility Equation (12).

With (16), (17) and the inverted form of the abbreviated version of Equations (1), we now have for the function W in (3)

$$\left.\begin{aligned} W = {} & \frac{N_{11}^2 + N_{22}^2 - 2\,\nu\,N_{11}\,N_{22}}{2\,(1-\nu^2)\,C} + \frac{M_{11}^2 + M_{22}^2 - 2\,\nu\,M_{11}\,M_{22}}{2\,(1-\nu^2)\,D} \\ & + \frac{(N_{12}+N_{21})^2}{4\,(1-\nu)\,C} + \frac{M_{12}^2 + M_{21}^2}{2\,(1-\nu)\,D} + \frac{\varrho\,(N_{12}+N_{21})\,(M_{12}-M_{21})}{4\,(1-\nu)\,C}\,. \end{aligned}\right\} \tag{18}$$

The step from lines of curvature coordinates to general orthogonal coordinates may be carried out in analogy to what has been done in [1] for the relations of FLÜGGE, LURJE, and BYRNE. Using the invariance relations for the states N_{jk} and M_{jk}, together with the further invariant

$$\left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)(N_{12}+N_{21}) + \frac{2}{R_{12}}\,(N_{11}-N_{22})$$

we obtain

$$\left.\begin{aligned} W = {} & \frac{N_{11}^2 + N_{22}^2 - 2\,\nu\,N_{11}\,N_{22}}{2\,(1-\nu^2)\,C} + \frac{(N_{12}+N_{21})^2}{4\,(1-\nu)\,C} \\ & + \frac{M_{11}^2 + M_{22}^2 - 2\,\nu\,M_{11}\,M_{22}}{2\,(1-\nu^2)\,D} + \frac{M_{12}^2 + M_{21}^2}{2\,(1-\nu)\,D} \\ & + \frac{M_{12}-M_{21}}{4\,(1-\nu)\,C}\left[\left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)(N_{12}+N_{21}) + \frac{2}{R_{12}}\,(N_{11}-N_{22})\right]. \end{aligned}\right\} \quad (19)$$

From this follow as stress strain relations which generalize (16) and (17) to general orthogonal coordinates

$$\left.\begin{aligned} & \varepsilon_{11} = \frac{N_{11} - \nu\,N_{22}}{(1-\nu^2)\,C} + \frac{1}{R_{12}}\,\frac{M_{12}-M_{21}}{2\,(1-\nu)\,C}\,, \qquad \varepsilon_{22} = \frac{N_{22} - \nu\,N_{11}}{(1-\nu^2)\,C} - \frac{1}{R_{12}}\,\frac{M_{12}-M_{21}}{2\,(1-\nu)\,C}\,, \\ & \varepsilon_{12} = \varepsilon_{21} = \frac{N_{12}+N_{21}}{2\,(1-\nu)\,C} + \left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)\frac{M_{12}-M_{21}}{4\,(1-\nu)\,C}\,, \\ & \varkappa_{11} = \frac{M_{11} - \nu\,M_{22}}{(1-\nu^2)\,D}\,, \qquad \varkappa_{22} = \frac{M_{22} - \nu\,M_{11}}{(1-\nu^2)\,D}\,, \\ & \varkappa_{12} = \frac{M_{12}}{(1-\nu)\,D} + \left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)\frac{N_{12}+N_{21}}{4\,(1-\nu)\,C} + \frac{1}{R_{12}}\,\frac{N_{11}-N_{22}}{2\,(1-\nu)\,C}\,, \\ & \varkappa_{21} = \frac{M_{21}}{(1-\nu)\,D} - \left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)\frac{N_{12}+N_{21}}{4\,(1-\nu)\,C} - \frac{1}{R_{12}}\,\frac{N_{11}-N_{22}}{2\,(1-\nu)\,C} \end{aligned}\right\} \quad (20)$$

where the quantities $\varepsilon_{j\,k}$ and $\varkappa_{j\,k}$ are as in [3],

$$\left.\begin{aligned} & \varepsilon_{11} = \frac{u_{1,\,1}}{\alpha_1} + \frac{\alpha_{1,\,2}\,u_2}{\alpha_1\,\alpha_2} + \frac{w}{R_{11}}\,, && \varepsilon_{22} = \frac{u_{2,\,2}}{\alpha_2} + \frac{\alpha_{2,\,1}\,u_1}{\alpha_1\,\alpha_2} + \frac{w}{R_{22}}\,, \\ & \varepsilon_{12} = \frac{u_{2,\,1}}{\alpha_1} - \frac{\alpha_{1,\,2}\,u_1}{\alpha_1\,\alpha_2} + \frac{w}{R_{12}} - \omega\,, && \varepsilon_{21} = \frac{u_{1,\,2}}{\alpha_2} - \frac{\alpha_{2,\,1}\,u_2}{\alpha_1\,\alpha_2} + \frac{w}{R_{12}} + \omega\,, \\ & \varkappa_{11} = \frac{\phi_{1,\,1}}{\alpha_1} + \frac{\alpha_{1,\,2}\,\phi_2}{\alpha_1\,\alpha_2} + \frac{\omega}{R_{12}}\,, && \varkappa_{22} = \frac{\phi_{2,\,2}}{\alpha_2} + \frac{\alpha_{2,\,1}\,\phi_1}{\alpha_1\,\alpha_2} - \frac{\omega}{R_{12}}\,, \\ & \varkappa_{12} = \frac{\phi_{2,\,1}}{\alpha_1} - \frac{\alpha_{1,\,2}\,\phi_1}{\alpha_1\,\alpha_2} - \frac{\omega}{R_{11}}\,, && \varkappa_{21} = \frac{\phi_{1,\,2}}{\alpha_2} - \frac{\alpha_{2,\,1}\,\phi_2}{\alpha_1\,\alpha_2} + \frac{\omega}{R_{22}} \end{aligned}\right\} \quad (21)$$

with $\phi_1 + w_1/\alpha_1 - u_1/R_{11} - u_2/R_{12} = 0$, and ϕ_2 defined analogously. Since $M_{12} - M_{21}$ is an invariant, the symmetry condition $M_{12} = M_{21}$ continues to hold for general orthogonal coordinates and the terms involving $(M_{12} - M_{21})$ may again be omitted in the expressions for $\varepsilon_{j\,k}$.

The following further consideration may also be of some interest. Returning again to lines of curvature coordinates and to the stress strain relations (8) and (9) written in the form

$$M_{12} = M_{21} = \frac{1}{2}\,(1-\nu)\,D\,(\varkappa_{12} + \varkappa_{21})\,, \quad (22)$$

$$(N_{12}, N_{21}) = \frac{1}{2}\,(1-\nu)\,C\,(\varepsilon_{12} + \varepsilon_{21}) + \frac{1}{4}\,(1-\nu)\,D\,(\varrho, -\varrho)\,(\varkappa_{12} + \varkappa_{21}) \quad (23)$$

we note that we may write these relations together with the relations $N_{11} = C(\varepsilon_{11} + \nu\,\varepsilon_{22})$, $M_{11} = D(\varkappa_{11} + \nu\,\varkappa_{22})$, etc., in the form

$$N_{jk} = \frac{\partial U}{\partial \varepsilon_{jk}}\,, \qquad M_{jk} = \frac{\partial U}{\partial \varkappa_{jk}} \tag{24}$$

provided we first observe that we may replace (22) by

$$M_{12} = M_{21} = \frac{1}{2}\,(1-\nu)\,D\,(\varkappa_{12} + \varkappa_{21}) + \frac{1}{4}\,(1-\nu)\,D\,\varrho\,(\varepsilon_{12} - \varepsilon_{21}) \tag{22'}$$

in view of the fact that (18) implies $\varepsilon_{12} - \varepsilon_{21} = 0$. With this, we have for lines of curvature coordinates

$$\left.\begin{aligned} U = {} & \frac{1}{2}\,C\,(\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\,\nu\,\varepsilon_{11}\,\varepsilon_{22}) + \frac{1}{4}\,(1-\nu)\,C\,(\varepsilon_{12} + \varepsilon_{21})^2 \\ & + \frac{1}{2}\,D\,(\varkappa_{11}^2 + \varkappa_{22}^2 + 2\,\nu\,\varkappa_{11}\,\varkappa_{22}) + \frac{1}{4}\,(1-\nu)\,D\,(\varkappa_{12} + \varkappa_{21})^2 \\ & + \frac{1}{4}\,(1-\nu)\,D\,\varrho\,(\varkappa_{12} + \varkappa_{21})\,(\varepsilon_{12} - \varepsilon_{21})\,. \end{aligned}\right\} \tag{25}$$

In the resulting stress strain relations, $\varepsilon_{12} = \varepsilon_{21}$ is to be considered as a consequence of the finite moment equilibrium Equation (10).

The corresponding function U for general orthogonal coordinates follows from the invariants for the ε_{jk} and $\varkappa_{jk}$ together with the supplementary invariant

$$\left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)(\varkappa_{12} + \varkappa_{21}) + \frac{2}{R_{12}}\,(\varkappa_{11} - \varkappa_{22})$$

as

$$\left.\begin{aligned} U = {} & \frac{1}{2}\,C\,(\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\,\nu\,\varepsilon_{11}\,\varepsilon_{22}) + \frac{1}{4}\,(1-\nu)\,C\,(\varepsilon_{12} + \varepsilon_{21})^2 \\ & + \frac{1}{2}\,D\,(\varkappa_{11}^2 + \varkappa_{22}^2 + 2\,\nu\,\varkappa_{11}\,\varkappa_{22}) + \frac{1}{4}\,(1-\nu)\,D\,(\varkappa_{12} + \varkappa_{21})^2 \\ & + \frac{1}{4}\,(1-\nu)\,D\,(\varepsilon_{12} - \varepsilon_{21})\left[\left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)(\varkappa_{12} + \varkappa_{21}) + \frac{2}{R_{12}}\,(\varkappa_{11} - \varkappa_{22})\right]. \end{aligned}\right\} \tag{26}$$

Introduction of this form of U into (24) gives as stress strain relations which generalize (23) and (22') to the case of arbitrary orthogonal coordinates

$$\left.\begin{aligned} (N_{12}, N_{21}) = {} & \frac{1}{2}\,(1-\nu)\,C\,(\varepsilon_{12} + \varepsilon_{21}) + \frac{1}{4}\,(1-\nu)\,D\,(+1, -1) \\ & \times \left[\left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)(\varkappa_{12} + \varkappa_{21}) + \frac{2}{R_{12}}\,(\varkappa_{11} - \varkappa_{22})\right] \\ = {} & \frac{1}{2}\,(1-\nu)\,C\,(\varepsilon_{12} + \varepsilon_{21}) \\ & + \left(+\frac{1}{2}, -\frac{1}{2}\right)\left[\frac{M_{11} - M_{22}}{R_{12}} + \left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right) M_{12}\right], \end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned} M_{12} = M_{21} = {} & \frac{1}{2}\,(1-\nu)\,D\,(\varkappa_{12} + \varkappa_{21}) + \frac{1}{4}\,(1-\nu)\,D\left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)(\varepsilon_{12} - \varepsilon_{21}) \\ = {} & \frac{1}{2}\,(1-\nu)\,D\,(\varkappa_{12} + \varkappa_{21}) \end{aligned}\right\} \tag{28}$$

while the remaining relations come out to be

$$N_{11} = C\,(\varepsilon_{11} + \nu\,\varepsilon_{22})\,, \tag{29}$$

$$M_{11} = D\,(\varkappa_{11} + \nu\,\varkappa_{22}) + \frac{1}{2}\,(1-\nu)\,D\,\frac{(\varepsilon_{12} - \varepsilon_{21})}{R_{12}} = D\,(\varkappa_{11} + \nu\,\varkappa_{22}) \tag{30}$$

with analogous expressions for N_{22} and M_{22}.

Equations (27) to (30) can be shown to coincide with the relations stated by KOITER [2] upon setting $\omega = 1/2\,(\tilde{\varepsilon}_{12} - \tilde{\varepsilon}_{21})$ and upon observing that our $\varkappa_{12} + \varkappa_{21}$, $\varepsilon_{12} + \varepsilon_{21}$, ω, $M_{12} = M_{21}$, and $1/2\,(N_{12} + N_{21})$ correspond to Koiter's τ, ψ, Ω, W, and S, respectively. A point of interest in this derivation of a system of relations analogous to Koiter's is that in the present formulation N_{12} and N_{21} come out as two separate quantities directly, instead of getting the sum $N_{12} + N_{21}$ from a stress strain relation, to be combined subsequently with the moment equilibrium equation giving $N_{12} - N_{21}$.

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Zusammenfassung

Es wird gezeigt, wie man gewisse bekannte Elastizitätsgleichungen der linearen Schalentheorie [2] umschreiben kann mit Hilfe von verallgemeinerten Dehnungskomponenten und verallgemeinerten Energiefunktionen, welche von dem erstgenannten Verfasser [3] eingeführt worden sind.

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