J. Soc. Indust. Appl. Math.
Vol. 13, No. 1, March, 1965
Printed in U.S.A.

ROTATING SHALLOW ELASTIC SHELLS OF REVOLUTION*

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Introduction. In what follows, we wish to show that the problem of the rotating thin elastic shell of revolution is intriguing in the following sense. It is possible to obtain a solution by means of a linear theory which looks so reasonable that one expects the nonlinear effects to be of secondary nature. However, when one considers the problem on the basis of a nonlinear theory, one finds that nonlinear effects are not at all of a secondary nature. Moreover, while a straightforward elementary analysis of the principal aspects of the nonlinear problem is possible, a consideration of the finer structure of the problem leads to an interior-layer problem in which not only the nature but also the location of the layer has to be determined in the course of the analysis.

For the sake of simplicity, our analysis is limited to the class of shells which are generally designated as shallow shells. Stresses and deformations in such shallow shells are governed by differential equations which are similar to the differential equations for finite deflections of flat plates as first obtained by von Kármán.

Statement of the problem. We consider a thin elastic shell of revolution with middle surface equation $z = z(r)$ which rotates with angular velocity ω about its axis. We designate radial, axial, and angular displacements by u, w, and β, stress resultants and couples by $N, N_\theta, Q, M, M_\theta$, and load intensity components by p_r and p_z in accordance with Fig. 1. We assume that the shell is shallow in the sense that $\sin(\phi+\beta) \approx \phi+\beta$, where $\phi = z'$ and $\beta = w'$, primes indicating differentiation with respect to r.

The differential equations of the problem consist of three equilibrium equations which may be written in the form

$$(rN)' - N_\theta + rp_r = 0, \tag{1a}$$

$$[rQ + (\phi + \beta)rN]' + rp_z = 0, \tag{1b}$$

$$(rM)' - M_\theta + rQ = 0, \tag{1c}$$

* Received by the editors September 16, 1964. Presented by invitation at the Symposium in Memory of Theodore von Kármán, SIAM 1964 National Meeting, May 13-14, Washington, D. C., under a grant-by the Air Force Office of Scientific Research. This research was supported by the Office of Naval Research of the United States Navy and Massachusetts Institute of Technology Lincoln Laboratory operated with the support of the United States Air Force.

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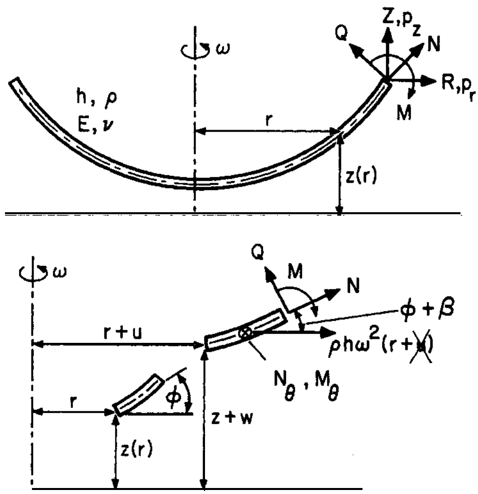

FIG. 1. *A rotating shell of revolution*

and of four stress-displacement relations which are taken as

(2a, b) $$\epsilon = A(N - \nu N_\theta), \qquad \epsilon_\theta = A(N_\theta - \nu N),$$

and

(2c,d) $$M = -D\left(\beta' + \frac{\nu}{r}\beta\right), \qquad M_\theta = -D\left(\nu\beta' + \frac{1}{r}\beta\right),$$

where

(3a, b) $$\epsilon = u' + \phi\beta + \tfrac{1}{2}\beta^2, \qquad r\epsilon_\theta = u,$$

and

(4a, b) $$D = \frac{Eh^3}{12(1-\nu^2)}, \qquad A = \frac{1}{Eh}.$$

We will limit ourselves here to problems for which axial surface and edge loads are absent and for which the only radial loads are the inertia forces of the rotating shell so that

(5a, b, c) $$p_z = 0, \qquad p_r = \rho h\omega^2 r, \qquad Q = -(\phi+\beta)N,$$

where ρ is the volume mass density and ω is the constant angular velocity of the shell.

To reduce the problem further, we introduce a stress function ψ, in terms of which

(6a, b) $$rN = \psi, \qquad N_\theta = \psi' + \rho h\omega^2 r^2,$$

and we assume for the purpose of the present discussion that D and A are independent of r. We then obtain from the moment equation **(1c)** a differential equation for β and ψ of the form

$$D\left(\beta'' + \frac{1}{r}\beta' - \frac{1}{r^2}\beta\right) - \frac{1}{r}(\phi + \beta)\psi = 0. \tag{7}$$

A second equation for β and ψ follows through the use of the compatibility equation $\epsilon = (r\epsilon_\theta)' + \phi\beta + \frac{1}{2}\beta^2$ which is implied by (3a, b) and the stress-strain relations (2a, b) in the form

$$A\left(\psi'' + \frac{1}{r}\psi' - \frac{1}{r^2}\psi\right) + \frac{1}{r}\left(\phi + \frac{1}{2}\beta\right)\beta = -(3+\nu)A\rho h\omega^2 r. \tag{8}$$

The nonlinear system (7) and (8) is to be solved in an interval $r_i \leqq r \leqq r_0$, where $0 \leqq r_i$, subject to the boundary conditions of absent edge forces and moments

$$\frac{\psi}{r} = 0, \qquad D\left(\beta' + \frac{\nu}{r}\beta\right) = 0, \tag{9a, b}$$

for $r = r_i$ and $r = r_0$. For the special case $r_i = 0$, *we* may alternately be interested in boundary conditions $u = \beta = 0$ for $r = 0$. The difference between the two sets of conditions is the difference between the cap with or without a zero diameter hole at the apex.

When $\phi = 0$, the system (7) and (8) reduces to a special case of the differential equations for finite bending of flat plates as first formulated by von Kármán. In what follows, we are particularly interested in the problem of the conical shell for which $\phi =$ constant, and in the problem of the, toroidal cap for which $\phi = (r - a)/R$.

Membrane solutions. Experience with related problems of shell theory indicates that for sufficiently thin shells, the state of stress and deformation in the rotating cap or ring should be effectively as if the shell had no bending stiffness; that is, the solutions of the complete problem (7) to (9) should be effectively the same as those for the special case $D = 0$. Setting $D = 0$, one of the two second order equations, (7), reduces to a zeroth order equation

$$(\phi + \beta)\psi = 0, \tag{10}$$

while the other, (8), remains as is. Concurrently, only one of the two sets of boundary conditions (9) remains, namely

$$\psi(r_i) = 0, \qquad \psi(r_0) = 0, \tag{11a, b}$$

as long as $r_i \neq 0$. When $r_i = 0$, the first of these conditions is to be replaced by $\lim_{r\to 0} r^{-1}\psi = 0$, or alternately by $\lim_{r\to 0}(r\psi' - \nu\psi) = 0$.

Linear membrane theory. Omitting the nonlinear term in the dependent variables β and ψ in (10), we have

$$\psi = 0, \tag{12}$$

except where $\phi = 0$, that is, except in regions where the shell is a flat plate, and where the linearized equation (10) supplies no information concerning ψ. Evidently, the solution (12) satisfies the boundary conditions (11).

Having (12), we find from the linearized form of (8) that

$$\beta = \beta_{LM} = -(3 + \nu)\frac{\rho\omega^2 r^2}{E\phi} \tag{13}$$

as long as $\phi \neq 0$. In view of (6), we have further that meridional and circumferential stress resultants are independent of ϕ and are given by

$$N = 0, \qquad N_\theta = \rho h\omega^2 r^2. \tag{14a, b}$$

In the region where $\phi = 0$, we have from (8) that

$$\psi'' + \frac{1}{r}\psi' - \frac{1}{r^2}\psi = -(3 + \nu)\rho h\omega^2 r, \tag{15}$$

so that

$$\psi = \rho h\omega^2\left[c_1 r + \frac{c_2}{r} - \frac{3 + \nu}{8} r^3\right], \tag{16}$$

and

$$N = \rho h\omega^2\left[c_1 + \frac{c_2}{r^2} - \frac{3 + \nu}{8} r^2\right], \tag{17a}$$

$$N_\theta = \rho h\omega^2\left[c_1 - \frac{c_2}{r^2} - \frac{1 + 3\nu}{8} r^2\right], \tag{17b}$$

which is the well-known solution for the rotating uniform disk.

The above analysis suggests in particular the consideration of a problem where a flat disk, say for $0 \leqq r \leqq r_t$, is joined to a shell for which $\phi \neq 0$, say for $r_t \leqq r \leqq r_0$, with the constants of integration c_1 and c_2 to be determined from one condition for $r = 0$ and from appropriate transition conditions for $r = r_t$. Evidently, there will be two such transition conditions, one expressing the fact that N must be continuous at $r = r_t$ and the other that ϵ_θ must be continuous. As it is not in general possible to satisfy all three conditions by means of the two constants c_1 and c_2 it follows that the problem of the shell with discontinuous meridional slope does not in general have a solution within the framework of linear membrane theory.

Before discussing the nonlinear membrane problem, we note the radically different stress distributions as given by the linear membrane solution (14), and by **(17)** for the corresponding problem of the flat disk with the boundary condition $\psi(r_0) = 0$ and with

$$\text{(18)} \qquad \text{(i)} \quad \lim_{r\to 0}\,(r\psi' - \nu\psi) = 0, \qquad \text{or} \qquad \text{(ii)} \quad \psi(r_i) = 0,$$

in particular, when $r_i \to 0$. Case (i) of the disk without hole at the apex gives

$$\text{(19a, b)} \qquad \begin{aligned} \frac{N}{\rho h\omega^2 r_0^2} &= \frac{3+\nu}{8}\left(1 - \frac{r^2}{r_0^2}\right), \\ \frac{N_\theta}{\rho h\omega^2 r_0^2} &= \frac{3+\nu}{8}\left(1 - \frac{1+3\nu}{3+\nu}\frac{r^2}{r_0^2}\right), \end{aligned}$$

while case (ii) for the disk with a hole of radius r_i gives

$$\text{(20a)} \qquad \frac{N}{\rho h\omega^2 r_0^2} = \frac{3+\nu}{8}\left(1 - \frac{r^2}{r_0^2} + \frac{r_i^2}{r_0^2} - \frac{r_i^2}{2}\right),$$

$$\text{(20b)} \qquad \frac{N_\theta}{\rho h\omega^2 r_0^2} = \frac{3+\nu}{8}\left(1 - \frac{1+3\nu}{3+\nu}\frac{r^2}{r_0^2} + \frac{r_i^2}{r_0^2} + \frac{r_i^2}{r^2}\right).$$

The distributions of stress in accordance with **(14), (19)** and (20) are illustrated in Fig. 2.

The following question now suggests itself in connection with these re-

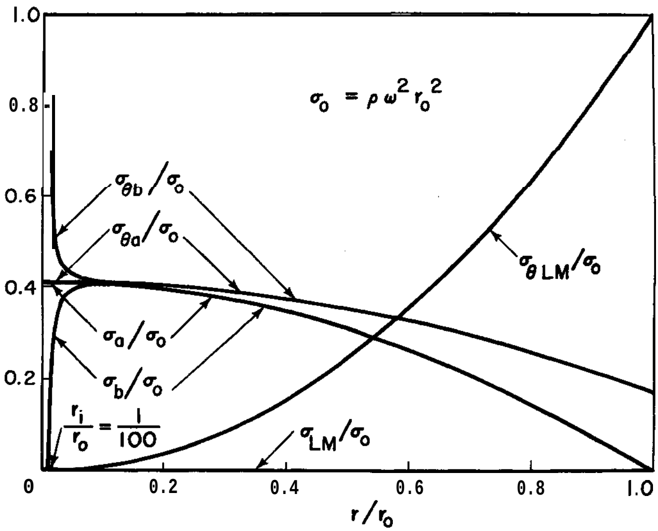

FIG. 2. Stresses in rotating shallow shells according to linear membrane theory compared with the corresponding stresses in rotating disks **(a)** *continuous at center, (b) with small circular hole at center.*

sults. Given a rotating shallow cap, for which the rotating disk is a limiting case, what significance, if any, is associated with the membrane stress distribution as given by **(14)** ?

Nonlinear membrane theory. We now consider that (10) may be solved in two different ways :

$$\psi = 0, \quad \text{or} \quad \beta = -\phi. \tag{21a, b}$$

When $\psi = 0$, we find from (8) and with β_{LM} defined by **(13),** that $\beta^2 + 2\phi\beta - 2\phi\beta_{LM} = 0$, so that

$$\beta = \beta_{NLM} = -\phi + \sqrt{\phi^2 + 2\phi\beta_{LM}}. \tag{22}$$

The plus sign in front of the square root is chosen as we expect that for sufficiently small ω, β_{NLM} should be nearly the same as β_{LM}.

From (22), we find that real values of β_{NLM} are obtained only as long as

$$-1 \leqq 2\frac{\beta_{LM}}{\phi}, \tag{23}$$

or, in view of (13), as long as

$$(6 + 2\nu)\frac{\rho\omega^2 r^2}{E\phi^2} \leqq 1. \tag{24}$$

Equation (24) shows in particular that, for the toroidal cap with $\phi = (r - a)/R$, there is no real solution of the equations of nonlinear membrane theory with $\psi = 0$ in a finite neighborhood of $r = a$. This is a qualitative distinction from the corresponding result of linear membrane theory where there is such a solution.

Equation (24) shows further that for any rotating shallow shell of revolution there is a critical angular velocity ω_{cr} above which the solution (21a) and (22) ceases to be possible throughout. This critical value of ω is given by

$$\omega_{cr} = \sqrt{\frac{E}{(6 + 2\nu)\rho}} \min\left|\frac{\phi}{r}\right|. \tag{25}$$

The condition (25) may alternately be expressed as a condition of a maximum possible hoop strain ϵ_θ in conjunction with the solution $\psi = 0$. Since when $\psi = 0$, $\epsilon_\theta = AN_\theta = \rho\omega^2 r^2/E$, we have from (24) that

$$\left(\frac{\epsilon_\theta}{\phi^2}\right)_{\max} = \frac{1}{6 + 2\nu}, \tag{26}$$

and, in particular, for a conical shell that, $(\epsilon_\theta)_{\max} = \phi^2/(6 + 2\nu)$.

An additional observation of interest is that when $\beta_{LM} = -\frac{1}{2}\phi$, which is the condition of criticality, we have $\beta_{NLM} = -\phi$, so that use of the linear theory for this case results in a one hundred per cent underestimation of the value of β.

We now consider the alternate solution $\beta = -\phi$ of (10). Introduction of this result into (8) leaves as differential equation for ψ,

$$A\left(\psi'' + \frac{1}{r}\psi' - \frac{1}{r^2}\psi\right) = \frac{\phi^2}{2r} - (3+\nu)A\rho h\omega^2 r. \tag{27}$$

The difference in signs of the two terms of the right side of (27) indicates that the deflection of the given surface in space into the base plane is associated with compressive circumferential strains while the effect of the rotational motion is to set up tensile circumferential strains.

The solution of (27) may be written in the form

$$\psi = \rho h\omega^2\left(c_1 r + \frac{c_2}{r} - \frac{3+\nu}{8}r^3\right) + \frac{1}{Ar}\int^r\left(\frac{r}{2}\int^r \frac{\phi^2}{r}\,dr\right)dr. \tag{28}$$

In summary, the differential equations of the nonlinear membrane theory of the rotating shell have the two classes of solutions

$$\psi = 0, \qquad \beta = -\phi + \sqrt{\phi^2 - (6+2\nu)\frac{\rho\omega^2 r^2}{E}}, \tag{29a, b}$$

and

$$\begin{aligned}\frac{\psi}{\rho h\omega^2} &= c_1 r + \frac{c_2}{r} - \frac{3+\nu}{8}r^3 \\ &\quad + \frac{E}{2\rho\omega^2 r}\int^r\left(r\int^r\frac{\phi^2}{r}\,dr\right)dr, \qquad \beta = -\phi,\end{aligned} \tag{30a, b}$$

with meridional and circumferential stress resultants given by

$$N = 0, \quad N_\theta = \rho h\omega^2 r^2, \tag{31a, b}$$

for the first class of solutions, and

$$N = \rho h\omega^2\left[c_1 + \frac{c_2}{r^2} - \frac{3+\nu}{8}r^2 + \frac{E}{2\rho\omega^2 r^2}\int^r\left(r\int^r\frac{\phi^2}{r}\,dr\right)dr\right], \tag{32a}$$

$$\begin{aligned}N_\theta = \rho h\omega^2\Bigg[c_1 &- \frac{c_2}{r^2} - \frac{1+3\nu}{8}r^2 \\ &- \frac{E}{2\rho\omega^2 r^2}\int^r\left(r\int^r\frac{\phi^2}{r}\,dr\right)dr + \frac{E}{2\rho\omega^2}\int^r\frac{\phi^2}{r}\,dr\Bigg],\end{aligned} \tag{32b}$$

for the second class of solutions.

It, remains to determine for any given problem in which range of values of r either the one or the other solution applies.

Conical shells For the class of shells for which ϕ is constant, the solution *(29)* is surely inapplicable for $r \geqq r_c$, where

$$r_c = \frac{\phi}{\omega}\sqrt{\frac{E}{(6+2\nu)\rho}}, \tag{33}$$

and therefore if the problem has a solution for $r_c < r$ it must be given by (30). Evidently, it is possible that the correct solution is given by (30) for values of r which are less than r_c, say for $r_t < r$, where $r_t \leqq r_c$. If $r_i < r_t$ then a transition will occur at $r = r_t$, the solution (29) being valid for $r < r_t$ and the solution (30) being valid for $r_t < r$. Setting ϕ constant, we have as expressions for N and N_θ in the region $r_t < r$:

$$(34a) \qquad N = \rho h\omega^2\left[c_1 + \frac{c_2}{r^2} - \frac{3+\nu}{8}r^2 + \frac{E\phi^2}{4\rho\omega^2}\left(\log r - \frac{1}{2}\right)\right],$$

$$(34b) \qquad N_\theta = \rho h\omega^2\left[c_1 - \frac{c_2}{r^2} - \frac{1+3\nu}{8}r^2 + \frac{E\phi^2}{4\rho\omega^2}\left(\log r + \frac{1}{2}\right)\right],$$

while for $r < r_t$, N and N_θ remain given by (31).

In the region where (34) is valid the slope function β is equal to $-\phi$, expressing the fact, that the portion $r_t < r$ of the shell is deformed into a flat disk. In the complementary region $r < r_t$, we have from (29) and with the definition (33) for r_c,

$$(35) \qquad \beta = \phi\left[\sqrt{1 - \frac{r^2}{r_c^2}} - 1\right].$$

This means that at the junction $r = r_t$, we have a discontinuity in β of the amount

$$(36) \qquad \Delta\beta = \beta(r_t^+) - \beta(r_t^-) = -\phi\sqrt{1 - \frac{r_t^2}{r_c^2}}.$$

For the formulation of a system of boundary and transition conditions we begin by assuming that $r_c < r_0$; we have then from (34a), as boundary condition for $r = r_0$,

$$(37a) \qquad c_1 + \frac{c_2}{r_0^2} - \frac{3+\nu}{8}r_0^2 + \frac{E\phi^2}{4\rho\omega^2}\left(\log r_0 - \frac{1}{2}\right) = 0.$$

As N = 0 for $r < r_t$, in accordance with (31a), we have from equilibrium considerations that N as given by (34a) must also vanish for $r = r_t$, so that

$$(37b) \qquad c_1 + \frac{c_2}{---} - \frac{3+\nu}{8}r_t^2 + \frac{E\phi^2}{4\rho\omega^2}\left(\log r_t - \frac{1}{2}\right) = 0.$$

It remains to establish a third condition for the determination of the three quantities c_1, c_2, and r_t. This third condition follows from the observation that the circumferential strain ϵ_θ must be continuous at $r = r_t$. In view of the fact that N = 0 for $r = r_t$, this means that N_θ must be continuous for $r = r_t$. From (31) and (34b) there follows then the further relation

$$(37c)\qquad c_1 - \frac{c_2}{r_t^2} - \frac{1+3\nu}{8} r_t^2 + \frac{E\phi^2}{4\rho\omega^2}\left(\log r_t + \frac{1}{2}\right) = r_t^2.$$

We may use (37a) and (37b) to express c_1 and c_2 in terms of r_t and in terms of the parameter $E\phi^2/\rho\omega^2$, which according to (33) may also be written as $(6 + 2\nu)r_c^2$. In this way we obtain frorn (37a) and (37b),

$$(38a, b)\quad c_1 = \frac{3+\nu}{2}\left(\frac{r_t^2}{2} - r_c^2 \log r_t\right), \qquad c_2 = \frac{3+\nu}{4} r_t^2\left(r_c^2 - \frac{1}{2} r_t^2\right).$$

Introduction of (38) into the remaining condition (37c) leads to the equation

$$(39)\qquad \left(1 - \frac{r_t^2}{r_0^2}\right)^2 = \left(4 \log \frac{r_0}{r_t} + 2\frac{r_t^2}{r_0^2} - 2\right)\frac{r_c^2}{r_0^2},$$

from which r_t/r_0 is to be determined as a function of r_c/r_0, or equivalently r_c/r_0 as a function of r_t/r_0, as shown in Fig. 3. We see that, indeed, r_t/r_0 is smaller than r_c/r_0 for all $0 < r_c/r_0 < 1$.

With r_t/r_0 as a function of r_c/r_0 in accordance with (39) and with c_1 and c_2 from (38), we obtain the following explicit expressions for N and N_θ in the region $r_t \leqq r$.

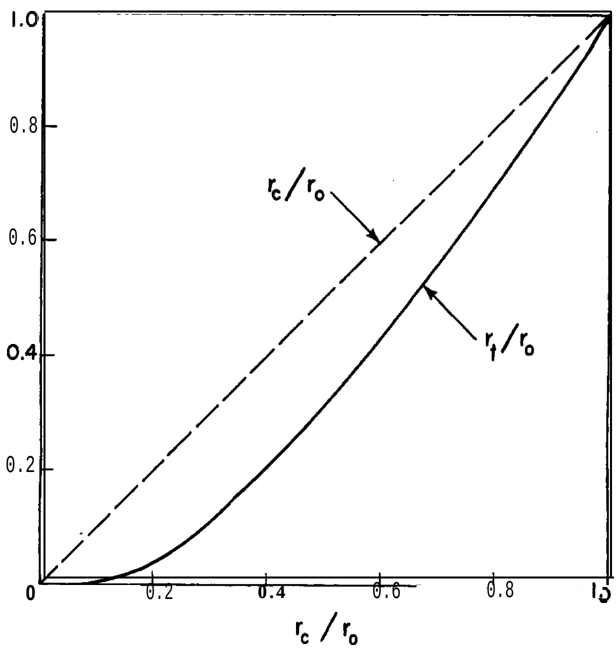

FIG. 3. *The critical and transition radii for a conical ring membrane*

$$\text{(40a)} \quad \begin{aligned} \frac{N}{\rho h \omega^2 r_0^2} &= \frac{3+\nu}{4}\left\{\frac{r_t^2}{r_0^2} - \frac{r_c^2}{r_0^2} - \frac{1}{2}\frac{r^2}{r_0^2}\right. \\ &\qquad \left. + \left(\frac{r_c^2 r_t^2}{r_0^4} - \frac{1}{2}\frac{r_t^4}{r_0^4}\right)\frac{r_0^2}{r^2} - 2\frac{r_c^2}{r_0^2}\log\frac{r_t}{r}\right\}, \end{aligned}$$

$$\text{(40b)} \quad \begin{aligned} \frac{N_\theta}{\rho h \omega^2 r_0^2} &= \frac{3+\nu}{4}\left\{\frac{r_t^2}{r_0^2} + \frac{r_c^2}{r_0^2} - \frac{1+3\nu}{6+2\nu}\frac{r^2}{r_0^2}\right. \\ &\qquad \left. - \left(\frac{r_c^2 r_t^2}{r_0^4} - \frac{1}{2}\frac{r_t^4}{r_0^4}\right)\frac{r_0^2}{r^2} - 2\frac{r_c^2}{r_0^2}\log\frac{r_t}{r}\right\}. \end{aligned}$$

Equations (40) are supplemented in the region $r < r_t$ by N and N_θ distributions in accordance with (31).

Considering the fact that when $r_c = r_0$ we also have $r_t = r_0$, we conclude, by means of a continuity argument, that when $r_c > r_0$ the solution everywhere in the membrane is given by (29) and (31); that is, the stress distribution as given by nonlinear membrane theory is the same as that given by linear membrane theory.

We have then that, for a given conical membrane, stresses and deformations according to nonlinear membrane theory are similar to those of linear membrane theory as long as the rotational speed ω is less than ω_{cr}. When ω is larger than ω_{cr} then the nonlinear membrane solution differs in an essential way from the solution by means of linear membrane theory. A flattened-out portion of the membrane appears for $r_t \leqq r$, with a discontinuity of the angular displacement β at $r = r_t$. As ω increases, r_t decreases. When $r_t = r_i$ then the membrane is completely flattened out. It persists in this flattened-out state for still further increases of ω. As ω increases towards infinity, the state of stress in the originally conical membrane approaches the state of stress in a flat disc rotating with the same angular velocity.

In order to see the nature of these results, we have in Fig. 4 shown the distributions of $\sigma_\theta = N_\theta/h$ and $\sigma = N/h$ for various values of r_c/r_0 for the case of a shell with $r_i/r_0 = 0.2$. We note that the result of linear membrane theory is the same for all. values of r_c/r_0, and that it coincides with the results of nonlinear membrane theory for $r_c/r_0 \geqq 1$. We note in particular that an analysis of the problem by means of the linear membrane theory in general leads to much higher stresses than the corresponding analysis of the problem by means of nonlinear membrane theory. Fig. 5 presents values of $\sigma_{\theta\max}$ and $\sigma_{\max}$ according to nonlinear membrane theory.

Toroidal cap. We consider next the class of shells for which

$$\text{(41)} \qquad \phi = \frac{r-a}{R}.$$

For this problem, it is clear that the solution (29) is inapplicable in a neighborhood of $r = a$, no matter how small ω is, In order to delineate the

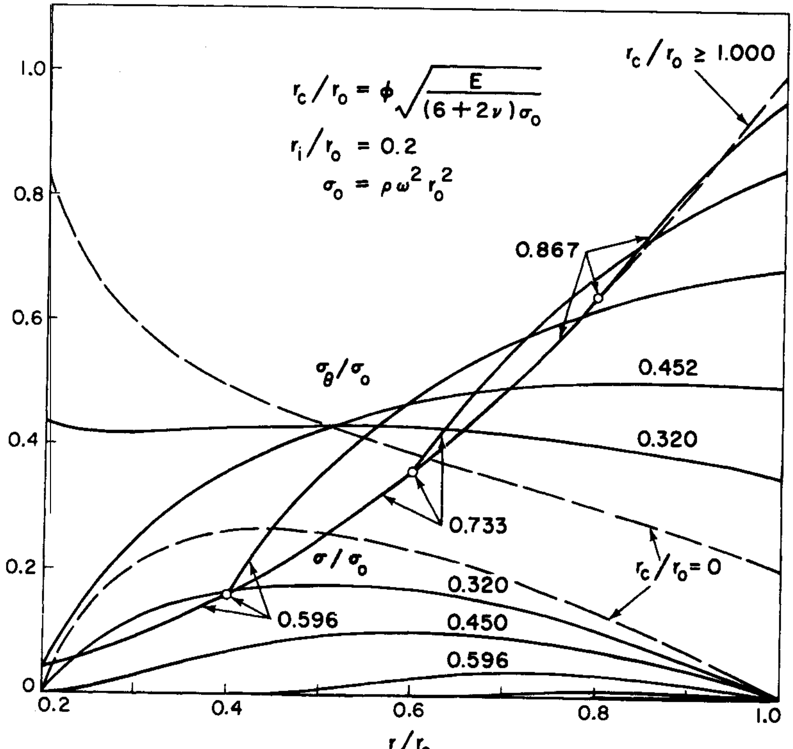

FIG. 4. Stress distributions in a shallow conical ring shell for different rotating speeds according to nonlinear membrane theory.

region of inapplicability of (29), we first determine for fixed ω the range of values of r for which **(29)** is surely not usable. The endpoints r_{ci} and r_{c0} of the interval in question are given by

(42) $$(6 + 2\nu)\frac{\rho\omega^2 r^2}{E} = \phi^2.$$

Combining **(41)** and *(42), we* find

(43a, b) $$r_{ci} = \frac{a}{1+k^2}, \qquad r_{c0} = \frac{a}{1-k^2},$$

where

(43c) $$k^2 = (6 + 2\nu)\frac{\rho\omega^2 R^2}{E}.$$

Having established that **(29)** cannot be applicable for $r_{ci} \leqq r \leqq r_{c0}$, we anticipate the possibility that the alternate solution (30) must be used in an interval $r_{ti} \leqq r \leqq r_{t0}$, where $r_{ti} \leqq r_{ci}$ and $r_{c0} \leqq r_{t0}$, with transition to

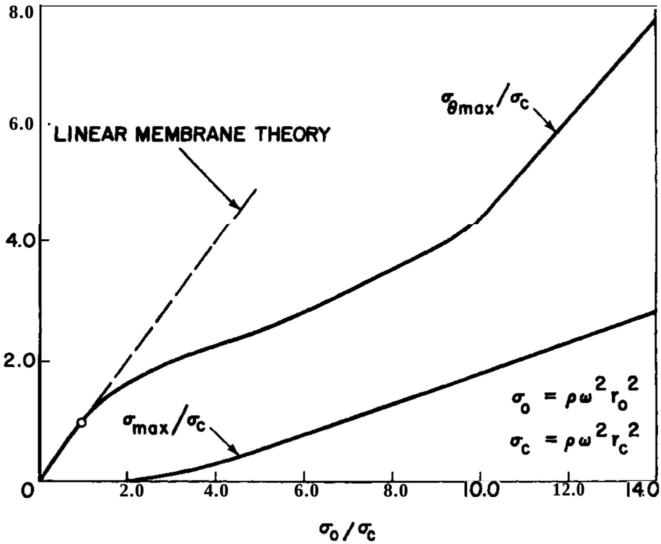

FIG. 5. *Maximum direct* stresses for a shallow conical membrane

the solution (29) occurring at $r = r_{ti}$ and $r = r_{t0}$. Without determining the values of r_{ti} and r_{t0}, we can state that as the angular velocity ω increases, we will encounter two different situations as follows. For sufficiently small values of ω, r_{ci} as well as r_{c0} will be inside the interval (r_i, r_0). With increasing values of ω either one or both of these will be outside (r_i, r_0). When both values are outside, the rotating toroidal cap will be completely flattened out. As the angular velocity increases further and further, the stress distribution will approach more and more the distribution for a rotating disk. Unlike the conical membrane, the toroidal cap membrane begins to flatten out in the neighborhood of the crown rather than of its outer edge, and does so for arbitrarily small values of ω.

Introducing ϕ from **(41)** into (32a, b) we have the following expressions for N and N_θ in the region $r_{ti} < r < r_{t0}$.

(44a)
$$\frac{N}{\rho h\omega^2} = c_1 + \frac{c_2}{r^2} - \frac{3+\nu}{8} r^2 + \frac{3+\nu}{24k^2}(3r^2 - 6a^2 - 16ar + 12a^2 \log r),$$

(44b)
$$\frac{N_\theta}{\rho h\omega^2} = c_1 - \frac{c_2}{r^2} - \frac{1+3\nu}{8} r^2 + \frac{3+\nu^2}{24k^2}(9r^2 + 6a^2 - 32ar + 12a^2 \log r),$$

while for $r_i < r < r_{ti}$ and $r_{t0} < r < r_0$, N and N_θ remain given by (31).

As N = 0 for $r < r_{ti}$ and $r > r_{t0}$, N as given by (44a) must also vanish at $r = r_{ti}$ and $r = r_{t0}$, so that

$$
(45a) \qquad c_1 + \frac{c_2}{r_{ti}^2} - \frac{3+\nu}{8} r_{ti}^2 + \frac{3+\nu}{24k^2}(r_{ti}^2 - 6a^2 - 16ar_{ti} + 12a^2 \log r_{ti}) = 0,
$$

$$
(45b) \qquad c_1 + \frac{c_2}{r_{t0}^2} - \frac{3+\nu}{8} r_{t0}^2 + \frac{3+\nu}{24k^2}(3r_{t0}^2 - 6a^2 - 16ar_{t0} + 12a^2 \log r_{t0}) = 0.
$$

The remaining two conditions needed to determine the four quantities c_1, c_2, r_{ti}, and r_{t0} again come from the continuity of the hoop stress resultant N_θ at r_{ti} and r_{t0}. From (31) and (44b) we have

$$
(46a) \qquad c_1 - \frac{c_2}{r_{ti}^2} - \frac{1+3\nu}{8} r_{ti}^2 + \frac{3+\nu}{24k^2}(9r_{ti}^2 + 6a^2 - 32ar_{ti} + 12a^2 \log r_{ti}) = r_{ti}^2,
$$

$$
(46b) \qquad c_1 - \frac{c_2}{r_{t0}^2} - \frac{1+3\nu}{8} r_{t0}^2 + \frac{3+\nu}{24k^2}(9r_{t0}^2 + 6a^2 - 32ar_{t0} + 12a^2 \log r_{t0}) = r_{t0}^2.
$$

We use (45a) and (45b) to express c_1 and c_2 in terms of r_{ti} and r_{t0} and in terms of the parameter k^2 defined by (43c) as follows:

$$
(46a) \qquad c_1 = \frac{3+\nu}{8(r_{t0}^2 - r_{ti}^2)}\left\{\left(1 - \frac{1}{k^2}\right)(r_{t0}^4 - r_{ti}^4) + \frac{1}{3k^2}[16a(r_{t0}^3 - r_{ti}^3) + 6a^2(r_{t0}^2 - r_{ti}^2) - 12a^2(r_{t0}^2 \log r_{t0} - r_{ti}^2 \log r_{ti})]\right\},
$$

$$
(46b) \qquad c_2 = -\frac{(3+\nu)r_{t0}^2 r_{ti}^2}{8(r_{t0}^2 - r_{ti}^2)}\left\{\left(1 - \frac{1}{k^2}\right)(r_{t0}^2 - r_{ti}^2) + \frac{1}{3k^2}\left[16a(r_{t0} - r_{ti}) + 12a^2 \log \frac{r_{ti}}{r_{t0}}\right]\right\}.
$$

Introduction of (46a, b) into *(45c,* d) leads to two transcendental equations of the form

(46c)
$$\frac{r_{t0}}{a} = -\frac{3}{2}\frac{(1-\gamma^2) + (1+\gamma^2)\log\gamma}{(1-\gamma)^3}$$

and

(46d)
$$k^2 = 1 - \frac{4}{1+\gamma}\left(\frac{a}{r_{t0}}\right) - \frac{\log\gamma}{1-\gamma^2}\left(\frac{a}{r_{t0}}\right)^2$$

for r_{t0}/a and for

(46e)
$$\gamma = \frac{r_{ti}}{r_{t0}}.$$

The solution of (46c, d) leads to values of r_{t0}/a and r_{ti}/a as functions of the load parameter k which is defined by (43c). This solution is obtained by first using (46c) to eliminate r_{t0}/a from (46d). The resulting equation is then solved to give γ as a function of k^2. Having this, we next use (46c) to obtain r_{t0}/a as function of k^2. After that, (46e) is used to obtain r_{ti}/a. Values of $r_{t0}/a, r_{ti}/a$ together with values r_{c0}/a and r_{ci}/a as given by (43a, b) are shown in Fig. 6, which also includes an alternate definition of k. We note that $r_{c0} < r_{t0}$ and $r_{ti} < r_{ci}$ for all $k^2 > 0$; that is, the flattened-out portion emanating from the crown of the toroidal cap is indeed wider than is required by the criticality condition (42).

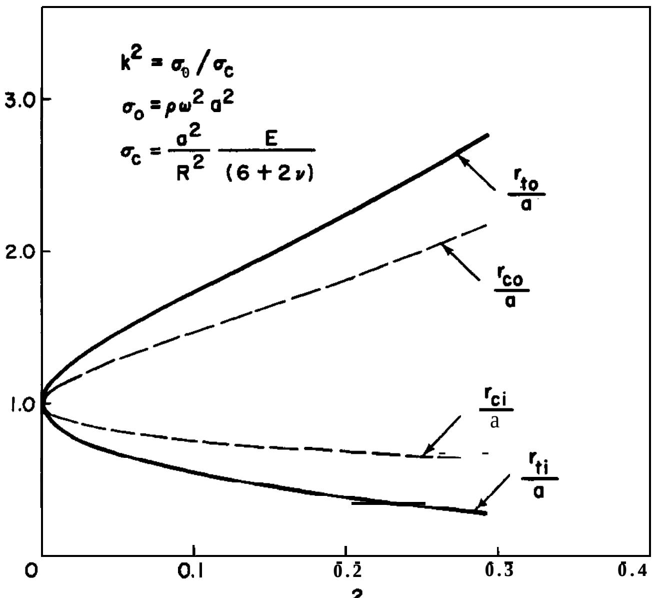

FIG. 6. *The critical and transition radii for a toroidal cap membrane*

We may introduce (46a, b) into (44a, b) to get explicit expressions for N and N_θ with r_{t0} and r_{ti} given by Fig. 6. The corresponding distributions of σ and σ_θ for various values of k^2 and for a shell with $r_i/a = 0.5$ and $r_0/a = 1.5$ are shown in Fig. 7. Fig. 8 shows how the maximum stresses vary as functions of k^2. It is interesting to note that while nonlinear membrane theory must be used for all $\omega^2 > 0$ in order to obtain real β, linear membrane theory (with nonexistent real β) gives the correct maximum hoop stress for sufficiently small values of k^2, say k^2 less than 0.05.

Interior layer analysis. Nonlinear membrane analysis, which is based on setting $D = 0$ in the differential equation (7) and in the boundary condition (9b), has been shown to lead to a solution of the problem for which β may be discontinuous for one or more values r_t of r. No such discontinuities can occur when D does not vanish. This suggests that for sufficiently small values of D, the smoothing out of the discontinuities for $D = 0$ may be

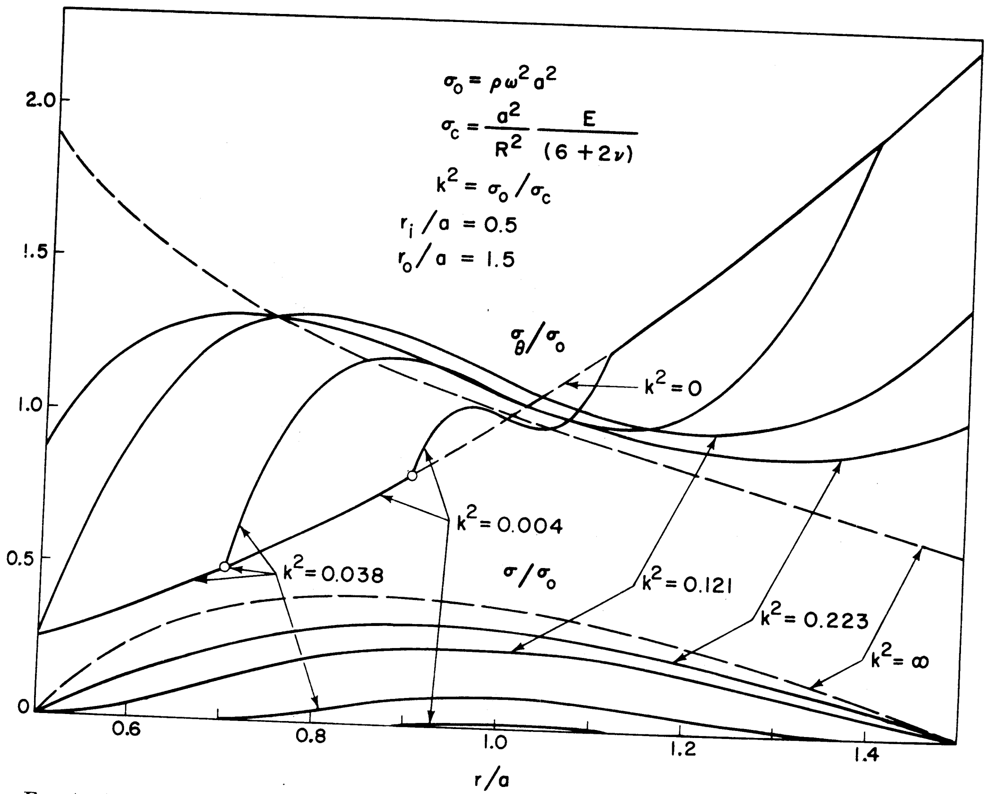

FIG. 7. *Stress distributions in a shallow toroidal cap for different rotating speeds according to nonlinear membrane theory.*

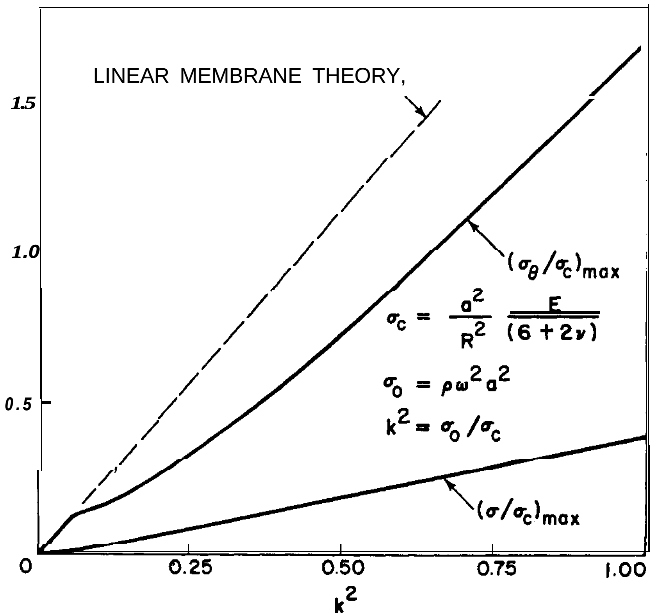

FIG. 8. *Maximum direct stresses* for *a shallow toroidal eambrmne*

confined to relatively narrow layers surrounding $r = r_t$. In the following we determine the circumstances under which this narrow layer will exist and the order of magnitude of the bending stresses in it when it does exist.

We begin by nondimensionalizing the differential equations of the problem by setting

(47a, b) $$r = r_0 x, \qquad \phi = \phi_0 q(x),$$

(47c, d) $$\psi = \rho h\omega^2 r_0^3 g(x), \qquad \beta = \beta_0 f(x) = \frac{\rho\omega^2 r_0^2}{E\phi_{00}} f(x).$$

We have then from (7) and (8):

(48) $$\lambda^4\left(f'' + \frac{1}{x}f' - \frac{1}{x^2}f\right) - \frac{1}{x}(q + \kappa f)g = 0,$$

(49) $$\left(g'' + \frac{1}{x}g' - \frac{1}{x^2}g\right) + \frac{1}{x}\left(q + \frac{1}{2}\kappa f\right)f = -(3+\nu)x,$$

where

(50a, b) $$\lambda^4 = \frac{h^2}{12(1-\nu^2)r_0^2\phi_0^2}, \qquad \kappa = \frac{\rho\omega^2 r_0^2}{E\phi_0^2},$$

and primes now indicate differentiation with respect to x.

The following qualitative conclusions appear from (48) and (49):

(1) When $\lambda^4 \ll 1$, membrane theory should in general be adequate. If, in addition, $\kappa \ll 1$, then linear membrane theory should in general be adequate.

(2) When $\lambda^4 = 0(1)$, the combined effect of nonlinear membrane and linear bending action in the shell must be considered. When $\kappa \ll 1$, linear bending theory should be adequate.

(3) When $\lambda^4 \gg 1$, we have from (48) that effectively $f = 0$ and (49) becomes the equation of the rotating disk for all values of the load parameter κ.

Knowing that nonlinear membrane theory may lead to a discontinuity in f at r_t, we now consider the case $\lambda^4 \ll 1$ in relation to this result. To determine the effect of bending stiffness near $r = r_t$, we set

(51a, b) $$r = r_t(1 + \mu y), \qquad \phi_t = \phi(r_t),$$

(52a, b) $$\beta = \phi_t f(y), \qquad \psi = \psi_t g(y),$$

where μ and ψ_t are yet to be chosen. We require that differentiations with respect to y do not change the order of magnitude of f and g, and we anticipate that μ will turn out to be small compared to unity. We may then write (7) and (8) approximately in the form

(52) $$\frac{D\phi_t}{r_t^2\mu^2} f'' - \frac{\phi_t \psi_t}{r_t}(1 + f)g = 0,$$

(53) $$\frac{A\psi_t}{r_t^2\mu^2} g'' + \frac{\phi_t^2}{r_t}\left(1 + \frac{1}{2}f\right)f = -(3 + \nu)\frac{\rho\omega^2 r_t}{E},$$

where now primes indicate differentiation with respect to y.

We require that f and g approach the appropriate nonlinear solution as μy goes from zero to values of order of magnitude unity, that is, as y tends to $\pm\infty$. In order to have a truly fourth order problem with the assumed order of magnitude relation concerning differentiation with respect to y, we set

(55a) $$\frac{A\psi_t}{r_t^2\mu^2} = \frac{\phi_t^2}{r_t}$$

and

(55b) $$\frac{D\phi_t}{r_t^2\mu^2} = \frac{\phi_t \psi_t}{r_t},$$

From this,

(56a) $$\psi_t = Ehr_t\mu^2\phi_t^2$$

and

$$\mu^4 = \frac{h^2}{12(1-\nu^2)r_t^2\phi_t^2} \tag{56b}$$

Since r_t and ϕ_t are of the same order of magnitude as r_0 and ϕ_0 respectively, we have $\mu = 0(\lambda) \ll 1$ as anticipated. The differential equations for f and g now become

$$f'' - (1+f)g = 0, \tag{57}$$

$$g'' + (1+\tfrac{1}{2}f)f = -\tfrac{1}{2}\kappa_t, \tag{58}$$

with κ_t defined by

$$\kappa_t = \frac{(6+2\nu)}{E\phi_t^2}\rho\omega^2 r_t^2. \tag{59}$$

We know from the earlier nonlinear membrane consideration (see (24)) that κ_t is at most unity. In fact, for a conical shell,

$$\kappa_t = \frac{r_t^2}{r_c^2}. \tag{60}$$

Equations (57) and (58) are supplemented by the limiting conditions

$$y \to \infty: \quad f \sim -1, \quad g \sim \tfrac{1}{4}(1-\kappa_t)y^2, \tag{61a}$$

$$y \to -\infty: \quad f \sim -1 + \sqrt{1-\kappa_t}, \quad g \sim 0. \tag{61b}$$

Together they form a boundary value problem for a layer of shell in the neighborhood of $r = r_t$.

Even without the explicit solution to this boundary value problem, we can readily obtain the order of magnitude of the stresses near r_c/r_t. Introducing (5la, b), (52a, b) and (55) into *(6a,* b) and (2c) we find

$$\sigma_D = \frac{N}{h} = \rho\omega^2 r_t^2\left[\frac{\mu^2}{\kappa_t}(6+2\nu)\right]g, \tag{62a}$$

$$\sigma_{\theta D} = \frac{N_\theta}{h} = \rho\omega^2 r_t^2\left[1 + \frac{\mu}{\kappa_t}(6+2\nu)g\right], \tag{62b}$$

$$\sigma_B = \pm\frac{6M}{h^2} = \pm\rho\omega^2 r_t^2\left[\frac{\mu}{\kappa_t}(6+2\nu)\sqrt{\frac{3}{1-\nu^2}}\right]f'. \tag{62c}$$

Equations (62) show that in the region where bending action is of importance, the bending stress is of a smaller order of magnitude than the maximum direct stresses if $\mu \ll \kappa_t$.

Concluding remarks. In addition to considering the problems of the conical ring membrane and the toroidal cap membrane as reported here, we have also considered a number of other cases. These are

(i) a spherical cap, for which $\phi = r/R$,

(ii) a shell the meridian of which is a quartic parabola, so that $\phi/\phi \quad r^3/r_0^3$,

(iii) a shell for which $\phi/\phi_0 = r_0/r$.

The spherical cap is of particular interest insofar as the criticality condition (25) applies to the entire membrane at once, ϕ/r being constant Accordingly, when $\omega < \omega_{cr}$, we have the possibility of two distinct solutions over the entire membrane, while for $\omega_{cr} < \omega$ only the completely flattened out state is possible. The question remains as to which of the two possible solutions for $\omega < \omega_{cr}$ is the appropriate one for all or part of the spherical cap. We think that the unflattened state is the physically correct state as it goes continuously over into the flattened-out state as ω goes beyond ω_{cr}

For the case $\phi/\phi_0 = r^3/r_0^3$ we find, in contrast to what happens for the conical ring and the toroidal cap, that the discontinuity moves outward from the apex r = 0 as the rotational speed increases. Moreover, the flat region of the deformed shell is now $r \leqq r_t$ rather than $r \geqq r_t$ as in the case of the conical ring, or $r_{ti} \leqq r \leqq r_{t0}$ as in the case of the toroidal cap.

The problem of the shell with $\phi/\phi_0 = r_0/r$ attracted our attention originally because the equations of the linear bending theory could be solved in terms of elementary functions. We have obtained numerical data for both the nonlinear membrane problem and the linear bending problem for this shell. It turns out that the nonlinear membrane behavior is similar to that of the conical shell while the results for the linear bending problem, though explicit, do not seem to be of sufficient, interest to warrant, including them in this account.

Our analysis of the nonlinear bending problem has been limited to one interesting aspect of it, the formulation of an interior-layer analysis together with results concerning various orders of magnitude involved in this analysis. We have also undertaken a narrow-layer analysis of the flattening out phenomenon at the apex of the toroidal cap shell, which is not included in this account.

While asymptotic considerations such as these are of intrinsic interest, it should however be mentioned that insofar as the symmetric problem of the rotating shell of revolution is concerned, it is possible today to obtain quantitative data by purely numerical methods.

Added in proof. Subsequent to the presentation of this paper, the authors became aware of a report by W. Flügge and P. M. Riplog entitled ***A large-deformation theory of shell membranes,*** designated as Technical Report No. 102 of the Engineering Mechanics Division of Stanford University and dated September, 1956. The contents of this report anticipate the contents of our work insofar as the formulation of transition conditions for the nonlinear membrane solutions is concerned, and insofar as a rotating conical membrane is considered. The work of Flügge and Riplog assumes general shells of revolution rather than shallow shells of revolution. The limitation to shallow shells means that we can explicitly evaluate the transition conditions of nonlinear membrane theory whereas without this limitation the evaluation is tied in with a proposed numerical solution of the differential equations of the problem.