XXXIII

The Dimpling of Spherical Caps

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Summary. The small finite deformation of a homogeneous, isotropic, thin elastic spherical cap subject to an axisymmetric external pressure distribution superimposed upon a uniform internal pressure distribution is analyzed in this note. For a certain range of loading and shell geometry, it is shown that the cap dimples in a region around the apex and this dimple-type deformation can be adequately described by an asymptotic solution of the relevant nonlinear two-point boundary-value problem consisting of two different inextensional bending solutions, each for a different part of the shell, a boundary-layer solution for the satisfaction of the boundary conditions and an interior layer solution for the transition from one inextensional bending solution to the other at a transition point to be determined in the solution process. The adequacy of the leading term asymptotic solution is confirmed by numerical solutions of the boundary-value problem.

1 INTRODUCTION

Consider a homogeneous, isotropic spherical shell whose rise H above the base plane (tangent to its apex) is large compared to its uniform thickness h so that $h/H \ll 1$. When subject to a uniformly distributed internal pressure $-p_0$, it is known from [1] that, except for a boundary-layer phenomenon near its edge(s), the corresponding deformed shell is also spherical if $\mu \equiv p_0 a^2/4EhH \ll 1$, where a and E are the radius of the spherical midsurface and the Young's modulus of the shell medium, respectively. Suppose the shell is subject to an additional axisymmetric external pressure distribution which varies in the meridional direction. It is possible to obtain the net (finite) deformation and stress distribution of the shell due to the two separate loadings by an analysis of the response of the undeformed shell to the combined loading. In particular, when the net inward loading is sufficiently large, the possibility of a snap-through instability can be investigated by methods such as those described in [2, 3, 4] and references therein.† However, when the internal and external pressure distributions are *of comparable magnitude* and the direction of the combined normal surface load is neither completely inward nor completely outward throughout the shell, it is possible to gain considerable insight to the finite deformation shell behavior by an asymptotic analysis of the type described in this note. Roughly, we learn from such an analysis that when the combined loading is directed inward in a region including the apex, the deformation of the shell varies in a qualitatively significant way with the magnitude of the dimensionless load parameter μ relative to that of h/H. While the deformed shell is expected to deviate insignificantly from its undeformed shape except for edge effect(s) for very small μ, it admits a dimple at the apex when $\mu = O(\sqrt{h/H})$ with the size of the dimple varying with the region of zero resultant axial force.

To bring out the essential features of our analysis without unnecessary complications, we limit our discussion here to the case of a clamped shallow spherical cap with a quadratically varying external pressure distribution superimposed upon a uniform internal pressure distribution. We work with a formulation of the small finite deformation elastostatics of the shell first developed in [1] which separates out the effect of the uniform internal pressure distribution. We obtain for $\mu = O(\sqrt{h/H})$ an asymptotic solution of our problem consisting essentially of two different inextensional bending solutions for two separate regions of the shell, one edge zone solution for the satisfaction of the boundary conditions and an interior layer solution for the transition from one inextensional bending solution to another with the unknown location of the layer determined in the course of the solution process. This type of asymptotic solution is akin to that encountered in [5] though they involve qualitatively different solution processes.

Limited in scope though our investigation may seem, it will be evident that the method of solution used here is also applicable to more general classes of problems such as non-shallow dome-type shells of revolution, shell frusta, more complex loadings, different types of edge conditions, etc.

2 FORMULATION

Consider a homogeneous isotropic, shallow spherical cap of constant thickness h subject to a quadratically varying axisymmetric external pressure

†Consistent with the problem treated in this note, these references do not consider the important effect of imperfections on shell buckling. In contrast, Ref. [9] which proposes still another method of solution does investigate the effect of imperfections.

distribution $p_1(1 - \xi^2/\xi_0^2)$ superimposed upon a uniform internal pressure distribution $-p_0$, where ξ is the angle between the meridional tangent at a point of the midsurface of the shell and the base plane, ξ_0 is the value of that angle at the only edge of the shell, and both p_1 and p_0 are positive constants. The small finite deformation elastostatics of such a shell has been shown to be governed by a pair of coupled nonlinear second-order ordinary differential equations for the meridional angle change Φ of the deformed shell and a stress function Ψ [1, 6]

$$\frac{A}{a\xi^2}L[\Psi] - \Phi\left(1 - \frac{1}{2\xi}\Phi\right) = 0, \tag{2.1}$$

$$\frac{D}{a\xi^2}L[\Phi] + \Psi\left(1 - \frac{1}{\xi}\Phi\right) - \tfrac{1}{2}p_0 a^2\Phi = -\tfrac{1}{2}p_1 a^2\xi\left(1 - \tfrac{1}{2}\frac{\xi^2}{\xi_0^2}\right) \tag{2.2}$$

for $0 < \xi < \xi_o$, where

$$A = \frac{1}{Eh}, \qquad D = \frac{Eh^3}{12(1-\nu^2)} \qquad \text{and} \qquad L[f] \equiv \xi^2\frac{d^2 f}{d\xi^2} + \xi\frac{df}{d\xi} - f.$$

In terms of Φ and Ψ, we have the following expressions for the transverse shear resultant Q, the inplane stress resultants N_ξ and N_θ, the stress couples M_ξ and M_θ, and the radial and axial midsurface displacement components u and w (Fig. 1):

$$Q = \tfrac{1}{2}p_0 a\xi\left[1 - \frac{p_1}{p_0}\left(1 - \tfrac{1}{2}\frac{\xi^2}{\xi_0^2}\right)\right] + \frac{1}{a\xi}\Psi(\Phi - \xi), \tag{2.3}$$

$$N_\xi = \frac{1}{a\xi}\Psi + \tfrac{1}{2}p_0 a, \qquad N_\theta = \frac{1}{a}\frac{d\Psi}{d\xi} + \tfrac{1}{2}p_0 a, \tag{2.4}$$

$$M_\xi = \frac{D}{a}\left(\frac{d\Phi}{d\xi} + \frac{\nu}{\xi}\Phi\right), \qquad M_\theta = \frac{D}{a}\left(\nu\frac{d\Phi}{d\xi} + \frac{1}{\xi}\Phi\right), \tag{2.5}$$

$$\frac{dw}{d\xi} = -a\Phi, \qquad u = A\left[\xi\frac{d\Psi}{d\xi} - \nu\Psi + \tfrac{1}{2}(1-\nu)p_0 a^2\xi\right]. \tag{2.6}$$

For a shell closed at the apex, we expect the displacement and stress measures to be bounded there. In particular, we have by symmetry

$$\Phi(0) = u(0) = 0. \tag{2.7}$$

We will only be concerned with the case of a clamped edge in this note so that

$$\Phi(\xi_0) = u(\xi_0) = 0, \tag{2.8}$$

though shells with other edge conditions can also be analyzed similarly.

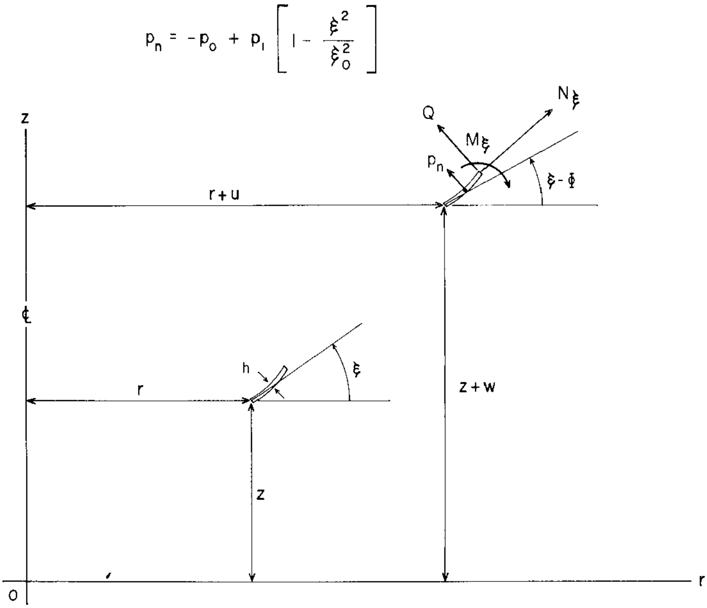


Fig. 1

The two differential equations (2.1)–(2.2) and the four boundary conditions (2.7)–(2.8) define a two-point boundary-value problem for Φ and Ψ. The stress and deformation measures of the shell will be known once we have the solution of this problem. For the purpose of an asymptotic analysis, we introduce appropriate dimensionless dependent and independent variables and write the governing equations in dimensionless form. With $H = \frac{1}{2}a\xi_0^2$ for a shallow spherical shell where $r_0 = a\xi_0$ is the radial distance from the axis of revolution to the outer rim of the spherical cap, we set

$$\xi = \xi_0 x, \qquad \Phi = \xi_0 \phi(x), \qquad \Psi = \tfrac{1}{2}p_0 a^2 \xi_0 \psi(x),$$

$$\gamma = \frac{p_1}{p_0}, \qquad \mu = \frac{p_0 a A}{2\xi_0^2} = \frac{p_0 a^2}{4EhH}, \qquad \varepsilon^4 = \frac{DA}{a^2\xi_0^4} = \frac{h^2}{48(1-\nu^2)H^2}. \tag{2.9}$$

In terms of these variables and parameters, we have the following two dimensionless differential equations for ϕ and ψ:

$$\mu\left[\psi'' + \frac{1}{x}\psi' - \frac{1}{x^2}\psi\right] - \phi\left(1 - \frac{1}{2x}\phi\right) = 0, \tag{2.10}$$

$$\frac{\varepsilon^4}{\mu}\left[\phi'' + \frac{1}{x}\phi' - \frac{1}{x^2}\phi\right] + \psi\left(1 - \frac{1}{x}\phi\right) - \phi = -\gamma x(1 - \tfrac{1}{2}x^2) \tag{2.11}$$

for $0 < x < 1$, the auxiliary equations

$$N_\xi = \tfrac{1}{2}p_0 a\left(\frac{1}{x}\psi + 1\right), \qquad N_\theta = \tfrac{1}{2}p_0 a(\psi' + 1),$$
$$u = a\xi_0^3\mu[x\psi' - \nu\psi + (1-\nu)x], \tag{2.12}$$

and the four boundary conditions

$$x = 0, 1: \qquad \mu[x\psi' - \nu\psi + (1-\nu)x] = \phi = 0. \tag{2.13}$$

In (2.10–13), primes indicate differentiation with respect to x. We have omitted the dimensionless form of other auxiliary quantities as they will not be needed in the subsequent development.

We emphasize that the scaling in (2.9) was chosen so that $\phi(x)$ and $\psi(x)$ are $O(1)$ at most. In particular, it is understood that p_1 is not large compared to p_0 so that $\gamma = O(1)$ at most. Otherwise, we should re-scale the variables before performing any asymptotic analysis. For a cap which is not nearly a flat plate, we must have $\varepsilon = O(\sqrt{h/H}) \ll 1$. If p_0 is not too large so that we have also $\mu \ll 1$ but ε^4/μ not large compared to one, the structure of the governing differential equations suggests the possibility that, except for layer phenomena, an inextensional bending solution dominates throughout the shell. In what follows, we establish by way of an asymptotic analysis that, when μ is of order ε, the inextensional bending deformation may take the form of a dimple centered at the apex when $\gamma > 1$ ($p_1 > p_0$) with the size of the dimple increasing with increasing γ. Except for an edge effect, the dimple spreads through the entire shell for $\gamma > 2$. That the shell in fact exhibits this particular mode of finite deformation is confirmed by numerical solutions of the boundary-value problem (2.10), (2.11), and (2.13).

3 INEXTENSIONAL BENDING SOLUTIONS

For $\mu \ll 1$, we expect, to a good first approximation, the solution of our two-point boundary value problem for ϕ and ψ to be effectively the same as that for the limiting case $\mu = 0$ and $\varepsilon^4/\mu = 0$. The differential equation (2.10) with $\mu = 0$ reduces to an algebraic equation

$$\phi_0\left(1 - \frac{1}{2x}\phi_0\right) = 0 \tag{3.1}$$

where we have used a subscript 0 to indicate the solution for $\mu = 0$. Equation (3.1) is satisfied by either $\phi_0^{(1)} = 2x$ or $\phi_0^{(2)} \equiv 0$ with $\phi_0^{(1)}$ corresponding to a snap-through type deformation (keeping in mind $dz/dr = \xi_0 x$ and $dw/dr = -\Phi = -\xi_0\phi$). Rather than choosing either one of the two acceptable solutions to hold for the entire shell, $0 < x < 1$, we consider the possibility of a different solution for a different part of the shell. More specifically, we consider a solution of the form

$$\phi_0(x) = \begin{cases} \phi_0^{(1)} \equiv 2x & (0 < x < x_t) \\ \phi_0^{(2)} \equiv 0 & (x_t < x < 1) \end{cases} \tag{3.2}$$

for some transition point x_t in $(0, 1)$ with the corresponding $\psi_0(x)$ obtained from (2.11):

$$\psi_0(x) = \begin{cases} \psi_0^{(1)} \equiv -2x + \gamma x(1 - \frac{1}{2}x^2) & (0 < x < x_t), \\ \psi_0^{(2)} \equiv -\gamma x(1 - \frac{1}{2}x^2) & (x_t < x < 1) \end{cases} \tag{3.3}$$

In other words, a region of the shell centered at the apex has undergone a snap-through type deformation while the rest of the shell experiences no change of meridional slope.

For $\mu = 0$, the solution (3.2) and (3.3) also satisfies the two boundary conditions on u and ϕ at the edge $x = 1$ and the two symmetry conditions at the apex $x = 0$. But both ϕ_0 and ψ_0 are discontinuous at x_t resulting in unbounded hoop stress resultant and stress couples. While we can eliminate the discontinuity in ψ_0 by choosing the as yet unspecified parameter x_t to be

$$x_t = \sqrt{\frac{2(\gamma - 1)}{\gamma}} \tag{3.4}$$

there is no way we can eliminate the discontinuity in ϕ. A discontinuity in ϕ is acceptable if the shell has no bending stiffness so that $D = 0$; for shells with a small bending stiffness factor, we anticipate that such a discontinuity may be removed by a layer-type solution in the neighborhood of x_t. A small bending stiffness factor is also expected to give rise to a small correction to the location of the transition point x_t.

For the actual case $0 < \mu \ll 1$, the solution (3.2) and (3.3) do not satisfy the DE (2.10) and the boundary condition on u at $x = 1$ while the discontinuity in ϕ at $x = x_t$ persists. The satisfaction of the boundary condition on u at $x = 1$ can be accomplished by an edge zone solution essentially as described in [10] since $\varepsilon \ll 1$. The satisfaction of the two differential equations can be accomplished by a regular perturbation series solution in powers of μ for ϕ and ψ. We may therefore focus our attention on the elimination of the discontinuity in our composite inextensional bending solution (3.2).

4 TRANSITION LAYER SOLUTION

For $0 < \mu \ll 1$ and $0 < \varepsilon^4/\mu \ll 1$, we expect the solution of the boundary-value problem (2.10), (2.11), and (2.13) to consist of three distinct components; an interior solution (which is predominantly the inextensional bending solution (3.2) and (3.3)) an edge zone solution adjacent to $x = 1$ and a transition layer solution in the vicinity of x_t. For simplicity, we will not concern ourselves here with the reasonably well-understood edge zone solution (see [1] and [10]) and will write the solution for ϕ and ψ as

$$\phi = F(x) + f(y) = \sum_{n=0}^{\infty} \phi_n(x)\mu^n + \sum_{n=0}^{\infty} f_n(y)\lambda^n,$$

$$\psi = G(x) + \lambda g(y) = \sum_{n=0}^{\infty} \psi_n(x)\mu^n + \sum_{n=0}^{\infty} g_n(y)\lambda^{n+1} \tag{4.1}$$

for some small parameter λ $(0 < \lambda \ll 1)$ to be specified later. The stretched variable y in the transition layer component of ϕ and ψ is defined with the help of hindsight as

$$y = \frac{c_t}{\lambda}(x - x_t) \quad (0 < y < \infty) \tag{4.2}$$

with $c_t = 1$ for $x > x_t$ and $c_t = -1$ for $x < x_t$. The layer solution of ψ is taken to be of order λ since we anticipate that $G(x)$ will be continuous across x_t as in the limiting case $\mu = 0$ discussed in Section 3. To bring out the essence of our analysis unencumbered by mathematical details, we will consider only the leading term solution consisting of the first term of all four expansions in (4.1) so that

$$\phi \cong \phi_0(x) + f_0(y), \quad \psi \cong \psi_0(x) + \lambda g_0(y) \tag{4.3}$$

where ϕ_0 and ψ_0 are as given in (3.2) and (3.3), respectively. It will be evident from observations in the next section that the determination of higher-order correction terms poses no conceptual difficulties.

Upon substituting (4.3) into (2.10) and (2.11), we get, except for higher-order terms,

$$\frac{\mu}{\lambda}\left[\ddot{g}_0 + \frac{c_t\lambda}{x}\dot{g}_0 - \frac{\lambda^2}{x^2}g_0\right] - \left(1 - \frac{1}{x}\phi_0 - \frac{1}{2x}f_0\right)f_0 = 0, \tag{4.4}$$

$$\frac{\varepsilon^4}{\mu\lambda^2}\left[\ddot{f}_0 + \frac{c_t\lambda}{x}\dot{f}_0 - \frac{\lambda^2}{x^2}f_0\right] - \underline{\left(1 + \frac{1}{x}\psi_0\right)}f_0 + \lambda g_0\left(1 - \frac{1}{x}\phi_0 - \frac{1}{x}f_0\right) = 0 \tag{4.5}$$

where dots indicate differentiation with respect to y. Equation (4.4) suggests that we take $\lambda = \mu$ in order to get a layer solution. We now omit terms of order μ or smaller in (4.4) to get

$$\ddot{g}_0 - \left(c_t - \frac{1}{2x_t} f_0\right) f_0 = 0. \tag{4.6}$$

To simplify (4.5) similarly, we make the important observation that

$$1 + \frac{1}{x}\psi_0 = \gamma x_t \mu y + O(\mu^2) \tag{4.7}$$

for both $x > x_t$ and $x < x_t$ so that the underlined term in (4.5), which appears to be $O(1)$, is in fact $O(\mu)$. Except for terms of higher order in μ, we may write (4.5) as

$$\frac{\varepsilon^4}{\mu^4}\ddddot{f}_0 - \gamma x_t y f_0 + g_0\left(c_t - \frac{1}{x} f_0\right) = 0.$$

For a layer solution, we must have $\mu = O(\varepsilon)$. With $\mu = c_0\varepsilon$ for some positive constant c_0, we get as a second equation for f_0 and g_0 from (4.5):

$$c_0^4 \ddddot{f}_0 - \gamma x_t y f_0 + \left(c_t - \frac{1}{x_t} f_0\right) g_0 = 0. \tag{4.8}$$

For the pair (4.6) and (4.8), we have $0 < y < \infty$ with $c_t = -1$ for $x < x_t$ and $c_t = 1$ for $x > x_t$. We denote the solution for $x > x_t$ by a superscript $(+)$ and that for $x < x_t$ by a superscript $(-)$. It is not difficult to see from the structure of the two ODE for f_0 and g_0 that

$$g_0^{(-)}(y) = g_0^{(+)}(y), \qquad f_0^{(-)}(y) = -f_0^{(+)}(y). \tag{4.9}$$

With (4.9), we can now formulate the continuity conditions for the leading term solution for ϕ and ψ and their first derivative.

The condition that ϕ be continuous at x_t requires

$$\phi_0^{(2)}(x_t) + f_0^{(+)}(0) = \phi_0^{(1)}(x_t) + f_0^{(-)}(0) = \phi_0^{(1)}(x_t) - f_0^{(+)}(0)$$

or

$$f_0^{(+)}(0) = x_t. \tag{4.10}$$

The continuity of ψ at x_t requires

$$\psi_0^{(2)}(x_t) + \mu g_0^{(+)}(0) = \psi_0^{(1)}(x_t) + \mu g_0^{(-)}(0)$$

which is satisfied to terms of order μ by our choice of x_t from Section 3.

(Actually, it is satisfied identically in view of the first equation of (4.9).) The continuity of ϕ' at x_t requires

$$\frac{1}{\mu}\frac{df_0^{(+)}}{dy}\bigg|_{y=0} = 2 - \frac{1}{\mu}\frac{df_0^{(-)}}{dy}\bigg|_{y=0}$$

which, in view of the second equation of (4.9), is satisfied to terms of order μ. Finally, the continuity of ψ' at x_t may be written with the help of (4.9) and (3.3) as

$$\frac{dg_0^{(+)}}{dy}\bigg|_{y=0} = -\tfrac{1}{2}\left[\frac{d\psi_0^{(2)}}{dx} - \frac{d\psi_0^{(1)}}{dx}\right]\bigg|_{x=x_t}$$

or

$$[g_0^{\cdot}(0)]^{(+)} = -2(\gamma - 1). \tag{4.11}$$

The two nontrivial continuity conditions (4.10) and (4.11) give two of the four boundary conditions for the fourth-order system (4.6) and (4.8). To have a layer solution, we need f and g tending to zero as y tends to infinity. Therefore, the transition layer problem takes the form

$$\begin{cases} \bar{g}^{\cdot\cdot} - \left(1 - \dfrac{1}{2x_t}\bar{f}\right)\bar{f} = 0, & (4.12) \\ & (0 < y < \infty) \\ c_0^4\bar{f}^{\cdot\cdot} - \gamma x_t y\bar{f} + \left(1 - \dfrac{1}{x_t}\bar{f}\right)\bar{g} = 0 & (4.13) \end{cases}$$

with

$$\bar{f}(0) = x_t, \qquad \bar{g}^{\cdot}(0) = -2(\gamma - 1), \tag{4.14}$$

$$\bar{f}(\infty) = 0, \qquad \bar{g}(\infty) = 0. \tag{4.15}$$

In terms of $\bar{f}$ and $\bar{g}$, we have $f_0 = \bar{f}$ and $g_0 = \bar{g}$ for $x > x_t$ and $f_0 = -\bar{f}$ and $g_0 = \bar{g}$ for $x < x_t$. Thus, a solution of the boundary value problem (4.12)–(4.15) makes polar dimpling a possible type of deformation under the prescribed external load.

5 CONCLUDING REMARKS

It should be pointed out once more that we have allowed p_1 to be greater than p_0 in our analysis but not by an order of magnitude in μ. We do not pursue a discussion of the case $p_1 \gg p_0$ in this note since for this case the stiffening of the shell by the uniform internal pressure distribution generally does not play a qualitatively significant role in the finite deformation shell behavior; but to do an asymptotic analysis for $p_1 \gg p_0$, we should use $\bar{\mu} = p_1 a^2/4HE$ instead of μ as the dimensionless load parameter. To the extent that the restriction $p_1 = O(p_0)$ is satisfied, we learned from the analysis of Sections 3 and 4 that the shell admits the dimple mode of deformation when $\mu = O(\varepsilon) \ll 1$. The possibility of polar dimpling for $\mu = O(\varepsilon^2)$ is discussed in [11]. A thorough analysis of the problem by the method of matched asymptotic expansions for the entire range of μ values is reported in [12].

It should also be noted that in deriving the leading term of the asymptotic solution for the dimple type finite deformation shell behavior and the necessary conditions for the existence of such a solution, we have used a combination of intuitive reasoning and mathematical deduction. The location of the transition point, x_t, obtained in this way corresponds only to a first approximation of the actual transition point x_T. For $0 < \mu = O(\varepsilon) \ll 1$, x_T may be expanded in powers of ε:

$$x_T = x_t + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \tag{5.1}$$

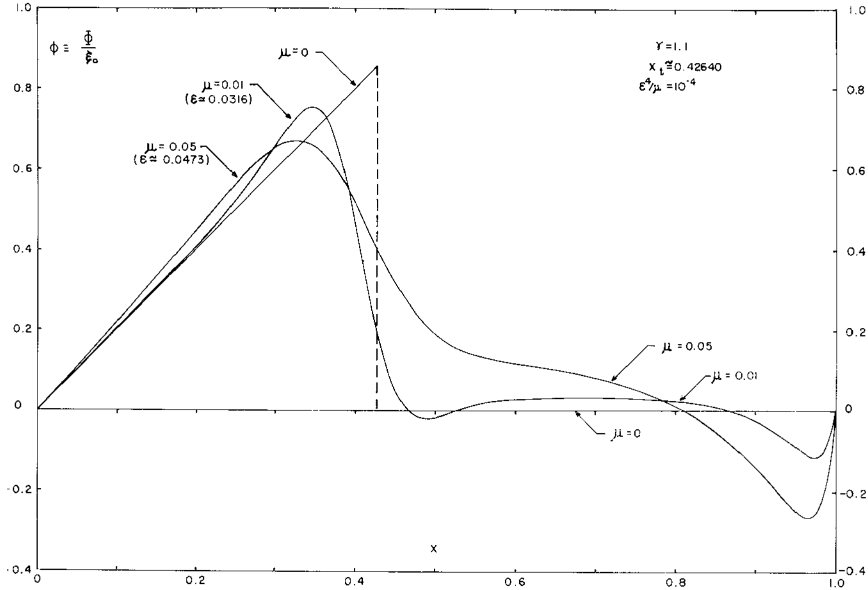
as the effect of the shell stiffness must play a role in the location of the transition point. A more systematic derivation of the asymptotic solution which also yields the higher-order correction terms in the expansion for ψ, ϕ, and x_T may now be implemented by using the series (5.1) and

$$\phi = \sum_{n=0}^{\infty} \phi_n(x)\varepsilon^n + \sum_{n=0}^{\infty} f_n(y)\varepsilon^n, \qquad \psi = \sum_{n=0}^{\infty} \psi_n(x)\varepsilon^n + \sum_{n=0}^{\infty} g_n(y)\varepsilon^{n+1} \tag{5.2}$$

with

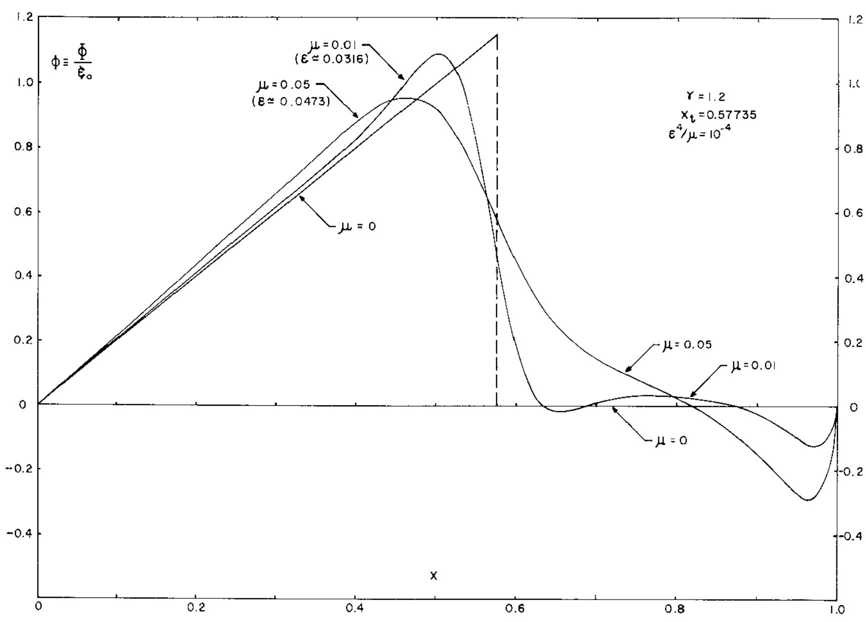
$$y = \frac{c_t}{\varepsilon}(x - x_T) \tag{5.3}$$

in (2.10), (2.11), and (2.13) in the usual way. (With $\mu = O(\varepsilon) \ll 1$, it seems more appropriate to use ε as the small parameter since it is fixed for a given shell and it controls the layer width of edge zone solutions in a linear shell theory.) The only novel feature in the solution process is in the way we determine the coefficients of the series (5.1) for x_T. Specifically, we choose x_k so that the sum of all terms in the equation for f_k similar to those underlined in (4.5) becomes small of order ε compared to the individual terms. Such a choice of $x_k, k = 1, 2, \ldots,$ has the effect of making $G(x)$ continuous at x_T and, with $\mu = O(\varepsilon)$, assuring us that f and g in (5.2) will in fact be significant only in a layer adjacent to x_T.



$\phi \equiv \frac{\Phi}{\xi_0}$
$\gamma = 1.1$
$x_t \cong 0.42640$
$\epsilon^4/\mu = 10^{-4}$
$\mu = 0$
$\mu = 0.01$
$(\epsilon \simeq 0.0316)$
$\mu = 0.05$
$(\epsilon \simeq 0.0473)$
$\mu = 0.05$
$\mu = 0.01$
$\mu = 0$
x

Fig. 2



$\phi \equiv \frac{\Phi}{\xi_0}$
$\mu = 0.01$
$(\epsilon \simeq 0.0316)$
$\mu = 0.05$
$(\epsilon \simeq 0.0473)$
$\gamma = 1.2$
$x_t = 0.57735$
$\epsilon^4/\mu = 10^{-4}$
$\mu = 0$
$\mu = 0.05$
$\mu = 0.01$
$\mu = 0$
x

Fig. 3

To see whether the exact solution of the nonlinear boundary value problem is adequately represented by the leading term asymptotic dimple solution derived in Sections 3 and 4, numerical solutions of (2.10), (2.11), and (2.13) were obtained for a wide range of values of γ, μ, and ε by a multiple shooting routine specifically developed for our problem. Some of the results were double-checked by an all-purpose spline-collocation routine (with error estimates) for general two-point boundary-value problems developed by U. Ascher *et al.* [7]. In all cases, we have taken $\nu = 0.3$. The extensive numerical results show that for $\mu = O(\varepsilon)$, dimpling occurs for $\gamma > 1.0$ but not for $\gamma < 1.0$ which is physically reasonable. In Figs. 2 and 3, we give profiles of the meridional angle change ϕ for $\gamma \equiv p_1/p_0 = 1.10$ and 1.20, respectively. For each γ, we have plotted the numerical solution for ϕ for $\mu = 0.01$ and 0.05 to show how the dimpling becomes more pronounced with decreasing $\varepsilon = O(\mu)$ (which control the transition layer width). With the asymptotic analysis giving

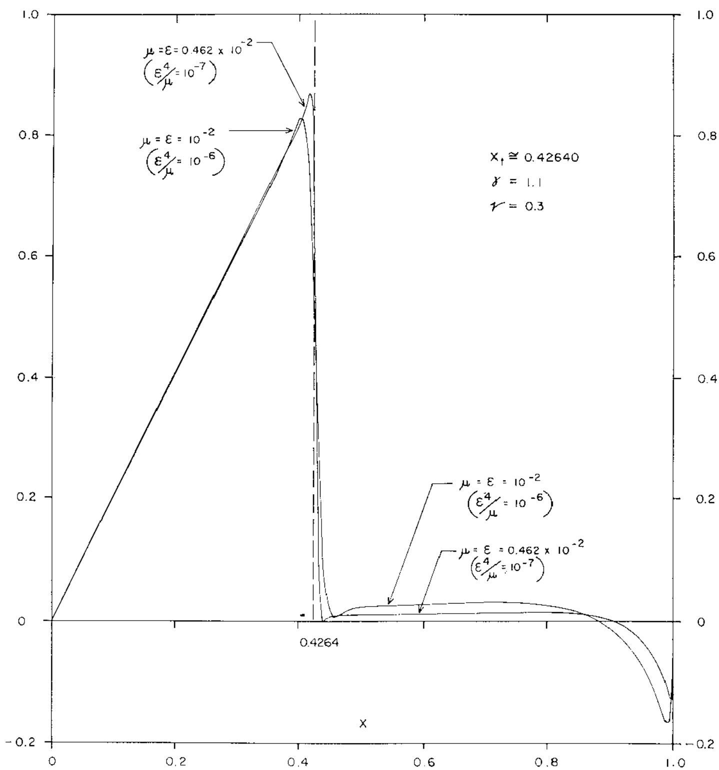


Fig. 4

us $\phi(x_t) = x_t + O(\varepsilon)$, the results also show the expected accuracy of x_t as an approximation for x_T. We have kept $\varepsilon^4/\mu = 10^{-4}$ in the results of Figs. 2 and 3; this particular ratio does give $\varepsilon = O(\mu)$ for the values of μ used. Results for smaller values of ε^4/μ (10^{-6} and 10^{-7}) are plotted in Fig. 4 for $\gamma = 1.1$ to show even more conspicuous dimplings, tending toward the limiting inextensional bending solution (3.2) as ε decreases. In contrast to the results in Figs. 2 and 3, we have kept $\varepsilon = \mu$ in Fig. 4.

Finally, it is evident from the foregoing development that a similar analysis can be carried out for more general classes of problems, e.g. different types of edge conditions, different kinds of external load distributions, nonshallow dome-type shells of revolution, etc. We have in fact obtained extensive results for shallow spherical shells with a hinged edge and/or with a point load at the apex. These results show once again that asymptotic solutions of the type obtained in Sections 3 and 4 provide an adequate description of the exact solution of the relevant boundary-value problem and effectively capture the essential qualitative features of the exact solution. For other dome-type shells of revolution, we note that when the shell is relatively flat at the apex, e.g. $dz/dr = r^n$ for $n > 1 (r = a\xi)$, the effect of bending stiffness will be important in the vicinity of the apex and, for $\varepsilon \ll 1$, another layer-type solution should be added to the asymptotic solution. Just as the edge zone solution, this layer solution may be omitted from the transition layer analysis of a dimple solution.

(After the first printing of this report, the work of Ranjan and Steele [8] on a related problem using a completely different method of analysis was brought to the author's attention.)

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