Perturbation and Asymptotic Solutions for Problems in the Theory of Urban Land Rent

By *Frederic Y. M. Wan*

Simple but accurate approximate solutions for R. M. Solow's theory of urban land rent with congestion cost of transportation are obtained by perturbation and asymptotic methods. Only the case of an absentee landlord is discussed though the techniques used can be modified in a straightforward manner to handle the more interesting (and mathematically more complicated) case of municipal ownership. Solow's model is also extended to have expenditures for housing and space as separate items in the budget equation. This extension eliminates the ambiguity in the value for the fraction of income after travel cost spent on ground rent, without affecting the structure of the mathematical problems associated with Solow's original theory.

1. Introduction

Every now and then a new road or freeway is built in a city to improve traffic within the city. In the process, the city loses some revenue from land rent. The merit of trading land rent for better transportation could in principle be decided by a cost-benefit analysis. But recently, R. M. Solow pointed out that a conventional cost-benefit analysis for urban land use may be seriously misleading [1, 2].

It is well known that when streets and other transportation facilities are congested, they will be inefficiently used. An additional trip by any traveler imposes delay costs on other travelers. However, congestion costs are usually not included in standard equilibrium models of urban geography [3]. One would therefore expect the equilibrium land rent predicted by the conventional theory of urban land to be unrealistic and its use in a cost-benefit analysis inappropriate. By extending the simplest standard model of urban geography to include the congestion cost of transportation [2], Solow showed that the resulting equilibrium ground rent distribution becomes significantly more convex than that of the standard theory [1].

*The research was supported by National Research Council of Canada, Grant No. A9259.

Published by Elsevier North-Holland, Inc.

Even if congestion has a role in the model, market land values would still reflect only private (differential) costs of transportation (which include only the time and expense of a particular traveler) and not the full social costs (which include also the time and expense imposed on other travelers) unless a corrective toll is charged on crowded roads, which is hardly ever the case. In the absence of congestion tolls, market land values will give a distorted picture of the relative social value of parcels of land in different locations, and cost-benefit analyses using such values would not give an accurate economic index for the tradeoff between land rent and better transportation. Indeed, such a cost-benefit analysis was shown to lead to too much land being taken away for streets (when congestion is important and corrective road tolls are not charged), especially near the center of the city [1].

With congestion costs included, to extract useful information from even the simplest acceptable urban land model requires some rather difficult mathematical analyses. While some very informative results were obtained from the extended model of [1] via a numerical solution of the relevant mathematical problems in nonlinear differential equations, the solution of many other cases of interest "strains the capacity of the computational routine" and was therefore not obtained. "In some of the sample solutions [obtained], the [decisive] last few digits may not be significant; the numerical integrations are not so accurate" (see [1]).

In this paper, simple but accurate approximate solutions will be obtained by methods of modern applied mathematics for several classes of problems associated with the urban land theory of [1] and [2]. For completeness, this model of urban geography will be summarized in the next section; readers are referred to [1] and [2] for a more detailed description and discussion. Perturbation solutions of the associated mathematical problems will then be obtained for cases where a moderate fraction, say $\frac{1}{4}$, of the household income after travel cost goes to housing, first for cities with a fixed geographical boundary and then for cities with an outer boundary determined by a prescribed residual (agricultural) rent. These perturbation solutions are inappropriate whenever the ground rent constitutes only a very small fraction of the household income, and a matched asymptotic solution will be obtained for these cases. In connection with the perturbation solutions, our main contribution is the identification of an appropriate small parameter for certain classes of problems, enabling us to bring the perturbation method to bear on these problems. For the case of a low ground rent budget, the construction of an appropriate approximate solution involves some novel features not previously encountered in matched asymptotic solutions of differential equations.

For the purpose of gaining a proper perspective of the conventional cost-benefit analysis, the mathematical techniques developed herein are useful only for the case of an absentee landlord [2]. However, these techniques can be modified in a straightforward manner to handle the more interesting (and mathematically more complicated) case of municipal ownership. Results for this latter case will be discussed elsewhere.

In a broader context, if relatively simple models of urban geography such as Solow's (and others cited in his papers) are useful in the study of urban land economics, the results of this paper suggest that analytical methods of applied mathematics should also be contemplated for the purpose of extracting accurate and useful information in relatively simple form from these models. Even for cases where a numerical solution is acceptable, both in accuracy and cost, it is always better to have a good approximate solution in terms of elementary functions for these cases. A simple analytical solution usually reveals more readily the effect of the various parameters in the problem.

The urban land model formulated in [1,2] does not distinguish housing as a separate commodity. Consequently, there was some question about an appropriate numerical value for the fraction of income after travel cost spent on ground rent (see [1]). Following a suggestion in [1], we consider in an appendix two different models which have expenditure for housing and space as separate items in the budget equation. These models eliminate the difficulty mentioned above and at the same time lead to the same mathematical problem as that associated with Solow's model.

2. Formulation

Consider an abstract circular city with a circular central business district (CBD) of radius R_i and with a concentric outer circular boundary of radius $R_0 > R_i$. A fraction of the land in the annular residential area $R_i < X < R_0$ is to be used for housing, and the rest for streets. The residential area of our idealized city is inhabited by N_0 identical households traveling only to and from the CBD along symmetrically distributed radial streets (with the polar symmetry giving us a spatially one dimensional problem). Each household has an identical annual income y to be used for housing rent, for a single composite consumption good and for transportation cost. Each household chooses a location (characterized by the radial distance X from the city center), an amount of land s for housing and an amount of consumption good c to maximize the same log-additive utility function $U(c,s) = \sigma \log s + (1-\sigma)\log c$, subject to the budget constraint $r(X)s + pc + t(X) = y$, where $r(X)$ is the annual rent per unit area of land at location X, $t(X)$ is the annual travel cost at the same location and p is the constant price of a unit of consumption good. With no loss of generality, we take $p = 1$ and write the budget equation as

$$rs + c = y\left(1 - \frac{t}{y}\right) \equiv yw. \tag{2.1}$$

In equilibrium,

The first order necessary conditions for a maximum $U(c,s)$ subject to the budget constraint give three relations among the four quantities c, s, r and t (or w). In terms of w, we have

$$\begin{aligned} &t = y(1-w), \qquad r = r_1 w^{\alpha+1}, \\ &s = \frac{y}{\alpha+1}\,\frac{w}{r} = \frac{y}{r_1(\alpha+1)} w^{-\alpha}, \qquad c = \alpha r s = \frac{\alpha y}{\alpha+1} w, \end{aligned} \tag{2.2}$$

where $\alpha + 1 = 1/\sigma$ and $r_1 = r(X = R_i)$ is the unknown unit ground rent at the edge of the CBD.

At this point, the conventional theory takes the travel cost to be just the distance cost, giving $t(X)=t_0+\tau_0(X-R_i)$, where t_0 is the travel cost within the CBD (taken to be a known constant in our model) and τ_0 is the fixed cost for moving one commuter one unit of distance within the residential area. Traveling within the CBD may be assumed free, so that we can set $t_0=0$, since a nonzero t_0 amounts to a reduced but still identical annual income for all households. With $t(X)$ given as an explicit function of location, (2.2) gives r, s and c up to the unknown constant r_1, which is then determined by the fraction of land in the residential area allocated for housing (with the balance for streets).

The new element in Solow's extension of the above standard model of urban land economics is to allow the travel cost of a household at location X to be the sum of the *distance cost* and the *congestion cost*, which is an increasing function of the number of households $N(X)$ living outside the circle of radius X and the fraction $b(X)$ of land used for housing in the ring between X and $X+dX$. As in [1] and [2], we consider here specific solutions only for the case (again with $t_0=0$)

$$t(X)=\int_{R_i}^{X}\left[\tau_0+\frac{aN(\xi)}{2\pi\xi[1-b(\xi)]}\right]d\xi$$

or

$$-\frac{dt}{dX}=y\frac{dw}{dX}=-\tau_0-\frac{aN}{2\pi X(1-b)}, \tag{2.3}$$

where a is the constant of proportionality which relates the congestion cost per unit distance to the traffic density at location X, since $N(X)$ gives the number of commuters traveling back and forth across the circle of radius X.

Equation (2.3) relates $w(X)$ to a new unknown $N(X)$. To get another equation for w and N, we observe that the amount of space occupied by the households in the annulus of area $2\pi X\,dX$ must equal the total amount of space in the ring allocated for housing, i.e.,

$$-s\,dN=2\pi Xb\,dX,$$

or

$$-\frac{dN}{dX}=2\pi Xbs^{-1}=\frac{2\pi r_1(\alpha+1)}{y}Xbw^{\alpha}. \tag{2.4}$$

Given τ_0, a, y, α and $b(X)$, the two differential equations (2.3) and (2.4) along with the auxiliary conditions

$$N(R_i)=N_0 \quad\text{and}\quad w(R_i)=1-\frac{t(R_i)}{y}=1 \tag{2.5}$$

(with the first condition expressing the fact that all households live outside the CBD) determine $N(X)$ and $w(X)$ up to the unknown constant r_1. The parameter

r_1 is then fixed by the fact that no one lives outside the city limit, so that

$$N(R_0)=0. \tag{2.6}$$

Except for the case $\alpha=1$, Eq. (2.4) is nonlinear, and an exact solution of the system (2.3)–(2.6) in terms of elementary or special functions is not possible for a general land use density pattern $b(X)$. We see from (2.2) that the constant σ is the fraction of income after travel cost spent on ground rent and should therefore be about 0.05 or smaller (see [1]). This corresponds to $\alpha+1 \geqslant 20$. However, it was pointed out in [1] that such a strict interpretation of σ might give a distorted picture because the model does not distinguish housing as a separate commodity. We should therefore consider also $\sigma=\frac{1}{4}$ (or $\alpha=3$) for cases where house and lot are perfectly correlated, i.e., anyone who buys a large and expensive lot is more or less committed to a large and expensive house. In each case, the boundary value problem (2.3)–(2.6) was solved numerically by an iterative method in [1]. The numerical scheme solves the initial value problem (2.3)–(2.5) for an assumed value of r_1. The correct solution is found if the remaining condition (2.6) is also satisfied by the particular choice of r_1. Otherwise, we modify r_1 and repeat the process. It was pointed out in [1] that the particular iterative procedure used there was impractical or costly for many cases of interest. In what follows, we shall develop several more efficient schemes for an approximate solution of (2.3)–(2.6), each scheme appropriate for a different class of problems.[1]

3. Perturbation solutions for moderate values of α

In the subsequent development, it will be convenient to work with a dimensionless form of a boundary value problem equivalent to (2.3)–(2.6). We introduce a dimensionless distance variable $x=X/R_i$, $1 \leqslant x \leqslant R \equiv R_0/R_i$, and think of all functions of location in the problem as functions of x. With x as the new independent variable, we use (2.3) to eliminate N from (2.4), obtaining the following dimensionless second order nonlinear ODE for w:

$$[p(x)w']'-\lambda q(x)w^{\alpha}=-\frac{t_d}{y}p'(x) \qquad (1<x<R), \tag{3.1}$$

where a prime indicates differentation with respect to x, and where

$$p(x)=\frac{x(1-b)}{(1-b_1)}, \qquad q(x)=xb, \qquad b_1=b(x=1),$$
$$\lambda=\frac{(\alpha+1)ar_1R_i^2}{y^2(1-b_1)}, \qquad t_d=\tau_0 R_i. \tag{3.2}$$

[1]In addition to the solution schemes discussed in the next few sections, we also found a numerical solution of (2.3)–(2.5), coupled with the secant method for the root r_1 of the nonlinear equation $N(R_0;r_1)=0$, to be an extremely effective and efficient method for an "exact" solution of the problem. Here R_0 is either given or to be determined simultaneous with r_1 by the residual rent condition at $X=R_0$ (see Sec. 5).

In view of (2.2) and (2.3), the three auxiliary conditions (2.5) and (2.6) may be expressed in terms of w and w':

$$w(1)=1, \qquad w'(1)=-\frac{t_d+t_c}{y}$$

and

$$w'(R)=-t_d/y, \tag{3.3}$$

where $t_c=aN_0/[2\pi(1-b_1)]$ is a measure of the annual congestion cost of travel, while $t_d=\tau_0 R_i$ is a measure of the annual distance cost of travel for a commuter. Therefore, the parameter

$$\varepsilon=\frac{t_d+t_c}{y} \qquad \text{with} \quad t_c=\frac{aN_0}{2\pi(1-b_1)}, \tag{3.4}$$

is a measure of the ratio of annual travel cost to annual income of an individual household and must therefore be considerably less than unity: $\varepsilon \ll 1$. If we write

$$\frac{t_d}{y}=\varepsilon\frac{t_d}{t_d+t_c}\equiv\varepsilon\eta \quad \text{and} \quad \lambda\equiv\varepsilon\nu, \tag{3.5}$$

then the boundary value problem (3.1)–(3.3) for $w(x)$ and λ can be stated as

$$\left[p(x)w'\right]'-\varepsilon\nu q(x)w^{\alpha}=-\varepsilon\eta p' \qquad (1<x<R), \tag{3.6}$$

$$w(1)=1, \quad w'(1)=-\varepsilon, \qquad w'(R)=-\varepsilon\eta. \tag{3.7}$$

It is clear from (3.5) that $\eta\equiv t_d/(t_d+t_c)\leqslant 1$ and that the unknown parameter ν may be written as

$$\nu=\frac{(\alpha+1)ar_1R_i^2}{y(1-b_1)(t_d+t_c)}=\frac{2(1-\eta)(\alpha+1)}{R^2-1}\left[\frac{r_1\pi(R_0^2-R_i^2)}{N_0y}\right]. \tag{3.8}$$

The denominator N_0y of the ratio in brackets is the total annual income of all inhabitants of the city, and the numerator $r_1\pi(R_0^2-R_i^2)$ is the upper bound of the total annual ground rent paid by all households. The ratio is therefore $O(1)$ at most and should be less than unity. Therefore, we have the crucial observation that the unknown parameter ν is $O(\alpha)$.

For moderate values of α, the form of the boundary value problem (3.6)–(3.7) suggests that we seek a parameter series solution in powers of ε:

$$\begin{aligned} w(x)&=w_0(x)+\varepsilon w_1(x)+\varepsilon^2 w_2(x)+\cdots \\ \nu&=\nu_0+\varepsilon\nu_1+\varepsilon^2\nu_2+\cdots \end{aligned} \tag{3.9}$$

With $\varepsilon \ll 1$ and $\alpha = O(1)$, we expect the first few terms of these series to provide an adequate approximation of $w(x)$ and ν. The nonlinear term of the ODE (3.6) has ε as a multiplicative factor; therefore, the determination of the coefficients $w_k(x)$ and ν_j of the two perturbation series in (3.9) involves only the solution of a sequence of *linear* boundary value problems which can be given exactly in terms of quadratures. To see this, we substitute (3.9) into (3.6) and (3.7) and collect terms of the same power of ε to get

$$\left.\begin{aligned}
&\{[p(x)w_0']'\} + \varepsilon\{[p(x)w_1']' - \nu_0 q(x) w_0^\alpha + \eta p'(x)\}\\
&\quad + \varepsilon^2\{[p(x)w_2']' - q(x)[\nu_1 w_0^\alpha + \alpha\nu_0 w_0^{\alpha-1} w_1]\} + O(\varepsilon^3) = 0,\\
&\{w_0(1) - 1\} + \varepsilon\{w_1(1)\} + \varepsilon^2\{w_2(1)\} + \cdots = 0,\\
&\{w_0'(1)\} + \varepsilon\{w_1'(1) + 1\} + \varepsilon^2\{w_2'(1)\} + \cdots = 0,\\
&\{w_0'(R)\} + \varepsilon\{w_1'(R) + \eta\} + \varepsilon^2\{w_2'(R)\} + \cdots = 0.
\end{aligned}\right\} \tag{3.10}$$

For these equations to hold for all ε, the coefficient of each power of ε must vanish. Therefore, we have the following sequence of *linear* boundary value problems:

The O(1) problem:

$$\begin{cases} [p(x)w_0']' = 0, \\ w_0(1) = 1, \qquad w_0'(1) = w_0'(R) = 0. \end{cases} \tag{3.11}$$

The solution of this problem is $w_0(x) = 1$.

The O(ε) problem:

$$\begin{cases} [p(x)w_1']' = \nu_0 q(x) w_0^\alpha - \eta p'(x) \\ w_1(1) = 0, \qquad w_1'(1) = -1, \qquad w_1'(R) = -\eta. \end{cases} \tag{3.12}$$

With $w_0(x)$ known from the $O(1)$ problem, we can immediately integrate the ODE for w_1 once to get

$$w_1'(x) = \frac{\nu_0 Q_0(x) - (1-\eta)}{p(x)} - \eta, \qquad q_0(x) = \int_1^x q(z)\,dz, \tag{3.13}$$

where the constant of integration has been chosen so that $w_1'(1) = -1$ [keeping in mind that $p(1) = 1$]. The condition $w_1'(R) = -\eta$ determines ν_0:

$$\nu_0 = \frac{1-\eta}{q_0(R)}. \tag{3.14}$$

Integrating (3.13) once more, we get

$$w_1(x) = \nu_0 Q_0(x) - \eta(x-1) - (1-\eta)P_0(x)$$
$$Q_0(x) = \int_1^x \frac{q_0(z)}{p(z)}\,dz, \qquad P_0(x) = \int_1^x \frac{dz}{p(z)}, \tag{3.15}$$

where the constant of integration has been chosen so that $w_1(1)=0$.

The $O(\varepsilon^2)$ problem:

$$\begin{cases} [p(x)w_2']' = q(x)[\nu_1 w_0^\alpha + \nu_0 \alpha w_0^{\alpha-1} w_1], \\ w_2(1) = w_2'(1) = w_2'(R) = 0. \end{cases} \tag{3.16}$$

Having found $w_0(x)$, $w_1(x)$ and ν_0, we can integrate the ODE for w_2 to get

$$p(x)w_2' = \nu_1 q_0(x) + \nu_0 q_1(x), \qquad q_1(x) = \alpha \int_1^x q(z) w_1(z)\,dz, \tag{3.17}$$

where the constant of integration has been chosen so that $w_2'(1)=0$. The condition $w_2'(R)=0$ determines ν_1:

$$\nu_1 = -\nu_0 \frac{q_1(R)}{q_0(R)}. \tag{3.18}$$

Integrate (3.17) once more to get

$$w_2(x) = \nu_1 Q_0(x) + \nu_0 Q_1(x), \qquad Q_1(x) = \int_1^x \frac{q_1(z)}{p(z)}\,dz, \tag{3.19}$$

where the constant of integration has been chosen so that $w_2(1)=0$.

It is now clear that we can continue the process to find $\{\nu_2, w_3\}$, $\{\nu_3, w_4\}$, and so on. As we shall see in the next section, the solution for $w_k(x)$, $k=0,1,2,\ldots$, are expressible in terms of elementary functions for many density pattern functions $b(x)$ of interest. For others, the problem is reduced to evaluating a number of integrals by some numerical method such as Simpson's rule. In either case, the solution of the problem is obtained in a single pass; no searching or iteration is involved in the solution scheme.

4. Two land use density patterns

Evidently, the solution of the boundary value problem (3.6)–(3.7) for $w(x)$ and ν (or r_1) depends on the land use density pattern function $b(x)$. It has been shown [2] that the (optimal) common achieved utility of each household increases with decreasing r_1. In principle, we could investigate the problem of finding that $b(x)$

which maximizes the optimal U. Instead, we shall be less ambitious and consider in this section only the case of a *constant* density pattern, $b(x) = b_I$, and a *hyperbolic* distribution pattern, $b(x) = 1-(1-b_1)/x$. Because of the simple form of $b(x)$, the truncated perturbation solutions are expressible in terms of elementary functions, and we limit ourselves to the truncated series

$$w(x) \cong w_0(x) + \varepsilon w_1(x) + \varepsilon^2 w_2(x) + \varepsilon^3 w_3(x) + \varepsilon^4 w_4(x),$$

$$\nu \cong \nu_0 + \varepsilon\nu_1 + \varepsilon^2\nu_2 + \varepsilon^3\nu_3. \tag{4.1}$$

With an approximate solution for ν, an approximate value of r_1 is then calculated by way of (3.8). It is not difficult to see that r_1 depends on the parameter b_1, and we may choose b_1 to minimize r_1 and thereby maximize the (optimal) utility for the particular class of $b(x)$. The distribution of $r_1(b_1)$ near the minimum value is shown in Table 1 for both one parameter families of

Table 1

Variation of r_1 with b_1 ($\sigma = \frac{1}{4}$, $y = 10^4$, $N_0 = 10^5$, $a = 0.02$, $\tau_0 = 0$ and $R = 5$)

b_1	ε	$r_1 \times 10^{7}$	$N(R_0)/N_0$	$\Delta w(R_0)/w(R_0)$
(a) Constant Land Use Density Pattern: $b(x) = b_I$				
0.85	0.21221	0.73929	0.00583	0.00090
0.75	0.12732	0.64713	0.00298	0.00027
0.74	0.12243	0.64619	0.00290	0.00026
0.73	*0.11789*	*0.64609*	0.00283	0.00024
0.72	0.11368	0.64675	0.00277	0.00023
0.71	0.10976	0.64811	0.00271	0.00022
0.70	0.10610	0.65012	0.00267	0.00020
0.50	0.06366	0.80087	0.00234	0.00019
(b) Hyperbolic Land Use Density Pattern: $b(x) = 1-(1-b_1)/x$				
0.750	0.12732	0.73894	−0.00115	0.00046
0.600	0.07958	0.60639	0.00230	0.00057
0.550	0.07074	0.58953	0.00246	0.00052
0.500	0.06366	0.57882	0.00254	0.00044
0.450	0.05787	0.57250	0.00259	0.00040
0.400	0.05395	0.56946	0.00261	0.00037
0.375	*0.05093*	*0.56894*	0.00262	0.00033
0.350	0.04897	0.56900	0.00263	0.000308
0.325	0.04716	0.56957	0.00263	0.000307
0.300	0.04547	0.57062	0.00264	0.00028
0.250	0.04244	0.57400	0.00265	0.00027
0.200	0.03979	0.57891	0.00266	0.00024

density pattern functions. All numerical results presented in this table are for the case $\sigma = \frac{1}{4}$ ($\alpha = 3$). Other parameter values are $R = 5$, $\tau_0 = 0$, $a = 0.02$, $y = 10^4$ and $N_0 = 10^5$, the same as those in an example worked out in [2]. Note that with $\tau_0 = 0$, the only contribution to travel cost is the congestion cost.

We also indicate in Table 1 the accuracy of the truncated perturbation series of (4.1). The approximate value of r_1 is used in the system (2.3)–(2.4) with the initial conditions (2.5) to generate a numerical solution of this initial value problem by a fourth order Runge-Kutta scheme. How well the end condition $N(R_0) = 0$ is satisfied provides a measure of the accuracy of the perturbation solution. From Table 1, we see that the error in $N(R_0)$ is less than 0.6% of N_0, the maximum value of $N(X)$, for both classes of density patterns in the range investigated. Another measure of accuracy for our perturbation scheme is the difference between the approximate value of $w(R_0)$ from the perturbation series and that generated by (2.3)–(2.5) using the approximate value for r_1 as input. From Table 1, we see that the ratio of this difference to the perturbation solution of $w(R_0)$ is less than 0.1% for all cases investigated. These results, along with others not presented herein, suggest that, considering the accuracy of the inputs, the truncated series (4.1) provide a more than adequate approximate solution for w and v whenever $\alpha = O(1)$ and b_1 is not nearly unity.

In contrast to the case of no congestion cost, i.e., $a = 0$ and $\tau_0 \neq 0$, for which r_1 is a monotone decreasing function as b_1 increases, the results in Table 1 show that r_1 attains a minimum for some value of b_1 in the interval $0 < b_1 < 1$ when congestion cost is included. The precise location of the minimum value of $r_1(b_1)$ can not be taken seriously, as the error inherent in our approximate solution limits the accuracy of r_1 as well as the value of b_1 which minimizes it. Nevertheless, a comparison of the results shows that the location of the minimum $r_1(b_1)$ in the interval $0 < b_1 < 1$ differs discernably for the two density patterns. The minimum value of r_1 (and therefore the maximum optimal utility) for the hyperbolic pattern is smaller and is attained at a smaller value of b_1 than that for the constant density pattern. Evidently, the larger fraction of space devoted to housing near the outer boundary in the case of a hyperbolic pattern makes it possible to have more land for streets near the CBD. However, because the distance cost has been ignored ($\tau_0 = 0$), these test cases do not represent typical situations. Results for other cases studied suggest that, in general, the more important the congestion cost (relative to the distance cost), the more land is needed for streets to minimize r_1.

5. Outer boundary and prescribed residual rent

If the city is free to expand radially outward with the outer boundary determined by the condition that the ground rent there is equal to a prescribed agricultural rent r_A, the perturbation scheme of Sec. 3 must be modified in a nontrivial way. Evidently, the outer city boundary R now depends on the small parameter ε, and we must also expand R in powers of ε:

$$R(\varepsilon) = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \cdots = \sum_{k=0}^{\infty} R_k \varepsilon^k. \tag{5.1}$$

At $x = R(\varepsilon)$, we have

$$w(R(\varepsilon);\varepsilon) = w|_{\varepsilon=0} + \varepsilon \frac{dw}{d\varepsilon}\bigg|_{\varepsilon=0} + \frac{1}{2}\varepsilon^2 \frac{d^2w}{d\varepsilon^2}\bigg|_{\varepsilon=0} + \cdots, \tag{5.2}$$

with

$$\begin{aligned} w|_{\varepsilon=0} &= w_0(R_0) \\ \frac{dw}{d\varepsilon}\bigg|_{\varepsilon=0} &= \left[\frac{\partial w}{\partial \varepsilon} + \frac{\partial w}{\partial R}\frac{dR}{d\varepsilon}\right]_{\varepsilon=0} = w_1(R_0) + R_1 w_0'(R_0), \\ \frac{d^2w}{d\varepsilon^2}\bigg|_{\varepsilon=0} &= \left[\frac{\partial^2 w}{\partial \varepsilon^2} + 2\frac{\partial^2 w}{\partial R\,\partial \varepsilon}\frac{dR}{d\varepsilon} + \frac{\partial^2 w}{\partial R^2}\left(\frac{dR}{d\varepsilon}\right)^2 + \frac{\partial w}{\partial R}\frac{d^2R}{d\varepsilon^2}\right]_{\varepsilon=0} \\ &= 2\left[w_2(R_0) + R_1 w_1'(R_0) + \tfrac{1}{2}R_1^2 w_0''(R_0) + R_2 w_0'(R_0)\right], \end{aligned} \tag{5.3}$$

etc. If we now introduce a normalized agricultural rent

$$\nu_A = \frac{(\alpha+1)a r_A}{y(1-b_1)(t_d+t_c)}, \tag{5.4}$$

then the auxiliary condition $r(R) = r_A$ can be written as

$$\begin{aligned} \nu_A &= \nu[w(R)]^{\alpha+1} \\ &= (\nu_0 + \varepsilon\nu_1 + \varepsilon^2\nu_2 + \cdots)\{w_0(R_0) + \varepsilon[w_1(R_0) + R_1 w_0'(R_0)] \\ &\quad + \varepsilon^2[w_2(R_0) + R_1 w_1'(R_0) + R_2 w_0'(R_0) + \tfrac{1}{2}R_1^2 w_0''(R_0)] + \cdots\}^{\alpha+1}, \end{aligned} \tag{5.5}$$

or

$$\begin{aligned} &\{\nu_0 w_0^{\alpha+1}(R_0) - \nu_A\} + \varepsilon\{\nu_1 w_0^{\alpha+1}(R_0) \\ &\quad + (\alpha+1)w_0^{\alpha}(R_0)\nu_0[w_1(R_0) + R_1 w_0'(R_0)]\} + \varepsilon^2\{\cdots\} + \cdots = 0. \end{aligned} \tag{5.6}$$

Similarly, the parametric series form of the last condition of (3.7) is now

$$\begin{aligned} &w_0'(R_0) + \varepsilon\{w_1'(R_0) + R_1 w_0''(R_0) + \eta\} \\ &\quad + \varepsilon^2\{w_2'(R_0) + w_1''(R_0)R_1 + w_0''(R_0)R_2 + \tfrac{1}{2}w_0'''(R_0)R_1^2\} \\ &\quad + \varepsilon^3\{\cdots\} + \cdots = 0. \end{aligned} \tag{5.7}$$

The condition (5.7) replaces the last condition of (3.10), and the resulting new set of (3.10) is to be supplemented by (5.6). Together, they give rise to the following sequence of problems for the determination of $\{w_j(x)\}$, $\{\nu_j\}$ and $\{R_j\}$:

The O(1) problem:

$$\begin{cases} [p(x)w_0']'=0, \\ w_0(1)=1, \quad w_0'(1)=0, \quad w_0'(R_0)=0, \\ \nu_0 w_0^{\alpha+1}(R_0)=\nu_A. \end{cases} \tag{5.8}$$

The solution of this problem is

$$w_0(x)=1 \quad \text{and} \quad \nu_0=\nu_A, \tag{5.9}$$

with R_0 (of (5.1), not the city's outer boundary of Sec. 2) still to be determined.

The O(ε) problem:

$$\begin{cases} [p(x)w_1']'=\nu_0 q(x)w_0^\alpha-\eta p'(x), \\ w_1(1)=0, \quad w_1'(1)=-1, \\ w_1'(R_0)+R_1 w_0''(R_0)=-\eta, \\ \dfrac{\nu_1}{\nu_0}w_0^{\alpha+1}(R_0)+(\alpha+1)w_0^\alpha(R_0)\left[w_1(R_0)+R_1 w_0'(R_0)\right]=0. \end{cases} \tag{5.10}$$

With $w_0(x)$ and ν_0 known from the solution of the $O(1)$ problem, $w_1'(x)$ is again determined by the ODE and the initial condition $w_1'(1)=-1$, and is again as given by Eq. (3.13). The third auxiliary condition can be simplified to read $w'(R_0)=-\eta$; it (together with the obvious constraint $R_0>1$) determines R_0. Integrate $w_1'(x)$ and use the condition $w_1(1)=0$ to get $w_1(x)$ as given by Eq. (3.15). Finally, the last auxiliary condition gives ν_1:

$$\begin{aligned} \frac{\nu_1}{\nu_0} &= -(\alpha+1)w_1(R_0) \\ &= -(\alpha+1)\left[\nu_0 Q_0(R_0)-\eta(R_0-1)-(1-\eta)P_0(R_0)\right]. \end{aligned} \tag{5.11}$$

We can continue the process to find $w_2(x)$, ν_2 and R_1 from the $O(\varepsilon^2)$ problem, etc. In general, the solution of the $O(\varepsilon^n)$ problem gives $w_n(x)$, ν_n and R_{n-1}.

Tables 2 and 3 give the perturbation solution (including the w_4, ν_4 and R_3 term in the series for w, ν and R, respectively) for the cases A, B, E, F and G, presented in Tables 1 and 3 in [1]. These perturbation solutions have been checked by the procedure described in Sec. 4,[2] as well as by the efficient iterative

[2]Namely, use the perturbation solution for r_1 as input and solve the initial value problem (2.3)–(2.5) numerically to see how well the condition $N(R_0;r_1)=0$ is satisfied and how close the resulting $w(R_0)$ is to that of the perturbation solution. Here R_0 is the city's outer boundary.

Table 2
Effect of Congestion Cost for a Linear Land Density Pattern
($N_0=3\times10^5$, $y=10^4$, $r_A=2\times10^7$, $\sigma=\frac{1}{4}$, $R_i=1$, $b_1=0.65$, $b_R=0.95$)[a]

	Rent $=r(X)\times10^{-7}$		Space per Family $=s(X)\times10^{-5}$	
X	A[b]	B[c]	A[b]	B[c]
1.00	5.842	5.940	4.28	4.21
1.50	4.561	4.359	5.15	5.31
2.00	3.503	3.286	6.28	6.56
2.50	2.641	2.523	7.76	8.00
2.959	2.000		9.56	
3.007		2.000		9.52

[a] $b_1\equiv b(R_i)$ and $b_R\equiv b(R_0)$. [b] $a=0$, $\tau_0=1200$. [c] $a=0.007$, $\tau_0=700$.

Table 3
Sample Equilibrium Solutions for Different Linear Land Density Patterns
($N_0=3\times10^5$, $y=10^4$, $r_A=2\times10^7$, $\tau_0=350$, $a=0.0105$, $\sigma=\frac{1}{4}$, $R_i=1$)

	Rent $=r(X)\times10^{-7}$			Space per Family $=s(X)\times10^{-5}$		
X	E[a]	F[b]	G[c]	E[a]	F[b]	G[c]
1.00	4.808	4.677	4.615	5.20	5.35	5.42
1.50	3.856	3.767	3.854	6.14	6.29	6.20
2.00	3.167	3.130	3.281	7.11	7.22	7.00
2.50	2.638	2.656	2.825	8.16	8.17	7.83
3.00	2.228	2.298	2.457	9.15	9.11	8.69
3.403	2.000			10.04		
3.50		2.044	2.166		9.95	9.55
3.617		2.000			10.11	
3.888			2.000			10.14

[a] $b_1=0.41$, $b_R=0.95$. [b] $b_1=0.41$, $b_R=0.82$. [c] $b_1=0.20$, $b_R=0.82$.

scheme described in footnote 1 of Sec. 2. In all cases, the perturbation solutions of this section have been confirmed to be extremely accurate.

It should be noted that the results in Tables 2 and 3 (and all the results obtained in this report, for that matter) are for the case of absentee landlords who spend the collected ground rent elsewhere, so that the constant annual household income y is identical to the known annual household wage income. Therefore, the solutions obtained here are necessarily different from those presented in [1]. Nevertheless, our solutions exhibit the same qualitative features as those described in [1]. The only noteworthy qualitative difference is the slightly larger difference in the ground rent at the inner boundary, case B attaining a value for r_1 nearly 2% higher than case A. The opposite is true in the case of city owned land with the ground rent distributed equally among the N_0 households as social dividends which form a part of the household annual income.

6. Matched asymptotic solution for large α[3]

Recall that

$$\lambda \equiv \varepsilon \nu = \frac{t_d + t_c}{y} \frac{(\alpha+1) a r_1 R_i^2}{y(1\text{-}b_1)(t_d + t_c)}, \tag{6.1}$$

so that $\nu = 0(\alpha + 1)$. Strictly speaking, σ is the fraction of income after travel cost spent on ground rent [see Eq. (2.2)], so that $\alpha + 1 \geqslant 20$ and λ is not necessarily small compared to unity even when we have $\varepsilon \ll 1$. Therefore, the perturbation solutions obtained in the last three sections are inappropriate for this range of α. Indeed, the accuracy of the truncated series used in Secs. 4 and 5 deteriorates with increasing α.

To seek an appropriate simple approximate solution for $\alpha \gg 1$, we observe that $w(x) < 1$ for $x > 1$; away from the CBD, we have $w^{\alpha} \ll 1$ for $\alpha \gg 1$. Thus, the nonlinear term of the ODE (3.1) is important only near the edge of the CBD, $x \cong 1$. This observation suggests that a solution by the technique of inner-outer expansion may be appropriate for this range of α.

The leading term $\bar{w}_0$ of the outer solution $\bar{w}(x)$ is obtained by neglecting the nonlinear term in the ODE (since $w^{\alpha} \ll 1$ away from the CBD) and solving the resulting equation

$$\left[p(x) \bar{w}_0' \right]' = -\varepsilon \eta p' \tag{6.2}$$

subject to the boundary condition

$$\bar{w}_0'(R) = -\varepsilon \eta. \tag{6.3}$$

(We should not try to satisfy the two auxiliary conditions at the edge of the CBD, as the outer solution alone is not adequate near $x = 1$.) We get from (6.2) and (6.3)

$$\bar{w}_0(x) = \varepsilon \eta (x - 1) + A_0, \tag{6.4}$$

where A_0 is an arbitrary constant to be determined later.

For the inner solution, it is not difficult to see that we should introduce the stretched coordinate

$$z = \alpha (x - 1), \qquad \text{or} \quad x = 1 + z/\alpha. \tag{6.5}$$

[3]The results of this section evolved from a discussion between the author and L. N. Howard of MIT. The author gratefully acknowledges Professor Howard's contributions to the problem.

Much less obvious is the nature of the contribution of the w^{α} term in the ODE (3.1) [or (3.6)], which should be significant (but not dominant) for the inner solution. Perturbation solutions for moderate values of α suggest that near the CBD, $w(x)$ does not differ much from unity and the difference decreases with increasing α. This suggests that we set

$$w(x)=1+\hat{w}(z)/\alpha, \tag{6.6}$$

which is to have important consequences in the subsequent development. With $(\ )'=\alpha(\ )^{\cdot}$, where a dot indicates differentiation with respect to z, the ODE (3.1) becomes

$$[p\hat{w}^{\cdot}]^{\cdot}-\hat{\lambda}q[1+\hat{w}/\alpha]^{\alpha}=-\varepsilon\eta p'(x)/\alpha, \tag{6.7}$$

where $\hat{\lambda}=\lambda/\alpha$. [Recall that $\lambda=O(\alpha)$, so that $\hat{\lambda}=O(1)$.] The ODE for the leading terms $\hat{w}_0$ and $\hat{\lambda}_0$ of $\hat{w}$ and $\hat{\lambda}$ are obtained by letting α tend to infinity while keeping z finite, so that (6.7) becomes

$$\hat{w}_0^{\cdot\cdot}-\hat{\lambda}_0 b_1 e^{\hat{w}_0}=0, \tag{6.8}$$

where we have made use of the fact that $p(x)=p(1+z/\alpha)\to p(1)=1$ and $q(x)\to q(1)=b_1$ as $\alpha\to\infty$. Equation (6.8) can be integrated once immediately to get

$$(\hat{w}_0^{\cdot})^2-2\hat{\lambda}_0 b_1 e^{\hat{w}_0}=\varepsilon^2-2\hat{\lambda}_0 b_1, \tag{6.9}$$

where the two auxiliary conditions at $x=1$ in (3.3), which now take the form

$$\hat{w}_0(z=0)=0 \quad\text{and}\quad \hat{w}_0^{\cdot}(z=0)=-\varepsilon, \tag{6.10}$$

have been used to fix the constant of integration.

Equation (6.9) is a first order autonomous equation, and its exact solution is

$$\hat{w}_0(z)=-2\ln\left\{B_0\sinh\left[\tfrac{1}{2}\mu(z+z_0)\right]\right\} \tag{6.11}$$

where

$$\mu^2=\varepsilon^2-2\hat{\lambda}_0 b_1, \qquad \mu B_0=\left(2\hat{\lambda}_0 b_1\right)^{1/2},$$

$$\mu z_0=2\ln\left[\frac{1+\left(1+B_0^2\right)^{1/2}}{B_0}\right]. \tag{6.12}$$

Table 4

Asymptotic Solution for Large α with $b(x)=b_1+(b_R-b_1)(x-1)/(R-1)$ ($y=10^4$, $N_0=3\times10^5$, $a=0.005$, $\tau_0=500$, $R_i=1$, $R_0=4.87$)

$\alpha+1=1/\sigma$	$r_1\times10^{-8}$			$W(R)$		$-W'(R)$	
	Exact[a]	Asympt.	% error	Exact[a]	Asympt.	Exact[a]	Asympt.
(a) $b_1=0.6370$ and $b_R=0.8722$							
10	2.5920	2.9065	12.13%	0.73617	0.71020	0.05000	0.05747
20	3.3804	3.4917	3.29%	0.75916	0.75436	0.05000	0.05101
40	4.3937	4.3926	0.03%	0.78076	0.78059	0.05000	0.05002
60	4.8894	4.8681	0.44%	0.78925	0.78937	0.05000	0.05000
80	5.1734	5.1478	0.49%	0.79356	0.79371	0.05000	0.05000
100	5.3574	5.3314	0.49%	0.79616	0.79629	0.05000	0.05000
(b) $b_1=0.6500$ and $b_R=0.8900$							
10	2.5955	2.9042	11.89%	0.73293	0.70732	0.05000	0.05765
20	3.3863	3.4889	3.03%	0.75730	0.75285	0.05000	0.05104
40	4.3879	4.3813	0.15%	0.77989	0.77984	0.05000	0.05002
60	4.8764	4.8513	0.51%	0.78870	0.78887	0.05000	0.05000
80	5.1563	5.1276	0.56%	0.79317	0.79333	0.05000	0.05000
100	5.3374	5.3088	0.54%	0.79586	0.79599	0.05000	0.05000

(c) $b_1 = 0.6630$ and $b_R = 0.9078$

10	2.6050	2.9066	11.58%	0.72933	0.70425	0.05000	0.05784
20	3.3983	3.4916	2.75%	0.75531	0.75124	0.05000	0.05106
40	4.3892	4.3764	0.29%	0.77898	0.77905	0.05000	0.05002
60	4.8708	4.8414	0.60%	0.78812	0.78835	0.05000	0.05000
80	5.1461	5.1145	0.61%	0.79279	0.79294	0.04999	0.05000
100	5.3250	5.2935	0.59%	0.79552	0.79568	0.05000	0.05000

[a]Obtained by the iterative scheme described in footnote 1, Sec. 2 of this paper.

Again, the initial conditions (6.10) have been used to fix the constant of integration. The remaining unknown constant $\hat{\lambda}_0$ in $\hat{w}_0(z)$ and the constant of integration A_0 in the leading term outer solution $\bar{w}_0(x)$ are determined by two matching conditions which specify the continuity of $w(x)$ and $w'(x)$ at some point in an overlapping region of validity for the inner and outer solutions. Omitting the details, the final explicit (leading term) composite solution for $w(x)$ is

$$w(x) \sim 1 - \frac{2}{\alpha} \ln \left\{ \frac{(1-\eta^2)^{1/2}}{\eta} \sinh\left[\tfrac{1}{2}\varepsilon\eta(z+z_0) \right] \right\}, \tag{6.13}$$

where

$$z_0 = \frac{1}{\varepsilon\eta} \ln\left[\frac{1+\eta}{1-\eta} \right], \qquad z = \alpha(x-1), \tag{6.14}$$

and where ε and η are as given in (3.4) and (3.5). For $\sigma = 0.05$ $(\alpha+1=20)$, the above solution for $w(x)$ is within 1% of the exact (numerical) solution for all test cases run, while $w'(x)$ is within 2% of the exact solution. Results for three of the test cases for a linear $b(x)$ (corresponding to those appearing in Table 5 of [1] but for the case of absentee landlords) are given in Table 4.

In contrast to the very good accuracy of the matched asymptotic solution for $w(x)$ and $w'(x)$, the leading term solution $\hat{\lambda}_0$ for $\hat{\lambda} = \lambda/\alpha$ obtained through the matching conditions mentioned above is accurate only for very large values of α. Though the error is $O(1/\alpha)$, it is still over 7% of the exact (numerical) solution for $\alpha+1=200$. To get a more accurate leading term asymptotic solution for $\hat{\lambda}$ (and therefore r_1), we integrate the original differential equation (3.1) once over the interval $[1, R]$ and rewrite the result as an expression for λ:

$$\lambda = \frac{\varepsilon(1-\eta)}{\int_1^R q(x) w^{\alpha}(x)\, dx}. \tag{6.15}$$

An approximate solution for $\lambda = \alpha\hat{\lambda}$ is obtained by using the very accurate leading term asymptotic solution given by (6.13) for $w(x)$ in (6.15). For $\alpha+1=20$, the error in the new approximate $\hat{\lambda}$ is about 3% of the exact value and is $O(1/\alpha)$ with increasing α.

Evidently, the leading term inner solution $\hat{w}_0(z)$ depends only on the value $b_1 \equiv b(x=1)$ and not on the actual distribution of $b(x)$. This is not surprising, since $\hat{w}_0(z)$ is only valid in a narrow region near the edge of the CBD. What is somewhat surprising is the fact that the same is true for the leading term outer solution (which can be shown to be identical to the outer solution itself for the present problem). Therefore, the leading term composite asymptotic solution for w is oblivious to the actual distribution of $b(x)$. But the optimal equilibrium solution of the land use problem can not be completely independent of the distribution of $b(x)$. This is reflected in the fact that the value r_1 (or λ), which enters into the rent function $r(X)$ and space allocation function $s(X)$, becomes more accurate when the effect of $b(x)$ is taken into account through equation (6.15).

The percentage error for the asymptotic solution for r_1 in Table 4 attains its lowest value for $\alpha+1=40$ in all cases presented. It increases slightly for larger values of α before stabilizing at about half of a percent. Apparently the sharper rent gradient associated with very large values of α reduces the accuracy of the numerical solution scheme used. Nevertheless, the small percentage error in all cases makes the simple asymptotic solution for large α very attractive for the purpose of delineating the effect of the various input quantities.

7. Concluding remarks

In the preceding development, we indicated how approximate solutions of the relevant mathematical problems in Solow's model of urban geography can be obtained by perturbation and asymptotic methods. In many cases, the appropriate (truncated) perturbation or (leading term) asymptotic solution turns out to be a very accurate approximation of the exact solution. For realistic land use patterns, the accurate approximate solutions can often be expressed in terms of elementary and special functions so that the effect of the various input parameters can be studied analytically. When α is not sufficiently large and/or ε not sufficiently small, so that neither a truncated perturbation solution nor a leading term asymptotic solution provides an adequate approximate solution, one of them should nevertheless be useful as a good initial trial solution for the iterative numerical solution scheme and will lead to faster convergence to the correct solution than an arbitrary initial guess.

In this paper, we have confined ourselves to the case of absentee landlords to bring out the essence of our methods. Similar but somewhat more complicated methods of solution can also be developed for the case of municipal ownership with the aggregate rent payment redistributed in equal shares as social dividends to all households in the city. Results for the case of city ownership will be presented in another report.

Appendix–Models with housing as a separate commodity

In Sec. 2, we discussed the difficulty associated with a proper interpretation of the parameter $\sigma=1/(\alpha+1)$ for the purpose of assigning to it an appropriate numerical value. The problem is related to the fact that the model proposed in [1] and [2] does not distinguish housing as a separate commodity. In what follows, we shall discuss two different models which distinguish expenditure for housing and lot.[4] In both cases, the relevant mathematical problem to be solved is still (3.6) and (3.7) but now there is no question about the meaning of and the appropriate numerical value for the parameter α.

For a typical household living at location X, consider a logarithmic-additive utility function

$$U(c,s,h)=\sigma_c \log c+\sigma_s \log s+\sigma_h \log h, \tag{A.1}$$

[4]These two approaches to separating housing expenditure and ground rent were suggested in [1].

where $c(X)$ and $s(X)$ are as previously defined and $h(X)$ is the housing expenditure. With c being numeraire, the consumer chooses c, s, h and X to maximize U subject to the budget constraint

$$c+rs+h+t=y. \tag{A.2}$$

In equilibrium,

The first order conditions for a maximum are

$$\frac{\sigma_c}{c}=\frac{\sigma_s}{sr}=\frac{\sigma_h}{h} \quad \text{and} \quad s\frac{dr}{dX}+\frac{dt}{dX}=0. \tag{A.3}$$

Straightforward calculations give

$$c=\frac{\sigma_c}{\sigma_s}rs, \qquad h=\frac{\sigma_h}{\sigma_s}rs, \qquad rs=\sigma_s(y-t)=\sigma_s yw(X) \tag{A.4}$$

and

$$r=r_1\left(1-\frac{t}{y}\right)^{1/\sigma_s}\equiv r_1\left[w(X)\right]^{1/\sigma_s}. \tag{A.5}$$

Further development beyond this point essentially repeats the steps in [2] leading to the nonlinear boundary value problem (3.6)–(3.7) with $(\alpha+1)^{-1}=\sigma_s$. From (A.4), it is clear that the parameters σ_s and σ_h in the utility function are the fractions of income after transportation cost used for space and housing, respectively. We can in fact identify $\sigma=\sigma_s+\sigma_h$, since [see also (2.2)]

$$rs+h=rs\left(1+\frac{\sigma_h}{\sigma_s}\right)=(\sigma_s+\sigma_h)(y-t). \tag{A.6}$$

Therefore, the value of α in (3.6) should be such that $(\alpha+1)^{-1}$ is just the ground rent (and not housing plus lot expenditure) fraction of the household budget, and that means $\alpha+1\gg 1$, since $\sigma_s\leqslant 0.05$.

A second model, which also distinguishes the individual contribution of space and housing but without both of them entering into the utility function explicitly, lumps these two items in a single variable H (called housing service), as far as the utility function is concerned. This housing service is produced from inputs of land and a single composite intermediate commodity m in variable proportions according to a definite technology. For a linear-logarithmic utility function, we now have

$$U(c,H)=\sigma\log H+(1-\sigma)\log c \tag{A.7}$$

with $H=H(s,m)$. A consumer chooses c, s, m and X to maximize (A.7) subject to the budget constraint

$$c+rs+m+t=y, \tag{A.8}$$

In equilibrium?)

where c is again numeraire. First order conditions for a maximum are

$$\frac{1-\sigma}{c} = \frac{\sigma}{rH}\frac{\partial H}{\partial s} = \frac{\sigma}{H}\frac{\partial H}{\partial m} \quad \text{and} \quad s\frac{dr}{dX} + \frac{dt}{dX} = 0. \tag{A.9}$$

(A.8) and (A.9) are four equations for the five unknowns c, r, s, m and t. These equations can be solved for r, s, c and m in terms of t [or $w(X)$] once the form of $H(s,m)$ is specified.

For a Cobb-Douglas type production function $H(s,m) = H_0 s^{\beta} m^{\gamma}$, we have from the first half of (A.9)

$$\frac{1-\sigma}{c} = \frac{\sigma\beta}{rs} = \frac{\sigma\gamma}{m}. \tag{A.10}$$

From Eq. (A.8), we have

$$c = \frac{1-\sigma}{\sigma\beta} rs, \qquad m = \frac{\gamma}{\beta} rs, \qquad rs = \bar{\sigma}_s (y - t), \tag{A.11}$$

where

$$\bar{\sigma}_s \equiv \frac{\sigma\beta}{1-\sigma(1-\beta-\gamma)}. \tag{A.12}$$

Further development again repeats the steps in [2] leading to the nonlinear boundary value problem (3.6) and (3.7) with $(\alpha+1)^{-1} = \bar{\sigma}_s$. The parameter σ would now be about $\frac{1}{4}$, and the value of $\bar{\sigma}_s$ is determined once we fix β and γ. If $H(s,m)$ is of constant returns to scale, then $\gamma = 1-\beta$ and $\bar{\sigma}_s = \sigma\beta$, with β to be estimated from data.

Acknowledgements

The author is grateful to Messrs. Richard Lee and Wolfgang Richter for doing the lion's share of the machine computation and to Professors L. N. Howard (MIT; see also Sec. 6 of the paper), R. M. Miura (UBC) and R. M. Solow (MIT) for several stimulating discussions.

References

1. R. M. Solow, Congestion cost and the use of land for streets, *Bell J. Econ. Manage. Sci.* **4** (2), 602–618 (1973).
2. R. M. Solow, Congestion, density and the use of land in transportation, *Swed. J. Econ.* **74** (1), 161–173 (Mar. 1972).
3. W. Alonso, *Location and Land Use*, Harvard U. P., Cambridge, 1964.

The University of British Columbia

(Received July 28, 1976)