F. Y. M. WAN
Associate Professor,
Department of Mathematics,
Massachusetts Institute of Technology,
Cambridge, Mass. Mem. ASME

Nonstationary Response of Linear Time-Varying Dynamical Systems to Random Excitation[1]

A direct time-domain method is used to analyze the titled problem. Special attention is paid to a system characterized by a general second-order equation with variable coefficients. The equation of flapping motion of a rigid rotor blade advancing in atmospheric turbulence belongs to this class. Steady-state mean-square response to ideal white noise and to exponentially correlated excitation is obtained by a perturbation series solution in a stiffness parameter. An upperbound of the same is derived. Explicit solution for the correlation matrix is obtained by the two-variable expansion method. Specialized to the rotor blade problem, the results have led to some new information concerning the blade behavior in a certain range of rotating speed. They have also served as useful check cases for computer programs developed for more general problems. Higher-order equations and their applications are also discussed.

1 Introduction

The dimensionless second-order stochastic differential equation

$$\ddot{y} + c(t)\dot{y} + [P^2 + k(t)]y = x(t), \tag{1}$$

where P is a constant and the excitation $x(t)$ is a random function with known statistics, often occurs in the analysis of time-varying mechanical systems. For example, in the study of the small amplitude flapping motion of a rigid rotor blade in air turbulence, $y(t)$ is the flapping angle of the blade, Fig. 1, and $x(t)$ is of the form $g(t)N(t)$ where $g(t)$ is a well-defined envelope function and $N(t)$ is a random function characterizing, say, the inflow ratio [1].[2] The *known* functions $c(t)$ and $k(t)$ in that case are periodic functions of the dimensionless time, t, with period 2π (see Section 2.3).

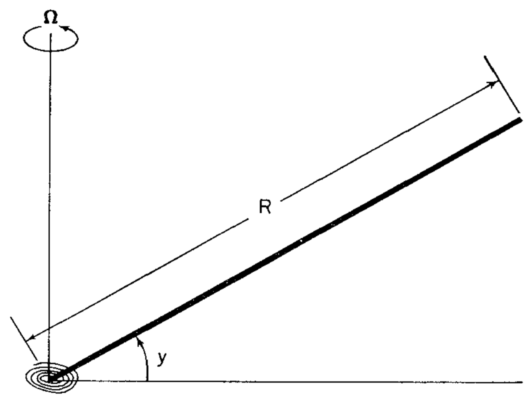

Fig. 1 Rigid rotor blade in spanwise flapping

The solution of the linear equation (1) can be written down immediately in terms of the corresponding impulse response function $h(t, z)$. If $y(0) = \dot{y}(0) = 0$, we have

$$y(t) = \int_0^t h(t, z)x(z)dz. \tag{2}$$

Unfortunately, $h(t, z)$ cannot be obtained in terms of simple or special functions for arbitrary $c(t)$ and $k(t)$. In general, we will

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[2] Numbers in brackets designate References at end of paper.

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have to determine $h(t, z)$ by numerical integration or some other approximate method. For a random excitation, the desired statistics of the response, $y(t)$, will still have to be computed by forming the ensemble average of different combinations of (2). The calculation for second or higher-order statistics will involve multiple integration. An alternate mixed time-frequency approach suggested in [2] and used in [11] is also tedious [3].

In practice, we are mainly interested in the first and second-order statistics of $y(t)$ (which are all we need for a Gaussian process). To determine these, we can use a different and more efficient method applicable to general linear time-varying dynamical systems of which (1) is a special case. The method is based on the observation that, for time-varying systems, the steady-state response of (1) will in general be a nonstationary random process even if $x(t)$ is stationary. In particular, the variance of y will change with t and the correlation will be a function of two distinct t_1 and t_2 and not just $t_2 - t_1$. The essential feature of our approach is to obtain (deterministic) differential equations for the desired time-varying statistics themselves. At the very least, the method will enable us to obtain the entire covariance matrix and the correlation matrix as functions of time by a single, straightforward algorithm, and the necessary calculations can be performed very efficiently by the computer [3].

Having the differential equations for the covariance and correlation functions, we are in a position to take advantage of the special properties of a given dynamical system to obtain approximate analytical solutions and their upperbounds by conventional methods. For example, the quantity P^2 in the rotor blade problem mentioned earlier is $1 + \omega^2/\Omega^2$ where ω^2 characterizes the elastic restraint at the root of the blade and Ω is the rotating speed. Evidently, the apparent stiffness parameter P increases as Ω decreases for a given blade [1]. For $P \gg 1$, we can obtain a parametric series solutions in powers of $1/P$ for the desired statistics. The approximate solutions in terms of elementary functions allow us to analyze qualitatively the effect of the various system parameters without a great deal of calculation.

Since the method does not seem to have been used in random vibration (see footnote three), it will be demonstrated in some detail through the system (1). Some new results will be given for the rotor blade problem. Aside from being useful data for an understanding of the behavior of rotor blades, these results have also served as check cases for computer programs developed to generate numerical solutions for the more general problem without any restriction on P and for more general blade models [3].

The corresponding treatment for a general linear dynamical system characterized by the vector equation

$$\dot{u} = A(t)u + G(t)N(t) \tag{3}$$

and its application to multidegree-of-freedom systems and flexible rotor blades will be discussed briefly.

2 Response to Shaped White Noise

2.1 Differential Equations for the Covariance Matrix. In order to take advantage of the matrix notation and to allow for later generalization, we set $y_1 = y$ and $y_2 = \dot{y}$ so that

$$\dot{y}_1 = y_2 \tag{4}$$

and write (1) as

$$\dot{y}_2 = -[P^2 + k(t)]y_1 - c(t)y_2 + g(t)N(t) \tag{5}$$

where $g(t)$ is a well-defined envelope function and $N(t)$ is a random process with zero mean. It follows immediately from (2) that the response y and $\dot{y}$ will also be of zero mean. Furthermore, for a white-noise input $N(t)$ with normalized spectral density, we have

$$\langle y_1(t)N(t)\rangle = 0, \qquad \langle y_2(t)N(t)\rangle = \tfrac{1}{2}g(t) \tag{6}$$

where $\langle \ldots \rangle$ is the ensemble-averaging operation.

With the covariance $D_{ij}(t)$ defined as $\langle y_i(t)y_j(t)\rangle$, $i, j = 1, 2$, and with the help of (4)–(6), we get

$$\begin{aligned} \dot{D}_{11} &= 2D_{12}, \qquad \dot{D}_{12} = -[P^2 + k(t)]D_{11} - c(t)D_{12} + D_{22} \\ \dot{D}_{22} &= -2[P^2 + k(t)]D_{12} - 2c(t)D_{22} + g^2(t) \end{aligned} \tag{7}$$

Since $D_{12} = D_{21}$, we may put the three equations of (7) in the form of a matrix equation

$$\dot{D} = AD + DA^T + GG^T \tag{8}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -[P^2 + k(t)] & -c(t) \end{bmatrix}; \qquad G = \begin{bmatrix} 0 & 0 \\ 0 & g(t) \end{bmatrix}$$

and where $(\ \ )^T$ is the transpose of $(\ \ )$. In terms of A and G, we may also write (4) and (5) as $\dot{u} = Au + Gv$ where $u = (y_1, y_2)^T$ and $v = (0, N)^T$.

Equation (8) along with the initial condition $D(0) = 0$ (which follows from $y(0) = \dot{y}(0) = 0$) determines the covariance matrix D completely.[3] In cases where an exact solution in terms of elementary or special functions is not possible, the system of first-order ordinary differential equations (8) is in a form suitable for numerical integration.

2.2 Perturbation Solution for $P^2 \gg 1$. The system (7) is an exact consequence of (1) and a shaped white-noise input. With a view toward a steady-state perturbation solution, we now consider the case where g, k, and c are analytic and are slowly varying functions in the sense that any significant change occurs only over a (dimensionless) time period of order unity or larger, while c and k are of order unity and c is positive.

Guided by hindsight, we will scale D_{ij} by setting $d_{11} = P^2D_{11}$, $d_{12} = P^2D_{12}$, and $d_{22} = D_{22}$. In terms of d_{ij}, the system (7) becomes

$$\begin{aligned} \dot{d}_{11} &= 2d_{12}, \\ P^{-2}\dot{d}_{12} &= d_{22} - [1 + P^{-2}k(t)]d_{11} - P^{-2}c(t)d_{12} \\ \dot{d}_{22} &= -2[1 + P^{-2}k(t)]d_{12} - 2c(t)d_{22} + g^2(t) \end{aligned} \tag{9}$$

A formal steady-state solution of (9) can be obtained in the form of a parametric series for d_{ij}

$$d_{ij} = \sum_{n=0}^{\infty} d_{ij}^{(n)}(t)P^{-2n} \tag{10}$$

with the coefficients $d_{ij}^{(n)}$ determined by

$$\begin{aligned} \dot{d}_{11}^{(0)} &= 2d_{12}^{(0)}, \qquad d_{22}^{(0)} = d_{11}^{(0)}, \\ \dot{d}_{22}^{(0)} &= -2d_{12}^{(0)} - 2c(t)d_{22}^{(0)} + g^2(t) \end{aligned} \tag{11}$$

and for $n \geq 1$,

$$\begin{aligned} &\dot{d}_{11}^{(n)} - 2d_{12}^{(n)} = 0, \\ &\dot{d}_{12}^{(n-1)} - d_{22}^{(n)} + d_{11}^{(n)} + k(t)d_{11}^{(n-1)} + c(t)d_{12}^{(n-1)} = 0 \\ &\dot{d}_{22}^{(n)} + 2d_{12}^{(n)} + 2k(t)d_{12}^{(n-1)} + 2c(t)d_{22}^{(n)} = 0 \end{aligned} \tag{12}$$

The system (11) may be rearranged to read

$$\dot{d}_{11}^{(0)} + c(t)d_{11}^{(0)} = \tfrac{1}{2}g^2(t), \qquad d_{22}^{(0)} = d_{11}^{(0)}, \qquad d_{12}^{(0)} = \tfrac{1}{2}\dot{d}_{11}^{(0)} \tag{13}$$

[3] An equation of the form (8) for the covariance matrix was given in [9] for a time invariant dynamical systems subject to shot noise. After the manuscript had been submitted, Dr. G. H. Gaonkar pointed out in a letter to the author that he too had used the direct time domain approach for the rotor blade problem in a recent paper [10]. A regular perturbation solution (valid for small $\gamma\mu/6$) and numerical integration were used to obtain solution for $P = 0(1)$ and an ideal white-noise excitation.

An exact solution of (13) and therefore (11) is always possible since it involves only the solution of a single first-order linear differential equation. In particular, we have in terms of the unscaled variables

$$D_{11}^{(0)} = \tfrac{1}{2}P^{-2}e^{-C(t)}\int^t e^{C(z)}g^2(z)dz, \qquad C(t) = \int_0^t c(z)dz$$
$$P^{-2}D_{22}^{(0)} = D_{11}^{(0)}, \qquad D_{12}^{(0)} = \tfrac{1}{2}\dot{D}_{11}^{(0)} \tag{14}$$

If P^2 is large compared to unity, (14) may be used as a good first approximation of the exact steady-state solution. Note that not all initial conditions are satisfied by (14) (except for special cases) even if we pick the lower limit of integration in the expression for $D_{11}^{(0)}$ to be zero. Higher-order correction terms do not improve the situation (see equation (16)). All initial conditions can be satisfied by combining (14) with suitable supplementary solutions corresponding to the transient response obtained by the two-variable expansion method [4]. But we will omit this step here since we are mainly interested in the steady-state response.

For the case $c(t) = c_0$ (a constant) and $k(t) = 0$, the leading term solution given by (14) becomes

$$D_{11}^{(0)} = \tfrac{1}{2}P^{-2}e^{-c_0t}\int^t e^{c_0z}g^2(z)dz, \tag{15}$$

the same as that obtained in [5] (which investigated only time invariant second-order systems) by a completely different method.

If $g(t)$ is a unit step function, the steady-state solution for $D_{11}^{(0)}$ obtained from (15) is the exact solution as given in [6].

The system (12) for the higher-order coefficients can also be reduced to solving a single first-order linear differential equation for each n

$$\dot{d}_{22}^{(n)} + c(t)d_{22}^{(n)} = \tfrac{1}{2}\{k(t)d_{11}^{(n-1)} + [\dot{d}_{12}^{(n-1)} + c(t)d_{12}^{(n-1)}]^{\cdot}\}$$
$$d_{11}^{(n)} = d_{22}^{(n)} - k(t)d_{11}^{(n-1)} - \dot{d}_{12}^{(n-1)} - c(t)d_{12}^{(n-1)}, \tag{16}$$
$$d_{12}^{(n)} = \tfrac{1}{2}\dot{d}_{11}^{(n)}$$

We see from (16) that the order of the system (12) is not high enough to allow for the satisfaction of all initial conditions. Therefore, the higher-order terms in expansion (10) correct only the leading term steady-state solution.

2.3 Application to Rotor Blades. For a rotor blade with small-to-moderate forward speed compared to tip rotating speed (the ratio is called the *advance ratio*) so that the effect of reverse flow is not important, the small amplitude flapping motion of the blade is governed by an equation of the form (1) with

$$g(t) = \frac{\gamma}{6}\left(1 + \frac{3\mu}{2}\sin t\right), \qquad k(t) = \frac{\gamma\mu}{6}\cos t - \frac{\mu^2\gamma}{8}\sin 2t,$$
$$c(t) = \frac{\gamma}{8}\left(1 + \frac{4\mu}{3}\sin t\right), \qquad C(t) = \frac{\gamma}{8}\left[t + \frac{4\mu}{3}(1 - \cos t)\right] \tag{17}$$

where μ is the advance ratio and γ is the so-called Lock number $(2 \le \gamma \le 16)$. In order that the reverse flow effect be negligible, we limit ourselves here to the range $0 \le \mu \le 0.6$ [3].

The derivation of (1) (which is simply a statement of the moment equilibrium of the blade [1]) is based on a model of a rigid blade attached to a fixed hub with elastically restrained central flapping hinges, on quasi-steady linearized aerodynamics (for the determination of the lift force) and on vanishing horizontal random turbulence velocity.

The leading term periodic solution $D_{11}^{(0)}$ given by (14) and (17) is shown in Fig. 2 over one period for $\gamma = 4$ and $\mu = 0.2, 0.4$, and 0.6. We see that the peak mean-square deflection of the blade in forward flight ($\mu > 0$) is substantially larger than that in hovering ($\mu = 0$). For $\mu = 0.6$, the former is more than one and a half times the latter. The single peak value is attained near the end of the forward sweep.

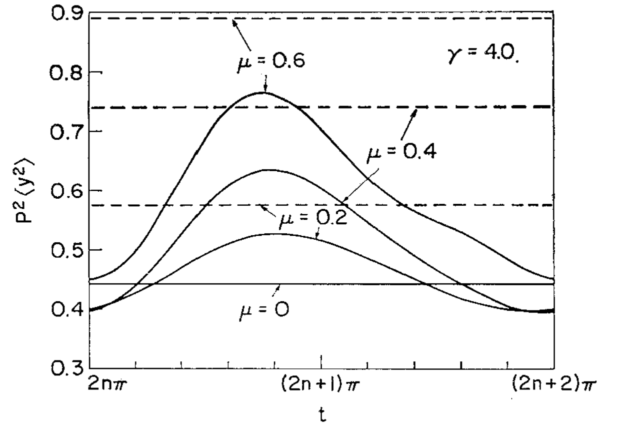

Fig. 2 Mean-square response of a rotor blade to white-noise excitation

The particular $g(t)$ used here corresponds to the case where $N(t)$ is the spatially uniform inflow ratio. Since very little is known about the effect of the random shedding of blade trailing vortices on the free atmospheric turbulence, except that it is likely to be substantial in the low advance ratio range, the results for the rotor blade problem given in this paper should only be viewed as a contribution to a qualitative understanding of a rotor blade in response to random load.

2.4 Peak Values and Bounds. In the study of the effect of various system parameters such as γ and μ in the rotor blade problem, one is often interested in how the peak values of D_{ij} depend on these parameters. Suppose the steady state $D_{11}^{(0)}$ attains a maximum at $\bar{t}$, then $\dot{D}_{11}^{(0)}(\bar{t}) = 0$. From (13), we have

$$D_{11}^{(0)}(\bar{t}) = \frac{g^2(\bar{t})}{2P^2c(\bar{t})} \tag{18}$$

Since $\bar{t}$ itself depends on the system parameters, a more useful piece of information is an upper bound on $D_{11}^{(0)}$. While we can obtain such a bound for the case $c(t) > 0$ for all t simply by maximizing the right-hand side of (18), we will take a different approach which applies also to correlated excitations. With $c(t) > 0$, we have from (14)

$$D_{11}^{(0)}(t) \le \max_{0\le z\le t}\left[\frac{g^2(z)}{2P^2c(z)}\right]e^{-C(t)}\int^t e^{C(z)}c(z)dz \tag{19}$$

so that at steady state

$$D_{11}^{(0)}(t) \le \frac{1}{2P^2}\max_{0\le z\le t}\left[\frac{g^2(z)}{c(z)}\right] \tag{20}$$

The peak value and bound of $D_{12}^{(0)}$ and $D_{22}^{(0)}$ follow from those of $D_{11}^{(0)}$ and equation (11).

For the steady-state solution of the rotor blade problem, we have for $\mu \le 0.6$

$$P^2D_{11}^{(0)}(t) \le \frac{\gamma}{9}\,\frac{(1+\frac{3}{2}\mu)^2}{1+\frac{4}{3}\mu} \tag{21}$$

For the case $\gamma = 4$ and $\mu = 0.2, 0.4$, and 0.6, this bound is given as a dashed line in Fig. 2.

2.5 Correlation Matrix. We denote by $R_{ij}(t; t_1)$ the correlation function $\langle y_i(t_1 + t)y_j(t_1)\rangle$ for a fixed $t_1 > 0$. The semicolon is used to distinguish it from the more conventional notation $R_{ij}(t_1, t_2)$ which means $\langle y_i(t_1)y_j(t_2)\rangle$. We will consider here only the case $t > 0$, since for fixed t and t_1 the results for $t < 0$ can be obtained from those for $t > 0$ with the help of the symmetry conditions $R_{ij}(t_2, t_1) = R_{ji}(t_1, t_2)$ (see also the Appendix).

From the definition of $R_{ij}(t; t_1)$ and equation (4), we have immediately

$$\dot{R}_{11} = R_{21}, \qquad \dot{R}_{12} = R_{22} \tag{22}$$

where dots indicate differentiation with respect to t with t_1 as a parameter. Also, since

$$\langle x(t_1 + t)y_1(t_1)\rangle = \int_0^{t_1} h(t_1, z)\langle N(t_1 + t)N(z)\rangle g(t_1 + t)g(z)dz = 0 \tag{23}$$

and similarly $\langle x(t_1 + t)y_2(t_1)\rangle = 0$ for $t > 0$, we have from (5) that

$$\begin{aligned} \dot{R}_{21} &= -[P^2 + k(t_1 + t)]R_{11} - c(t_1 + t)R_{21} \\ \dot{R}_{22} &= -[P^2 + k(t_1 + t)]R_{12} - c(t_1 + t)R_{22} \end{aligned} \tag{24}$$

By the continuity of R_{ij} at $t = 0$, we have $R_{ij}(0; t_1) = D_{ij}(t_1)$ (see the Appendix). The latter serves as the initial conditions for the homogeneous system (22) and (24). The system may be put in the form of a single matrix equation

$$\dot{R} = A(t_1 + t)R, \qquad R(0; t_1) = D(t_1) \tag{25}$$

which is suitable for numerical integration.

The differential equations for R_{ij} separate into two identical second-order systems, one for R_{21} and R_{11} and the other for R_{12} and R_{22}. For $P^2 \gg 1$, a uniformly valid asymptotic solution for R_{ij} may be obtained by the two-variable expansion method [4] with the slow and fast time being t and t^*, respectively, and with R_{ij} expanded in powers of $1/P$

$$R_{ij}(t; t_1) = \sum_{n=0}^{\infty} R_{ij}^{(n)}(t, t^*; t_1)P^{-n} \tag{26}$$

where

$$t^* = P^*t = Pt[1 + \omega_2 P^{-2} + \omega_3 P^{-3} + \ldots] \tag{27}$$

We omit the details of the two-variable expansion method and give here only the results

$$\begin{aligned}
P^2 e^{\frac{1}{2}C(t,t_1)}R_{11} &= d_{11}(t_1)\cos P^*t \\
&\quad + P^{-1}[d_{12}(t_1) + \tfrac{1}{4}d_{11}(t_1)B_{11}(t, t_1)]\sin P^*t + 0(P^{-2}) \\
Pe^{\frac{1}{2}C(t,t_1)}R_{21} &= -d_{11}(t_1)\sin P^*t \\
&\quad + P^{-1}[d_{12}(t_1) + \tfrac{1}{4}d_{11}(t_1)B_{21}(t, t_1)]\cos P^*t + 0(P^{-2}) \\
Pe^{\frac{1}{2}C(t,t_1)}R_{12} &= d_{22}(t_1)\sin P^*t \\
&\quad + P^{-1}[d_{12}(t_1) - \tfrac{1}{4}d_{22}(t_1)B_{12}(t, t_1)]\cos P^*t + 0(P^{-2}) \\
e^{\frac{1}{2}C(t,t_1)}R_{22} &= d_{22}(t_1)\cos P^*t \\
&\quad - P^{-1}[d_{12}(t_1) - \tfrac{1}{4}d_{22}(t_1)B_{22}(t, t_1)]\sin P^*t + 0(P^{-2})
\end{aligned} \tag{28}$$

where $d_{ij}(t)$ are defined as in Section 2.2 and

$$C(t, t_1) = \int_0^t c(t_1 + z)dz \tag{29}$$

$$B_{ij}(t, t_1) = (-)^{i-1}c(t_1 + t) + (-)^{j-1}c(t_1) + B(t, t_1)$$

with

$$B(t, t_1) = \int_0^t [4\omega_2 - 2k(t_1 + z) + \tfrac{1}{2}c^2(t_1 + z)]dz \tag{30}$$

and $C(t, 0) = C(t)$ (see equation (14)). It is interesting that $k(t)$ contributes to the foregoing results while it does not contribute to the first approximation of the $D_{ij}(t)$.

To the extent that we have omitted $0(1/P^2)$ terms, we may replace P^*t by Pt. Nevertheless, ω_2 must be chosen in such a way that $0(P^{-2})$ terms in the expansions are bounded by $\exp[-\frac{1}{2}C(t, t_1)]$ as $t \to \infty$.

The value for ω_2 depends of course on the actual form of $k(t)$ and $c(t)$. For a time invariant system with $c(t) = c_0$ and $k(t) = 0$, we have

$$\begin{aligned} \omega_2 &= -\tfrac{1}{8}c_0^2, \qquad B(t, t_1) = 0 \\ B_{12}(t, t_1) &= B_{21}(t, t_1) = 0, \quad B_{11}(t, t_1) = B_{22}(t, t_1) = 2c_0 \end{aligned} \tag{31}$$

Up to terms of order $1/P^2$, our results are the same as the exact solution [6].

For the rotor blade problem of Section 2.4, we have

$$\begin{aligned} \omega_2 &= -\frac{\gamma^2}{512}\left(1 + \frac{8\mu^2}{9}\right), \\ C(t, t_1) &= \frac{\gamma}{8}\left\{t - \frac{4\mu}{3}[\cos(t_1 + t) - \cos t_1]\right\} \end{aligned} \tag{32}$$

and

$$\begin{aligned} B(t, t_1) = &-\frac{\gamma^2\mu}{48}\left\{[\cos(t_1 + t) - \cos t_1] + \frac{\mu}{6}[\sin 2(t_1 + t) - \sin 2t_1]\right\} \\ &+ \frac{\gamma\mu}{3}\left\{[\sin(t_1 + t) - \sin t_1] - \frac{3\mu}{8}[\cos 2(t_1 + t) - \cos 2t_1]\right\} \end{aligned} \tag{33}$$

From (32) and the expression for $R_{11}(t; t_1)$, it appears that for a fixed t_1 and $0 \leq \mu \leq 0.6$, whether the flapping motion is a quasi-narrow-band process depends on the magnitude of the Lock number γ alone. Since γ is a measure of the aerodynamic damping, the situation is similar to the response of a time invariant dynamical system to shaped white-noise excitation. The same may not be true for larger values of μ. It should be kept in mind, however, that for $\mu > 0.6$, the effect of reverse flow can no longer be neglected.

3 Response to Shaped Correlated Noise

3.1 Differential Equations for the Covariance Matrix. If the random part, $N(t)$, of the excitation, $x(t)$, in (5) is not white noise, the general scheme used in Section 2 does not apply because $\langle y_k(t)x(t)\rangle$, $k = 1, 2$, does not necessarily vanish. One way of obtaining a system of differential equations for the determination of the variances and covariance of $y_1 = y$ and $y_2 = \dot{y}$ is to associate the correlated noise with the response of another linear dynamical system to white noise. By this, we mean that the correlated noise and the steady-state output of the supplementary linear system have the same first and second-order statistics. By setting up stationary or nonstationary shaping filters of first or higher order with white-noise input, almost any correlation function for $N(t)$ can be generated [7, 8].[4] Once this supplementary linear system is found, the problem can be reduced to one similar to that studied in Section 2.

To illustrate, we consider in this section a noise process, $N(t)$, with zero mean, unit variance and a correlation function

$$\langle N(t_2)N(t_1)\rangle = e^{-\alpha|t_2 - t_1|} \tag{34}$$

[4] We should also keep in mind that the statistics on $N(t)$ are often obtained from only a few of its sample histories.

Note that $x(t)$ may not be stationary even though $N(t)$ is stationary. Again, we have only to consider the case $t_2 > t_1$. For this case, we can associate the random function $N(t)$ with the steady-state output of the first-order linear time invariant system characterized by the differential equation

$$\dot{y}_3 + \alpha y_3 = \sqrt{2\alpha}\, w(t) \tag{35}$$

where $w(t)$ is a white-noise process with zero mean and normalized spectral density. We can show by way of the integral representation of y_k that

$$\langle y_1(t)w(t)\rangle = \langle y_2(t)w(t)\rangle = 0$$

$$\langle y_3(t)w(t)\rangle = \sqrt{\tfrac{1}{2}\alpha} \qquad \tfrac{1}{2}\langle y_3^2(t)\rangle^{\cdot} + \alpha\langle y_3^2(t)\rangle = \alpha \tag{36}$$

With these results, we obtain immediately from (35), (4), and (5)

$$\begin{aligned} \dot{D}_{13} &= D_{23} - \alpha D_{13} \\ \dot{D}_{23} &= -[P^2 + k(t)]D_{13} - [c(t) + \alpha]D_{23} + g(t)D_{33} \end{aligned} \tag{37}$$

where we have identified y_3 with N and where $D_{ij} = \langle y_i(t)y_j(t)\rangle$. These two equations along with the initial conditions $D_{13}^{(0)} = D_{23}^{(0)} = 0$ determine the covariance D_{13} and D_{23}, since D_{33} has already been obtained from the last equation of (36).

From (5), (35), and (36), we get also

$$\begin{aligned} \dot{D}_{11} &= 2D_{12}, \\ \dot{D}_{12} &= -[P^2 + k(t)]D_{11} - c(t)D_{12} + D_{22} + g(t)D_{13} \\ \dot{D}_{22} &= -2[P^2 + k(t)]D_{12} - 2c(t)D_{22} + 2g(t)D_{23} \end{aligned} \tag{38}$$

Having D_{13} and D_{23} from (37), the system (38) determines the remaining D_{ij}.

We can again put the equations for D_{ij}, $i, j = 1, 2, 3$, in the form of a matrix equation

$$\dot{D} = AD + DA^T + GG^T \tag{39}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -[P^2 + k(t)] & -c(t) & 0 \\ 0 & 0 & -\alpha \end{bmatrix}; \qquad G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2\alpha} \end{bmatrix} \tag{40}$$

Equation (39) is in a form suitable for numerical integration.

Note that the original stochastic differential equations for y_k can also be expressed in terms of A and G and two vectors $u = (y_1, y_2, y_3)^T$ and $v = (0, 0, w)^T$:

$$\dot{u} = Au + Gv \tag{41}$$

3.2 Leading Term Perturbation Solution for Large P^2. The parametric series solution in powers of P^{-2} described in Section 2.2 is again appropriate for the systems (37) and (38) whenever $P^2 \gg 1$. We give here only the result for the leading term of the parametric series solution which should be an adequate approximation for the steady-state solution for $P^2 \gg 1$.

From (37), we get

$$\begin{aligned} D_{13} &= P^{-2}g(t)[1 + 0(P^{-2})], \\ D_{23} &= P^{-2}(\dot{g} + \alpha g)[1 + 0(P^{-2})] \end{aligned} \tag{42}$$

After some rearrangement, the differential equations for the leading term solution of the remaining D_{ij} read as follows:

$$\begin{aligned} &\dot{\hat{d}}_{11} + c(t)\hat{d}_{11} = (g^2)^{\cdot} + [c(t) + \alpha]g^2 \\ &\hat{d}_{12} = -\tfrac{1}{2}\{c(t)\hat{d}_{11} - (g^2)^{\cdot} - [c(t) + \alpha]g^2\}, \qquad \hat{d}_{22} = \hat{d}_{11} - g^2 \end{aligned} \tag{43}$$

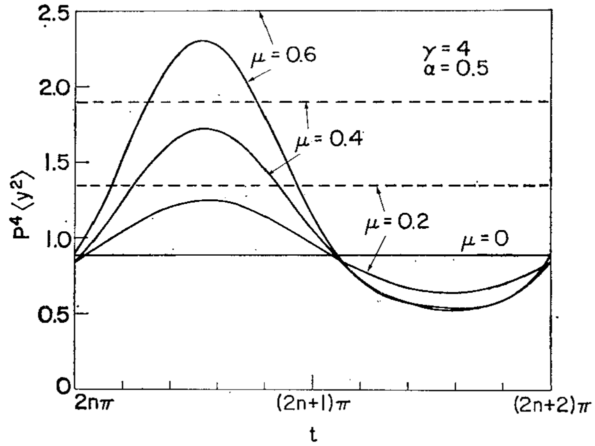

Fig. 3 Mean-square response of a rotor blade to an exponentially correlated excitation

where $\hat{d}_{11} = P^4D_{11}^{(0)}$, $\hat{d}_{12} = P^4D_{12}^{(0)}$ and $\hat{d}_{22} = P^2D_{22}^{(0)}$. The differential equation for $\hat{d}_{11}$ can be integrated and the steady-state solution is

$$D_{11}^{(0)} = P^{-4}\left[g^2(t) + \alpha e^{-C(t)} \int^t e^{C(z)}g^2(z)dz\right] \tag{44}$$

where $C(t)$ is as defined in (14). $D_{12}^{(0)}$ and $D_{22}^{(0)}$ are then obtained from the remaining equations of (43).

Note that D_{11}, D_{12}, and D_{22} for our shaped correlated noise are an order of magnitude smaller (by a factor $1/P^2$) than the corresponding quantities for the shaped white-noise excitation.

For a time invariant system with $c(t) = c_0$ and $k(t) = 0$ and for a unit step envelope function $g(t)$, the result given by (44) agrees with the exact solution up to terms of order $1/P^2$.

3.3 Application to Rotor Blades. For the model of rotor blade discussed earlier, we have as the steady-state solution

$$D_{11}^{(0)} = \frac{1}{P^4}\left(\frac{\gamma}{6}\right)^2 \left[\left(1 + \frac{3\mu}{2}\sin t\right)^2 + \alpha e^{-C(t)} \int^t e^{C(z)}\left(1 + \frac{3\mu}{2}\sin z\right)^2 dz\right] \tag{45}$$

where $C(t)$ is given by (17).

The leading term periodic solution $D_{11}^{(0)}$ given by (45) is shown in Fig. 3 over one period for $\gamma = 4$, $\alpha = 0.5$, and $\mu = 0.2$, 0.4, and 0.6. Note that the maximum mean-square deflection increases with μ. For $\mu = 0.6$, it is almost three times that for a blade in hovering ($\mu = 0$). Just as in the case of a white-noise excitation, the single peak value is attained during the forward sweep. This is in contrast to the results for $P = 0(1)$, reference [3], which show two relative maxima per revolution with the higher peak value occurring during the back stroke.

As $\alpha \to 0$, we have from (43) and (44) that

$$D_{11}^{(0)} \to P^{-4}g^2(t), \qquad D_{12}^{(0)} \to \tfrac{1}{2}P^{-4}(g^2)^{\cdot}, \qquad D_{22}^{(0)} \to 0 \tag{46}$$

so that D_{22} is of the same order as D_{11}. Except for higher-order terms, $g(t)/P^2$ is simply the deterministic solution of (1) with $x(t) \equiv g(t)$. We note that $\alpha \simeq 0$ means a turbulence scale length much longer than the blade length, a situation prevalent in free air turbulence at high altitude. The long relaxation time of the excitation is a direct opposite of the uncorrelated nature of a shaped white-noise excitation. Together, these two extreme cases delimit the results for finite values of α.

3.4 Peak Values and Bounds. Suppose that $D_{11}^{(0)}$ reaches a maximum at $t = \bar{t}$, then $\dot{D}_{11}^{(0)}(\bar{t}) = 0$. We get from (43)

$$D_{11}^{(0)}(\bar{t}) = \left[\frac{1}{c(t)}\{(g^2)^{\cdot} + [\alpha + c(t)]g^2\}\right]_{t=\bar{t}} \tag{47}$$

A more useful result is an upper bound on $D_{11}^{(0)}$ obtained from (44) by the technique which gave us (20)

$$D_{11}^{(0)}(t) \leq \frac{1}{P^4} \max_{0 \leq z \leq t} \left\{ g^2(z) \left[1 + \frac{\alpha}{c(z)} \right] \right\} \tag{48}$$

For the steady-state response of the rotor blade model discussed in the last section, we have

$$D_{11}^{(0)}(t) \leq \frac{1}{P^4} \left(\frac{\gamma}{6} \right)^2 \left(1 + \frac{3\mu}{2} \right)^2 \left[1 + \frac{8\alpha/\gamma}{1 + 4\mu/3} \right] \tag{49}$$

This bound is given as a dashed line in Fig. 3 for the three D_{11} curves.

3.5 Correlation Matrix. As in Section 2.6, we define $R_{ij}(t; t_1) = \langle y_i(t_1 + t) y_j(t_1) \rangle$, $i, j = 1, 2, 3$, for positive t and t_1. With $\langle w(t_1 + t) y_k(t_1) \rangle = 0$ for $k = 1, 2, 3$, it is not difficult to obtain a system of differential equations for the determination of R_{ij} as functions of t with t_1 as a parameter. The procedure is analogous to that used in Section 2.6. We can write this system of equations as a single matrix equation

$$\dot{R} = A(t_1 + t)R \tag{50}$$

where $A(t)$ is as defined by equation (40). Furthermore, we have

$$R(0; t_1) = D(t_1) \tag{51}$$

(see the Appendix). The solution of the initial value problem for R can be obtained by numerical integration in general.

For the purpose of obtaining an approximate solution for $P^2 \gg 1$, we note that the system (50) consists of two groups of equations. One of these includes three uncoupled equations of the form

$$\dot{R}_{3j} + \alpha R_{3j} = 0; \qquad (j = 1, 2, 3) \tag{52}$$

The solution of (52) which satisfies the associated initial condition is

$$R_{3j}(t; t_1) = D_{3j}(t_1) e^{-\alpha t} \tag{53}$$

The other group involves three systems of two coupled equations

$$\begin{aligned} \dot{R}_{1j} &= R_{2j}, \\ \dot{R}_{2j} &= -[P^2 + k(t_1 + t)]R_{1j} - c(t_1 + t)R_{2j} + g(t_1 + t)R_{3j} \end{aligned} \tag{54}$$

$j = 1, 2, 3$. Since R_{3j} has already been obtained in (53), the two equations of (54) for each j determine R_{1j} and R_{2j}. Except for the inhomogeneous terms involving R_{3j}, (54) is the same as the corresponding equations for the case of shaped white-noise input discussed in Section 2.6. The solution can be obtained in a similar way. We list here only the results for the $j = 1$ set

$$\begin{aligned} P^4 e^{\frac{1}{2}C(t,t_1)} R_{11} &= \tilde{d}_{11}(t_1) \cos P^*t \\ &\quad + P^{-1} [\tfrac{3}{2}\alpha g^2(t_1) + \tfrac{1}{4}\tilde{d}_{11}(t_1) B_{12}(t, t_1)] \sin P^*t \\ &\quad + g(t_1 + t) g(t_1) e^{-\alpha t + \frac{1}{2}C(t,t_1)} + 0(P^{-2}) \\ P^3 e^{\frac{1}{2}C(t,t_1)} R_{21} &= -\tilde{d}_{11}(t_1) \sin P^*t \\ &\quad + P^{-1}[\tfrac{3}{2}\alpha g^2(t_1) - \tfrac{1}{4}\tilde{d}_{11}(t_1) B_{22}(t, t_1)] \cos P^*t \\ &\quad + P^{-1}[\dot{g}(t + t_1) g(t_1) - \alpha g(t + t_1) g(t_1)] e^{-\alpha t + \frac{1}{2}C(t,t_1)} \\ &\quad + 0(P^{-2}) \end{aligned} \tag{55}$$

where

$$\tilde{d}_{11}(t_1) = \hat{d}_{11}(t_1) - g^2(t_1) = \alpha e^{-C(t_1)} \int_0^{t_1} e^{C(z)} g^2(z) dz \tag{56}$$

and where P^*, C, B_{12}, and B_{22} are as defined by (28)–(30).

For a time invariant system with a unit step envelope function, the results given by (55) agree with the exact solution up to terms of order $1/P^2$.

Note that unlike the case of a white-noise excitation where the relaxation time of the response (in units of Ω^{-1}) is dictated by the magnitude of the damping coefficient $c(t)$, the same relaxation time for an exponentially correlated excitation is the larger of $1/\alpha$ and $8/\gamma$. In fact, as $\alpha \to 0$, we have

$$R_{11} \to P^{-4} g(t_1 + t) g(t_1), \qquad R_{21} \to P^{-4} g(t_1 + t) g(t_1) \tag{57}$$

which may not tend to zero as $t \to \infty$.

4 Expected Rate of Threshold Crossing

The second-order statistics obtained in Sections 2 and 3 are useful for computing the expected rate of upward threshold crossing per unit time [9]

$$\nu_+(\xi, t) = \int_0^\infty \dot{y} p(\xi, \dot{y}) d\dot{y} \tag{58}$$

where $p(y, \dot{y})$ is the joint probability density of y and $\dot{y}$ at time t and where ξ is the threshold value for y. This quantity is of interest in connection with the problem of structural fatigue failure.

Assuming that the response is a Gaussian process (of zero mean), we write their joint probability density function as

$$p(y, \dot{y}) = \frac{1}{2\pi} [D_{11} D_{22}(1 - r^2)]^{-1/2} \times \exp \left[-\frac{1}{1 - r^2} \left\{ \frac{y^2}{D_{11}} - \frac{2ry\dot{y}}{\sqrt{(D_{11} D_{22})}} + \frac{\dot{y}^2}{D_{22}} \right\} \right] \tag{59}$$

where $r^2 = D_{12}^2 / D_{11} D_{22}$. Upon inserting (59) into (58) and carrying out the integration, we get

$$\begin{aligned} \nu_+(\xi, t) = \frac{1}{2\pi} \left[\frac{D_{22}(1 - r^2)}{D_{11}} \right]^{1/2} &\left\{ \exp \left[-\frac{\xi^2}{2D_{11}(1 - r^2)} \right] \right. \\ &+ r\xi \left[\frac{\pi}{2(1 - r^2) D_{11}} \right]^{1/2} \exp \left(-\frac{\xi^2}{2D_{11}} \right) \\ &\left. \times \left[1 + \text{erf} \left(\frac{r\xi}{\sqrt{[r D_{11}(1 - r^2)]}} \right) \right] \right\} \end{aligned} \tag{60}$$

where

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz \tag{61}$$

The expression (60) reduces to the known expression [9] for a stationary $y(t)$ since we have $r = 0$ in that case.

For the case of upward zero crossings, (60) becomes

$$\nu_+(0, t) = \frac{1}{2\pi} \left[\frac{D_{22}(1 - r^2)}{D_{11}} \right]^{1/2} \tag{62}$$

When $P^2 \gg 1$, we have, except for the limiting case $\alpha = 0$, $r^2 = 0(P^{-2})$ (see Sections 2 and 3) so that $(1 - r^2) \cong 1$. With the results of Section 3 for an exponentially correlated excitation, we may write (62)

$$\nu_+(0, t) = \frac{P}{2\pi} \left[\frac{\alpha b(t)}{g^2(t) + \alpha b(t)} \right]^{1/2} [1 + 0(P^{-2})] \tag{63}$$

where

$$b(t) = e^{-C(t)} \int^t e^{C(z)} g^2(z) dz \tag{64}$$

For $0 < \alpha \ll 1$, we see that $\nu_+(0, t)$ is $0(P\alpha/2\pi)$ away from the zero of $g(t)$. In the neighborhood of the zeros of $g(t)$, we have $\nu_+(0, t) = 0(P/2\pi)$ which is an order of magnitude larger. Thus, for $0 < \alpha \ll 1$, the distribution of $\nu_+(0, t)$ as a function of t has peaks at the zeros of $g(t)$. For the case of a rotor blade with $\mu =$

0.6, $g(t)$ does not vanish but $\nu_+(0, t)$ has a distinct peak at $t = 3\pi/2$ for moderately small α (e.g., $\alpha = 0.1$). Therefore, most of the upward zero crossings fall within a narrow range around $t = 3\pi/2$ and the expected total number of upward zero crossings over one blade revolution is a small fraction of P.

For $\alpha \gg 1$, $\nu_+(0, t)$ tends to $P/2\pi$ uniformly for all t, which is just the result for a shaped white excitation based on the results of Section 2. The expected total upward zero crossings over one revolution of a rotating blade is therefore P. (Strictly speaking, (63) applies only when $\alpha \ll P$.)

5 Remarks on Higher-Order Systems

The idea of obtaining (deterministic) differential equations for the (first and) second-order statistics of the response of a linear time-varying dynamical system to random excitation has also been applied to more general systems characterized by an initial-value problem

$$\dot{u} = A(t)u + G(t)N(t), \qquad u(0) = 0 \tag{65}$$

where $A(t)$ and $G(t)$ are known matrix functions of t and the vector $N(t)$ is a vector random function. A fourth-order system of the form (65) appears in the analysis of coupled torsion-flapping of rotor blades in air turbulence. Higher-order systems are encountered in the study of the corresponding problem for flexible blades [12].

Ensemble-averaging both sides of (65), we get

$$\langle u(t)\rangle = A(t)\langle u(t)\rangle + G(t)\langle N(t)\rangle \tag{66}$$

with $\langle u(0)\rangle = 0$. This is an initial-value problem in the ordinary sense for $\langle u(t)\rangle$ and can be solved by known methods.

If all components of $N(t)$ are white noises of zero means with $\langle N(t_1)N^T(t_2)\rangle = Q\delta(t_2 - t_1)$ where Q is a nonnegative definite matrix, it can be shown that

$$\dot{D} = AD + DA^T + GQG^T \tag{67}$$

$$\dot{R}(t; t_1) = A(t + t_1)R(t; t_1) \qquad (t > 0) \tag{68}$$

(see the Appendix for a derivation of the equation for R). Depending on the structure of A and G, approximate solutions of these equations may be obtained by various analytical and numerical methods.

If some or all of the elements of $N(t)$ are not white noise, but each can be associated with the response of another linear dynamical system subject to white noise (in the sense that their first and second-order statistics are identical), the original dynamical system can be augmented to have the form (65) with $N(t)$ being a white-noise process. This conversion was demonstrated for the second-order system (1) in Section 3.

An investigation of systems of the form (65) in connection with rotor blade problems just mentioned will be reported elsewhere [12].

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APPENDIX

Continuity of Correlation Functions

Consider the matrix differential equation

$$\dot{u} = A(t)u + G(t)N(t) \tag{69}$$

with

$$\langle N(t)\rangle = 0, \qquad \langle N(t_2)N^T(t_1)\rangle = Q\delta(t_2 - t_1) \tag{70}$$

where Q is a nonnegative definite constant matrix. The correlation matrix of the response $u(t)$ is given by

$$R(t_1, t_2) = \langle u(t_2)u^T(t_1)\rangle \tag{71}$$

For $t_2 \neq t_1$, we have from (69)

$$\frac{dR}{dt_2} = \left\langle \frac{d}{dt_2} u(t_2)u^T(t_1) \right\rangle = A(t_2)R + G(t_2)\langle N(t_2)u^T(t_1)\rangle \tag{72}$$

With

$$u(t_1) = \int_0^{t_1} h(t_1, z)G(z)N(z)dz, \tag{73}$$

where $h(t_1, z)$ is the impulse response (matrix) function of (69), we have

$$\langle N(t_2)u^T(t_1)\rangle = \int_0^{t_1} QG^T(z)h^T(t_1, z)\delta(t_2 - z)dz$$
$$= QG^T(t_2)h^T(t_1, t_2) \tag{74}$$

keeping in mind that $h(t_1, t_2) = 0$ for $t_2 > t_1$. Therefore, (72) becomes

$$\frac{dR}{dt_2} = A(t_2)R + G(t_2)QG^T(t_2)h^T(t_1, t_2). \tag{75}$$

(75) contains the results for correlation functions stated in Section 5 as a special case.

We see from (75) that, as a function of t_2, the derivative of R_{ij} with respect to t_2 has at worst a (finite) jump discontinuity across t_1. Every element of R is therefore continuous across t_1. In particular, we have $R(t_1, t_1) = D(t_1)$.

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