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A note on the linear theory of shallow shear-deformable shells

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Introduction

The starting point of what follows is an earlier consideration of the equations of linear shallow shell theory including the effects of transverse shear deformability and of moment components normal to the shell surface [2]. Given the relatively greater technical importance of the first-named effect, and given certain recent advances in the formulation of the problem of shear-deformable plates [3], we now undertake a reconsideration of the appropriate special case of the results in [2], in such a way that the final form of the equations of this tenth-order theory shell theory contains both the eight order classical results, and the sixth order theory of shear-deformable plates as special cases. While we limit consideration to the problem of shells which are two-dimensionally homogeneous and isotropic it will be apparent that analogous results may be obtained without these assumptions of homogeneity and isotropy.

Basic equations

We have as equilibrium equations of linear shallow shell theory

$$N_{11,1}+N_{21,2}=0\,,\quad N_{12,1}+N_{22,2}=0\,,\quad N_{12}=N_{21}\,, \tag{1}$$

$$Q_{1,1}+Q_{2,2}+z_{,11}N_{11}+z_{,12}(N_{12}+N_{21})+z_{,22}N_{22}+q=0\,, \tag{2}$$

$$M_{11,1}+M_{21,2}=Q_1\,,\quad M_{12,1}+M_{22,2}=Q_2\,, \tag{3}$$

and we take stress strain relations in the form

$$\varepsilon_{11}=B(N_{11}-\nu N_{22})\,,\quad \varepsilon_{12}=\varepsilon_{21}=\tfrac{1}{2}(1+\nu)B(N_{12}+N_{21})\,,\quad \gamma_1=AQ_1\,, \tag{4}$$

$$M_{11}=D(\varkappa_{11}+\nu\varkappa_{22})\,,\quad M_{12}=M_{21}=\tfrac{1}{2}(1-\nu)D(\varkappa_{12}+\varkappa_{21})\,, \tag{5}$$

with corresponding expressions for ε_{22}, γ_2 and M_{22}. Equations (1) to (5) are complemented by strain displacement relations,

$$\varepsilon_{11}=u_{1,1}-z_{,11}w\,,\quad \varepsilon_{12}+\varepsilon_{21}=u_{1,2}+u_{2,1}-2z_{,12}w\,, \tag{6}$$

$$\gamma_1=w_{,1}+\Phi_1\,,\quad \varkappa_{11}=\Phi_{1,1}\,,\quad \varkappa_{12}=\Phi_{2,1}\,, \tag{7}$$

with corresponding expressions for ε_{22}, γ_2, $\varkappa_{21}$, $\varkappa_{22}$.

In the above $z=z(x_1,x_2)$ represents the middle surface of the shell, the u_i and w are tangential and transverse displacement components, the Φ_i are rotational displacements, the N_{ij} and Q_i are tangential and transverse stress resultants and the M_{ij} are stress couples.

We note that while the transverse component w and the tangential components N_{ij} are effectively equivalent to base plane perpendicular and parallel components, respectively, there is no such equivalent for the transverse components Q_i and the tangential components u_i.

Reduction of differential equations

We satisfy (1) in terms of an Airy stress function by setting

$$N_{11} = K_{,22}, \quad N_{22} = K_{,11}, \quad N_{12} = N_{21} = -K_{,12}, \tag{8}$$

and we note that (6) and (7) imply a compatibility equation of the form $\varepsilon_{11,22} - (\varepsilon_{12} + \varepsilon_{21})_{,12} + \varepsilon_{22,11} + Lw = 0$ where

$$L \equiv z_{,22}(\quad)_{,11} - 2z_{,12}(\quad)_{,12} + z_{,11}(\quad)_{,22}\,. \tag{9}$$

Therewith, and with (4), we obtain as one of three differential equations governing the problem of the shallow shear-deformable shell, the same as in the theory without transverse shear deformability,

$$B\nabla^2\nabla^2 K + Lw = 0\,. \tag{10}$$

In order to obtain the remaining two differential equations we begin by combining (5) and (7) in the form

$$\begin{aligned} M_{11} &= -D(w_{,11} + \nu w_{,22}) + AD(Q_{1,1} + \nu Q_{2,2})\,, \\ M_{12} &= -(1-\nu)Dw_{,12} + \tfrac{1}{2}(1-\nu)AD(Q_{2,1} + Q_{1,2})\,, \end{aligned} \tag{11}$$

with a corresponding expression for M_{22}, and we use (11), in conjunction with (3), so as to have

$$\begin{aligned} Q_1 &= -D(\nabla^2 w)_{,1} + AD[\nabla^2 Q_1 - \tfrac{1}{2}(1+\nu)(Q_{1,2} - Q_{2,1})_{,2}]\,, \\ Q_2 &= -D(\nabla^2 w)_{,2} + AD[\nabla^2 Q_2 + \tfrac{1}{2}(1+\nu)(Q_{1,2} - Q_{2,1})_{,1}]\,. \end{aligned} \tag{12}$$

Equations (12) evidently imply the relation $(1 - AD\nabla^2)(Q_{1,1} + Q_{2,2}) = -D\nabla^2\nabla^2 w$ and this, in conjunction with (2), (8), and (9), gives as the second of three differential equations.

$$D\nabla^2\nabla^2 w - (1 - AD\nabla^2)(LK + q) = 0\,. \tag{13}$$

We may note, as before [2], that equations (10) and (13) in conjunction with (12) have earlier been given by Naghdi [1], and that it remains to reduce the twelfth order system (10), (12), and (13) to one of tenth order. We here accomplish this reduction, essentially as in [2] and in [4 (p. 52)], by the introduction of a function χ with defining relation

$$\chi = \tfrac{1}{2}(1-\nu)AD(Q_{1,2} - Q_{2,1})\,. \tag{14}$$

Having (14) we obtain from (12) as differential equation for χ, and as the third differential equation of the system of three,

$$\tfrac{1}{2}(1-\nu)AD\nabla^2\chi - \chi = 0\,, \tag{15}$$

where it remains to express the quantities Q_i and M_{ij} in terms of w, K *and* χ. We accomplish this by deducing from (2), in conjunction with (8) and (9), $Q_{1,11} = -Q_{2,21} - (LK+q)_{,1}$ and therewith $\nabla^2 Q_1 = (Q_{1,2} - Q_{2,1})_{,2} - (LK+q)_{,1}$, with a corresponding expression for $\nabla^2 Q_2$. This, in conjunction with the defining relation (14) then gives, with the further defining relation

$$v = w + AD[\nabla^2 w + A(LK + q)], \tag{16}$$

in place of equations (12),

$$Q_1 = -D(\nabla^2 v)_{,1} + \chi_{,2}, \quad Q_2 = -D(\nabla^2 v)_{,2} - \chi_{,1}. \tag{17}^*$$

and in place of equations (11)

$$\begin{aligned} M_{11} &= -D(v_{,11} + \nu v_{,22}) + (1-\nu)AD\,\chi_{,12}, \\ M_{22} &= -D(v_{,22} + \nu v_{,11}) - (1-\nu)AD\,\chi_{,12}, \\ M_{12} &= -(1-\nu)D\,v_{,12} + \tfrac{1}{2}(1-\nu)AD(\chi_{,22} - \chi_{,11}). \end{aligned} \tag{18}$$

We recover the previously known results for shear-deformable plates upon setting $L \equiv 0$, and for shells with absent transverse shear deformability upon setting $A \equiv 0$ and therewith $\chi \equiv 0$.

References

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Abstract

The solution of the tenth order problem of two-dimensionally homogeneous isotropic shallow shear-deformable elastic shells is expressed in terms of three functions, one of them solution of a second order equation and two of them solutions of two simultaneous fourth order equations, in such a way that known results for shear-deformable plates and for non-shear-deformable shells appear directly as special cases.

Zusammenfassung

Die Lösung des Problems der zwei-dimensional isotropischen flachen elastischen Schale wird ausgedrückt durch drei Funktionen. Zwei von diesen sind Lösungen von zwei simultanen Gleichungen der vierten Ordnung und die dritte ist Lösung einer Gleichung der zweiten Ordnung. Das neue in diesem Resultat, welches schon einmal in 1969 in einer etwas anderen Art und in einem allgemeineren Zusammenhang gegeben wurde, besteht darin, daß in der neuen Fassung bekannte Ergebnisse für schubverformbare Platten und für nichtschubverformbare Schalen unmittelbar als spezielle Fälle erhalten werden.

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*) We can here, with the help of (13), replace $\nabla^2 v$ by $\nabla^2 w + A(LK + q)$ and in this way avoid the appearance of fifth derivatives of w in the expressions for Q_i.