Studies in Applied Mathematics. Vol. LIV, No. 4, December 1975. Copyright © by The Massachusetts Institute of Technology

An Eigenvalue Problem for a Semi-infinite Pretwisted Strip

By Frederic Y. M. Wan

1. Introduction

'For an isotropic, homogeneous and linearly elastic sheet subjected to self-equilibrating edge stresses over a portion of its boundary and free of surface and edge load otherwise, an explicit rate of decay of the stresses in the sheet was given in [1]. In particular, the stresses in a semi-infinite rectangular strip subjected to self-equilibrating in-plane normal and shear stress resultants along its short edge become exponentially small at a distance large compared to the strip width away from the loaded edge. For a strip with a small amount of pretwist, one would expect the rate of stress decay to be qualitatively similar to that of a flat strip. However, it was noted in [2] that a Saint-Venant type solution for a pretwisted rectangular strip subjected to equal and opposite sheet end bending moments, first obtained in [3], becomes unrealistic whenever a dimensionless pretwist parameter μ is large compared to unity. With μ proportional to the amount of pretwist and inversely proportional to the sheet thickness, the pretwist parameter may be large even for a slightly pretwisted strip, if the sheet is sufficiently thin. Therefore, an analysis of the (self-equilibrating) edge load problem for the semi-infinite pretwisted strip should be instructive. For a strip with a relatively small amount of pretwist so that $\mu \ll 1$, it is apparent that a straightforward perturbation solution [4] would give four families of eigenfunctions for the relevant eigenvalue problem, all with a decay length which is essentially that of a flat strip with a small correction term of order μ. As we shall see, the situation for $\mu \gg 1$ is qualitatively different from that of the flat strip. Of the four possible families of solutions in this case, three have a decay length short (of order $\mu^{-1/3}$) compared to the strip width. The decay length of the remaining family is much longer, by a factor μ, than the strip width. Therefore, a Saint-Venant type solution for the sheet bending problem is of dubious value beyond the range of μ considered in [3]. In particular, the detailed distribution of applied edge loads is likely to be significant for a pretwisted strip sufficiently thin so that $\mu \gg 1$.

The combination of one long and three short decay lengths for our problem is distinctly different from the results for complete shells of revolution, for which a Fourier sine and/or cosine expansion of the solutions in the polar angle is possible [5], An asymptotic solution for the present problem requires some finesse.

2. The **eigenvalue problem for a pretwisted strip**

We consider here a slightly pretwisted strip of uniform properties and constant thickness h, whose elastostatic behavior is governed by Marguerre's linear shallow shell theory with the middle surface of the shell in Cartesian coordinates given by $z = kxy$, where k is a positive constant. For a semi-infinite isotropic strip of width $2b$ and with Young's modulus E and Poisson's ratio ν, we introduce the dimensionless quantities

$$W=\frac{w}{b},\qquad f=\frac{\sqrt{12(1-\nu^2)}}{Eh^2b}F,$$
$$\xi=\frac{x}{b},\qquad \eta=\frac{y}{b},\qquad \mu=\frac{kb^2}{h}\sqrt{12(1-\nu^2)}\equiv\varepsilon^{-3}, \tag{1}$$

where w and F are the midsurface axial displacement and the Airy stress function, respectively, and write the two governing partial differential equations for a strip free of surface load as

$$\nabla^2\nabla^2W=-\mu f'^{\cdot},\qquad \nabla^2\nabla^2 f=\mu W'^{\cdot}, \tag{2}$$

where primes and dots indicate differentiation with respect to ξ and η, respectively, and $\nabla^2(\,)=(\,)''+(\,)^{\cdot\cdot}$. In terms of f and W, we have also

$$\frac{b\sqrt{12(1-\nu^2)}}{Eh^2}\{N_{yy},N_{yx}\}=\{f'',-f'^{\cdot}\},\qquad \frac{b^2}{D}Q_y=\nabla^2W^{\cdot},$$
$$\frac{b}{D}M_{yy}=-(W^{\cdot\cdot}+\nu W''),\qquad \frac{b}{D}M_{yx}=-(1-\nu)W'^{\cdot} \tag{3}$$

for the stress resultants and couples needed in later development.

We are interested here in solutions of (2) of the form

$$\{W,f\}=\{\psi(\eta),\phi(\eta)\}e^{-\lambda\xi},\qquad \mathrm{Re}(\lambda)>0, \tag{4}$$

which give rise to no edge resultant and couple along the edges $y=\pm b\,(\eta=\pm 1)$ of the shell. The differential equations (2) require that ψ and ϕ satisfy the ODE

$$\psi^{\cdot\cdot\cdot\cdot}+2\lambda^2\psi^{\cdot\cdot}+\lambda^4\psi=\mu\lambda\phi^{\cdot}, \qquad \phi^{\cdot\cdot\cdot\cdot}+2\lambda^2\phi^{\cdot\cdot}+\lambda^4\phi=-\mu\lambda\psi^{\cdot} \tag{5, 6}$$

Along the edges $\eta=\pm 1$, the homogeneous Kirchhoff-Bassett stress boundary conditions, $N_{yy}=N_{yx}=M_{yy}=Q_y+M_{yx,x}=0$, require

$$\eta=\pm 1: \qquad \phi=\phi^{\cdot}=\psi^{\cdot\cdot}+\nu\lambda^2\psi=\psi^{\cdot\cdot\cdot}+(2-\nu)\lambda^2\psi^{\cdot}=0. \tag{7}$$

The ODE (5) and (6) and the boundary conditions (7) constitute an eigenvalue problem with λ as the eigenvalue and with μ as a parameter. We are interested here in the solution of this problem for $\mu\gg 1$. We mention in passing that for $\mu\ll 1$, a straightforward parametric series solution [4]

$$\{\phi,\psi,\lambda\}=\sum_{n=0}^{\infty}\{\hat{\phi}_n,\hat{\psi}_n,\hat{\lambda}_n\}\,\mu^n \tag{8}$$

shows that the eigenvalues and eigenfunctions are essentially those for a flat strip with small correction terms of order μ.

3. Eigenfunctions with long decay length

For $\mu\gg 1$, we set $\bar{\lambda}=\lambda\mu$ and write the boundary value problem (5)-(7) as

$$\psi^{\cdot\cdot\cdot\cdot}+2\mu^{-2}\bar{\lambda}^2\psi^{\cdot\cdot}+\mu^{-4}\bar{\lambda}^4\psi=\bar{\lambda}\phi^{\cdot}, \tag{9}$$

$$\phi^{\cdot\cdot\cdot\cdot}+2\mu^{-2}\bar{\lambda}^2\phi^{\cdot\cdot}+\mu^{-4}\bar{\lambda}^4\phi=-\bar{\lambda}\psi^{\cdot} \tag{10}$$

$$\eta=\pm 1: \qquad \phi=\phi^{\cdot}=\psi^{\cdot\cdot}+\nu\mu^{-2}\bar{\lambda}^2\psi=\psi^{\cdot\cdot\cdot}+(2-\nu)\,\mu^{-2}\bar{\lambda}^2\psi^{\cdot}=0. \tag{11}$$

A solution of (9)-(11) may be obtained in the form of parametric series

$$\{\phi,\psi,\bar{\lambda}\}=\sum_{n=0}^{\infty}\{\phi_n,\psi_n,\bar{\lambda}_n\}\,\mu^{-2n}. \tag{12}$$

The leading terms ϕ_0,ψ_0 and $\bar{\lambda}_0$ are determined by the eigenvalue problem

$$\psi_0^{\cdot\cdot\cdot\cdot}=\bar{\lambda}_0\phi_0^{\cdot}, \qquad \phi_0^{\cdot\cdot\cdot\cdot}=-\bar{\lambda}_0\psi_0^{\cdot}, \tag{13}$$

$$\eta=\pm 1: \qquad \phi_0=\phi_0^{\cdot}=\psi_0^{\cdot\cdot}=\psi_0^{\cdot\cdot\cdot}=0. \tag{14}$$

The solution of (13) may be written as

$$\psi_0 = C_1 \sin 2\gamma\eta + C_2 \cos\gamma\eta \sinh\sqrt{3}\,\gamma\eta + C_3 \sin\gamma\eta \cosh\sqrt{3}\,\gamma\eta$$

$$+ B_1 \cos 2\gamma\eta + B_2 \cos\gamma\eta \cosh\sqrt{3}\,\gamma\eta + B_3 \sin\gamma\eta \sinh\sqrt{3}\,\gamma\eta + B_0,$$

$$\phi_0 = C_0 - C_1 \cos 2\gamma\eta - C_2 \sin\gamma\eta \sinh\sqrt{3}\,\gamma\eta + C_3 \cos\gamma\eta \cosh\sqrt{3}\,\gamma\eta$$

$$+ B_1 \sin 2\gamma\eta - B_2 \sin\gamma\eta \cosh\sqrt{3}\,\gamma\eta + B_3 \cos\gamma\eta \sinh\sqrt{3}\,\gamma\eta,$$

where $\gamma=\frac{1}{2}\sqrt[3]{\bar\lambda_0}$ The constants of integration B_i and C_i (at least some of them) and the eigenvalue parameter $\bar\lambda_0$ are to be determined by eight homogeneous boundary conditions (14).

The functions in ψ_0 multiplied by the C_i's are odd in η, and those multiplied by the B_i's are even in η. The opposite is true for ϕ_0. The eight conditions (14) may therefore be arranged into two uncoupled sets of four homogeneous linear algebraic equations, one for the B_i's and the other for the C_i's. For our purpose (as stated in the Introduction), we will consider only the second set. The four equations for the C_i's are equivalent to the three conditions

$$C_1 \cos 2\gamma + C_2 \sin\gamma \sinh\sqrt{3}\,\gamma - C_3 \cos\gamma \cosh\sqrt{3}\,\gamma = 0,$$

$$2C_1 \sin 2\gamma - C_2\left(\cos\gamma \sinh\sqrt{3}\,\gamma + \sqrt{3}\,\sin\gamma \cosh\sqrt{3}\,\gamma\right)$$

$$- C_3\left(\sin\gamma \cosh\sqrt{3}\,\gamma - \sqrt{3}\,\cos\gamma \sinh\sqrt{3}\,\gamma\right) = 0, \tag{16}$$

$$2C_1 \sin 2\gamma - C_2\left(\cos\gamma \sinh\sqrt{3}\,\gamma - \sqrt{3}\,\sin\gamma \cosh\sqrt{3}\,\gamma\right)$$

$$- C_3\left(\sin\gamma \cosh\sqrt{3}\,\gamma + \sqrt{3}\,\cos\gamma \sinh\sqrt{3}\,\gamma\right) = 0,$$

and the following expression for C_0:

$$C_0 = C_1 \cos 2\gamma + C_2 \sin\gamma \sinh\sqrt{3}\,\gamma - C_3 \cos\gamma \cosh\sqrt{3}\,\gamma. \tag{17}$$

For the equations (16) to have a nontrivial solution, the determinant of the coefficient matrix must vanish. This requirement implies that the eigenvalue $\bar\lambda_0$ must be a root of the transcendental equation

$$\cos 2\gamma \left[\cosh(2\sqrt{3}\,\gamma) + \cos(2\gamma)\right]\ 2\,. \tag{18}$$

The eigenvalues are therefore given by

$$\bar\lambda_0^{(n)} = (2\gamma_n)^3, \qquad n=1,2,3\,,\ldots, \tag{19}$$

where γ_n are the roots of (18) for which $\bar{\lambda}_0$ has a positive real part. For each $\bar{\lambda}_0^{(n)}$, the homogeneous boundary value problem (13), (14) has a nontrivial solution.

With $\lambda \sim \bar{\lambda}_0/\mu$ and $\mu \gg 1$, the solution of the boundary value problem (5)-(7) and the corresponding stresses in the strip become exponentially small only at a distance away from the loaded edge much larger than the strip width. Inasmuch as μ increases with decreasing h for a strip with fixed width and pretwist [see (l)], the geometrically induced slow decay solutions found above may be considered as a counterpart of the slow decay solution induced by a certain type of anisotropy in flat sheets, found in [6]. But in contrast to the fourth order plane problem encountered in [6], the present shell problem involves an eigenvalue problem for an eighth order system of ODE. The study of this eighth order problem requires a completely different set of analytical tools.

4. Eigenfunctions with short decay length

While the results of last section are all we need to show that the decay length for the stresses in a pretwisted strip may be much longer than the strip width, it is still of some interest to investigate how the decay length of other solutions of the eigenvalue problem (5)-(7) behaves for large μ. Since we can prescribe four independent boundary conditions along the edge x =0 of the pretwisted strip, we expect that there should be three other families of eigenfunctions associated with three other sets of distinct eigenvalues (whose real part is positive). In what follows, we will show that there are in fact three families of eigenvalues whose dependence on μ differs in an essential way from those encountered in the last section.

From the structure of the ODE (5), (6) we expect that any new solution of our problem must be such that the contribution of the last term on the left side of each equation can no longer be neglected. This in turn suggests that we set

$$\tilde{\lambda} = \lambda\varepsilon, \qquad \varepsilon = \mu^{-1/3} \tag{20}$$

and write (5)-(7) as

$$\varepsilon^4\psi'''' + 2\varepsilon^2\tilde{\lambda}^2\psi'' + \tilde{\lambda}^4\psi = \tilde{\lambda}\phi',$$

$$\varepsilon^4\phi'''' + 2\varepsilon^2\tilde{\lambda}^2\phi'' + \tilde{\lambda}^4\phi = -\tilde{\lambda}\psi', \tag{21}$$

$$\eta = \pm 1: \qquad \phi = \phi' = \varepsilon^2\psi'' + \nu\tilde{\lambda}^2\psi = \varepsilon^2\psi''' + (2-\nu)\tilde{\lambda}^2\psi' = 0. \tag{22}$$

Evidently, an asymptotic solution in ε is appropriate for the above eigenvalue problem. In fact, with the rescaling (20), a straightforward application of the results of [7] gives[1]

$$\tilde{\lambda} \sim (n\pi)^{1/3}\left\{1, \tfrac{1}{2}(1 \pm \sqrt{3}\,i)\right\}, \tag{23}$$

$n = 1, 2, 3, \ldots,$ as well as the corresponding eigenfunctions. The latter will be

[1]The other roots of $\tilde{\lambda}^3 = \pm n\pi$ are rejected, as they have negative real parts. **Also, we** again considered only the case of ψ odd and ϕ even in η.

omitted, as they are not particularly relevant to our discussion here. We have therefore the following asymptotic behavior for the eigenvalue parameter:

$$\lambda=\tilde{\lambda}\mu^{1/3}\sim(n\pi\mu)^{1/3}\left\{1,\tfrac{1}{2}(1\pm\sqrt{3}\,i)\right\}. \tag{24}$$

The decay length for these three families of solutions is therefore short compared to the strip width.

It should be noted that eigenvalues problems such as (5)-(7) can often be transformed into those considered in [7] by simple preliminary rescalings The method outlined in [7] becomes applicable to the rescaled problems, though not to the original problems.

Appendix. Higher order corrections for solutions with long decay length

The perturbation scheme of Sec. 3 also allows us to calculate higher order correction terms. We will illustrate this with a discussion of the 0 (μ^{-2}) problem, i.e., the determination of ψ_1, ϕ_1 and $\bar{\lambda}_1$ in the expansions (12) for a given choice of $\bar{\lambda}_0$.

Inserting (12) into (9)-(11) leads to the following boundary value problem for the determination of ψ_1 and ϕ_1:

$$\begin{aligned}\psi_1^{\cdot\cdot\cdot\cdot}-\bar{\lambda}_0\phi_1^{\cdot}&=\bar{\lambda}_1\phi_0^{\cdot}-2\bar{\lambda}_0^2\psi_0^{\cdot\cdot},\\ \phi_1^{\cdot\cdot\cdot\cdot}+\bar{\lambda}_0\psi_1^{\cdot}&=-\bar{\lambda}_1\psi_0^{\cdot}-2\bar{\lambda}_0^2\phi_0^{\cdot\cdot},\end{aligned} \tag{A.1}$$

$$\eta=\pm1: \qquad \phi_1=\phi_1^{\cdot}=\psi_1^{\cdot\cdot}+\nu\bar{\lambda}_0^2\psi_0=\psi_1^{\cdot\cdot\cdot}+(2-\nu)\bar{\lambda}_0^2\psi_0^{\cdot}=0. \tag{A.2}$$

Again, we will consider only solutions of ϕ_1 and ψ_1 which are even and odd in η, respectively.

The general solution of the two ODE (A.1) is

$$\begin{aligned}\phi_1={}&A_0-A_1\cos(2\gamma\eta)+A_3\cos(\gamma\eta)\cosh(\sqrt{3}\,\gamma\eta)\\ &-A_2\sin(\gamma\eta)\sinh(\sqrt{3}\,\gamma\eta)+\bar{\lambda}_1\phi_{1p}(\eta)+2\bar{\lambda}_0^2\phi_{0p}(\eta),\end{aligned} \tag{A.3}$$

$$\begin{aligned}\psi_1={}&A_1\sin(2\gamma\eta)+A_3\sin(\gamma\eta)\cosh(\sqrt{3}\,\gamma\eta)\\ &+A_2\cos(\gamma\eta)\sinh(\sqrt{3}\,\gamma\eta)+\bar{\lambda}_1\psi_{1p}(\eta)+2\bar{\lambda}_0^2\psi_{0p}(\eta).\end{aligned} \tag{A.4}$$

Without writing out the particular solutions $\psi_{0p}, \psi_{1p}, \phi_{0p}$ and ϕ_{1p} explicitly, we merely note that the constant $\bar{\lambda}_1$ does not appear in these four functions.

For a given choice of $\bar{\lambda}_0$ (and therefore ψ_0 and ϕ_0), the four boundary conditions (A.2) give four conditions on the five constants A_i and $\bar{\lambda}_1$. We note that the condition on $\psi_1^{\cdot\cdot\cdot}$ is equivalent to $A_0+\nu\bar{\lambda}_0^2\psi_0^{\cdot}=0$ and thereby determines

A_0. The remaining three conditions become

$$A_1\cos(2\gamma)+A_2\sin\gamma\sinh(\sqrt{3}\,\gamma)-A_3\cos\gamma\cosh(\sqrt{3}\,\gamma)$$
$$=A_0+\bar{\lambda}_1\phi_{1p}(1)+2\bar{\lambda}_0^2\phi_{0p}(1), \tag{A.5}$$

$$2A_1\sin(2\gamma)-A_2\left[\cos\gamma\sinh(\sqrt{3}\,\gamma)+\sqrt{3}\,\sin\gamma\cosh(\sqrt{3}\,\gamma)\right]$$
$$-A_3\left[\sin\gamma\cosh(\sqrt{3}\,\gamma)-\sqrt{3}\,\cos\gamma\sinh(\sqrt{3}\,\gamma)\right]$$
$$=-\frac{\bar{\lambda}_1}{\gamma}\dot{\phi}_{1p}(1)-\frac{2\bar{\lambda}_0^2}{\gamma}\dot{\phi}_{0p}(1), \tag{A.6}$$

$$2A_1\sin(2\gamma)-A_2\left[\cos\gamma\sinh(\sqrt{3}\,\gamma)-\sqrt{3}\,\sin\gamma\cosh(\sqrt{3}\,\gamma)\right]$$
$$-A_3\left[\sin\gamma\cosh(\sqrt{3}\,\gamma)+\sqrt{3}\,\cos\gamma\sinh(\sqrt{3}\,\gamma)\right]$$
$$=\frac{\bar{\lambda}_1}{2\gamma^2}\ddot{\psi}_{1p}(1)+\frac{\bar{\lambda}_0^2}{\gamma^2}\ddot{\psi}_{0p}(1)+\frac{\nu\bar{\lambda}_0^2}{2\gamma^2}\psi_0(1). \tag{A.7}$$

Since $\bar{\lambda}_0$ is an eigenvalue of (16), the coefficient matrix of (A.5)-(A.7) is of rank 2. For this system to have a solution, the augmented matrix must also be of the same rank. This condition determines $\bar{\lambda}_1$ completely. The two conditions (A.5) and (A.6) determine A_2 and A_3 in terms of A_1. Finally, A, may be fixed by a normalization condition.

Acknowledgment

The author is grateful to Eric Reissner, who, together with the author, jointly initiated the investigation of the validity of Saint Venant type solutions for pretwisted strips, and with whom the author has had many fruitful discussions on the problem; to James K. Knowles, who called the author's attention to references [6] and [7] and shared his thoughts so generously with the author; and to Joe Keller for his helpful comments on the results. The research reported herein was supported by National Research Council of Canada Grant No. A9259.

References

1. J. K. Knowles, On Saint-Venant's principle in the two-dimensional linear theory of elasticity, ***Arch. Rat.*** Mech. ***Anal.*** 21, 1-22 (1966).
2. F. Y. M. Wan, St. Venant flexure of pretwisted rectangular plates, Int. J. ***Eng. Sci.*** 7, 351-360 (1969).

3. L. Maunder and E. Reissner, Pure bending of pretwisted rectangular plates, J. *Mech. Phys. Solids 5, 261-266 (1957).*
4. R. Bellman, *Perturbation Techniques in Mathematics, Physics and Engineering,* Holt, Rinehart and Winston, 1966.
5. J. G. Simmonds, Saint-Venant's principle for semi-infinite shells of revolution, to appear in *Proc. 10th Annual Meeting of Soc. of Eng. Sci.* (North Carolina State University, 1973).
6. C. O. Horgan, On Saint-Venant's principle in plane anisotropic elasticity, J. *Elasticity 2, 169-180 (1972).*
7. G. H. Handelman, J. B. Keller and R. E. O'Malley, Jr., Loss of boundary conditions in the asymptotic solution of linear ordinary differential equations, I: Eigenvalue problems, ***Comm.*** *Pure Appl. Math. 21, 243-261 (1969).*

UNIVERSITY OF BRITISH COLUMBIA

(Received May 20, 1975)