Ingenieur-Archiv 42 (1973) S. 245--258

Laterally Loaded Elastic Shells of Revolution

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Summary: An alternate form of the Chernin type equations for laterally loaded shells of revolution is obtained. Except for terms of order of the inherent error in shell theory, the final two coupled second order ordinary differential equations are remarkably similar to the Reissner-Meissner type equations for problems involving axi-symmetric stress distributions. Unlike all previous versions, our two equations can be further reduced (just as in the case of axi-symmetric stress distributions) to a single second order equation for a complex stress function without any additional approximation for uniform cylindrical, spherical, conical and toroidal shells. The side force and tilting moment problem for a shell frustum is shown to be the static geometric analogue of the problem of asymmetric bending and twisting of a ring shell sector. An efficient method for the evaluation of the overall influence coefficients is discussed. The stress state of a complete uniform spherical shell subject to concentrated side force and tilting moment at the two poles is analyzed.

Übersicht: Es wird eine neue Form der Cherninschen Gleichung fur rotationssymmetrische, seitlich belastete Schalen abgeleitet. Abgesehen von Gliedern, deren Größenordnung dem allgemeinen Fehler der Schalentheorie entspricht, haben die zwei gekoppelten gewöhnlichen Differentialgleichungen zweiter Ordnung eine merkwürdige Ähnlichkeit zu den von Reissner und Meissner abgeleiteten Gleichungen für Probleme mit axialsymmetrischem Spannungszustand. Zum Unterschied zu früheren Fassungen der Cherninschen Gleichungen können die jetzigen zu einer einzigen Gleichung zweiter Ordnung auf eine komplexe Spannungsfunktion zusammengefaßt werden, wie dies im Falle rotationssymmetrischer Spannung auch möglich ist. Dabei sind keine zusätzlichen Näherungen für gleichförmige Cylinder-, Kugel-, Kegel- und Ringflächenschalen notwendig. Es zeigt sich, daß das Problem der Beanspruchung einer Stumpfschale durch Seitenkraft und Momente statisch und geometrisch analog ist zum Problem der Beanspruchung eines Ringschalensektors durch unsymmetrische Biegung und Drillung. Es wird ein Verfahren zur Bestimmung der Einflußkoeffizienten angegeben. Außerdem wird der Spannungszustand in einer gleichförmigen vollständigen Kugelschale analysiert, die durch eine einzelne Seitenkraft und Momente an den zwei Polen beansprucht wird.

1. Introduction

With reference to cylindrical coordinates (r, θ, z), the middle surface of a shell of revolution may be described by the parametric equations $r = r(\xi)$ and $z = z(\xi)$. We are concerned here with linear elastostatic problems for shells of revolution involving stress distributions whose dependence on the polar angle 8 is of the form cos 8 or sin θ. A typical loading which gives rise to such a stress state is wind load. When the deformation of the shell also has the same dependence on θ, Chernin [I] succeeded in an exact reduction of this class of problems to the solution of two simultaneous second order differential equations for a stress function and a displacement function. A different reduction which generalizes Chernin's results to include the possibility of non-periodic displacement states and variable shell properties was given recently in [2] where reference to other related papers can also be found.[1]

Except for terms of order h/R and h^2/l^2 (where R is a representative magnitude of the principal radii of curvature R_ξ and R_θ, 2 is a characteristic dimension of the shell and h is the shell thickness), the final two governing differential equations for a stress function ψ and a strain function Φ obtained in [2] are

$$\Phi'' + \frac{(D\,r/\alpha)'}{D\,r/\alpha}\,\Phi' - \left\{4\left(\frac{r'}{r}\right)^2 - \frac{[(1+\nu_b)\,D\,r'/\alpha]'}{D\,r/\alpha} + (1-\nu_b)\left(\frac{z'}{r}\right)^2\right\}\Phi - \frac{z'}{D\,r/\alpha}\,\psi$$
$$= \frac{P_x}{\pi}\,\frac{(z/r)'}{D/\alpha} + \frac{T_y}{\pi}\,\frac{r'}{D\,r^2/\alpha} - \frac{1}{D\,r/\alpha}\left\{\frac{\nu_b\,D}{r\,\alpha}\left[\frac{\Omega_x}{\pi}(r\,r' + z\,z') - \frac{U_y}{\pi}\,z'\right]\right\}' +$$
$$+ \frac{(1+\nu_b)\,r'}{r^3}\left\{\frac{\Omega_x}{\pi}(r\,r' + z\,z') - \frac{U_y}{\pi}\,z'\right\},$$

[1] A brief sketch of the main result of [2] is given in the appendix.

$$\left.\begin{aligned} &\psi'' + \frac{(A\,r/\alpha)'}{A\,r/\alpha}\,\psi' - \left\{4\left(\frac{r'}{r}\right)^2 - \frac{[(1-\nu_s)\,A\,r'/\alpha]'}{A\,r/\alpha} + \underline{(1+\nu_s)\left(\frac{z'}{r}\right)^2}\right\}\psi + \frac{z'}{A\,r/\alpha}\,\Phi \\ &= -\frac{\Omega_x}{\pi}\,\frac{(z/r)'}{A/\alpha} - \frac{U_y}{\pi}\,\frac{r'}{A\,r^2/\alpha} + \frac{1}{A\,r/\alpha}\left\{\frac{\nu_s\,A}{r\,\alpha}\left[\frac{P_x}{\pi}\,(r\,r' + z\,z') - \frac{T_y}{\pi}\,z'\right]\right\}' + \\ &+ \frac{(1-\nu_s)\,r'}{r^3}\left\{\frac{P_x}{\pi}\,(r\,r' + z\,z') - \frac{T_y}{\pi}\,z'\right\} - \frac{(A\,r^2\,\hat{p}_0)'}{A\,r/\alpha} + (1-\nu_s)\,r'\,\alpha\,\hat{p}_\theta, \end{aligned}\right\} \quad (1.1)$$

where primes indicate differentiation with respect to $\xi, \alpha = (r'^2 + z'^2)^{1/2}$, D and $1/A$ are the bending and stretching stiffness factors, ν_b and ν_s are the corresponding effective Poisson's ratios[2] and where only load terms appear on the right hand side. A principal feature of these equations is that they are very similar to the classical Reissner-Meissner equations for problems involving axi-symmetric stress distributions [3]. However, a closer examination reveals that a certain desirable structure of the Reissner-Meissner equations is missing from (1.1). More specifically, the Reissner-Meissner equations can be further reduced to a single second order equation for a complex stress function in the case of a homogeneous cylindrical, spherical, conical or toroidal shell of constant thickness. Because of the appearance of Poisson's ratio in the underlined terms, such a further reduction of (1.1) (or of Chernin's equation) without additional simplifications is not possible for conical and toroidal shells.

In this paper, we derive a new set of two equations for Φ and ψ. These equations are equivalent to (1.1) to within the accuracy of shell theory, but permit a further reduction to a single complex equation for the only four shells of revolution of constant R_ξ without any additional approximation. The derivation is only a minor variation of the one used in [2].

We will then consider problems for shell frusta with displacement components prescribed along two parallel circular edges. To be concrete, we will describe the nature of our results by way of the side force and tilting moment problem, special cases of which have been studied previously in [4 5, 6]. In the formulation of [1] or [2], the displacement components are to be determined as integrals of the strain and curvature change measures. Therefore, it appears that all supplementary conditions for the differential equations would be given in terms of definite integrals of Φ and ψ. We will show that the usual displacement conditions can in fact be transformed into two independent conditions on the values of Φ and ψ and their derivatives at each of the two edges of the shell, and two integral conditions. The four non-integrated conditions reduce the problem with prescribed edge displacements to a two point boundary value problem for Φ and ψ. The solution of this problem determines the stress and strain distributions completely in terms of the resultant side force P_x and resultant tilting moment T_y[3]. If P_x and T_y are not known, they are then determined by the remaining two integral conditions in terms of the overall displacement and rotation.

The solution of the two point boundary value problem is in general quite complicated. The evaluation of the two integral conditions for P_x and T_y will not be feasible in general except by numerical methods. We will show that, except for o (DAR_θ^2/l^4) terms, the overall load deformation relations can be arranged to depend only on the value of ψ at the two edges, and therefore a direct evaluation of the integrals can often be avoided. Moreover, for a sufficiently thin shell, we need only a particular solution for ψ in most cases.

By transforming the prescribed displacement conditions into conditions on Φ and ψ as described above, we will have established also the fact that the side force and tilting moment problem is the static geometric analogue of the problem of asymmetric bending and twisting of shell sectors as formulated in [2]. Therefore, known results for special cases of the former problem [4 5, 6] and the results obtained herein, translated according to the rules of the static geometric duality, will become results for the latter.

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[2] For a homogeneous and isotropic shell, $A = 1/Eh$, $D = Eh^3/12(1-\nu^2)$ and $\nu_s = \nu_b = \nu$.

[3] (x, y, z) is a system of cartesian coordinate axes with the z-axis being the axis of revolution of the middle surface of the shell.

2. Exact and Simplified Stress and Strain Function Formulation

We consider here stress resultants (N, Q), couples (M), strains (ε), curvatures changes (k, λ) and external surface load components (p) (and edge load components) of the form

$$\left.\begin{aligned} &(N_\xi, N_\theta, Q_\xi, M_\xi, M_\theta, \varkappa_\theta, \varkappa_\xi, \lambda_\theta, \varepsilon_\theta, \varepsilon_\xi, p_\xi, p_n) = (n_\xi, n_\theta, q_\xi, m_\xi, m_\theta, k_\theta, k_\xi, l_\theta, e_\theta, e_\xi, \hat{p}_\xi, \hat{p}_n)\cos\theta\,,\\ &(N_{\xi\theta}, N_{\theta\xi}, Q_\theta, M_{\xi\theta} = M_{\theta\xi}, \varkappa_{\xi\theta}, \varkappa_{\theta\xi}, \lambda_\xi, \varepsilon_{\xi\theta} = \varepsilon_{\theta\xi}, p_\theta) = (n_{\xi\theta}, n_{\theta\xi}, q_\theta, m, k_{\xi\theta}, k_{\theta\xi}, l_\xi, e, \hat{p}_\theta)\sin\theta\,. \end{aligned}\right\} \quad (2.1)$$

We note that λ_ξ and λ_θ are the normal components of the curvature change vectors [7,8].

Differential equations governing the linear elastostatic behavior of an isotropic elastic shell of revolution, taken in the form of equilibrium, constitutive and compatibility equations, can be satisfied exactly by expressing the &dependent portions of the stress and strain measures, $n_{\xi\theta}, n_{\theta\xi}$, etc. in terms of a stress function ψ and a strain function ϕ as specified in section (4) of [2] (also, see appendix). The two unknowns ϕ and ψ are determined by two coupled second order differential equations of the form

$$\phi'' - \frac{A(1+\nu_s)}{R_\theta(1+\varepsilon_1^2)}\psi'' = f_1\,, \qquad \psi'' + \frac{D(1-\nu_b)}{R_\theta(1+\varepsilon_1^2)}\phi'' = f_2\,, \quad (2.2)$$

where $1/R_\theta = -z'/r\alpha$, $\varepsilon_1^2 = o\,(D\,A/R^2)$ and where only ψ', ψ, ϕ', ϕ and load terms appear in f_1 and f_2.

The *exact* equations (2.2) for ψ and ϕ are rather complicated. We limit ourselves here to the case $D\,A = o(h^2)$ and $\nu_s = \nu_b$, and to shell properties and loadings not varying significantly over a distance of the order of the shell thickness. We may then omit all h/R and h^2/l^2 terms from (2.2) to get (1.1) as we did in [2].

Alternately, we may first solve (2.2) for ϕ'' und ψ'' to get

$$\left[1 + \frac{\varepsilon_4^2}{(1+\varepsilon_1^2)^2}\right]\psi'' = f_2 - \frac{D(1-\nu_b)}{R_\theta(1+\varepsilon_1^2)} f_1\,, \qquad \left[1 + \frac{\varepsilon_4^2}{(1+\varepsilon_1^2)^2}\right]\phi'' = f_1 + \frac{A(1+\nu_s)}{R_\theta(1+\varepsilon_1^2)} f_2\,, \quad (2.3)$$

where $\varepsilon_4^2 = o\,(D\,A/R_\theta^2)$. We then omit all h/R and h^2/l^2 terms from these equations to get, instead of (1.1), the following two differential equations for ϕ and ψ:

$$\begin{aligned} &\phi'' + \frac{(D\,r/\alpha)'}{D\,r/\alpha}\phi' - \left[4\left(\frac{r'}{r}\right)^2 - \frac{\{(1+\nu_b)\,D\,r'/\alpha\}'}{D\,r/\alpha} + 2\left(\frac{z'}{r}\right)^2\right]\phi - \frac{z'}{D\,r/\alpha}\psi = \\ &= \frac{1}{D\,r/\alpha}\left[\frac{P_x r}{\pi}\left(\frac{z}{r}\right)' + \frac{T_y}{\pi}\frac{r'}{r}\right] - \frac{1}{D\,r/\alpha}\left[\frac{\nu_b D}{r\alpha}\left\{\frac{\Omega_x}{\pi}(r\,r' + z\,z') - \frac{U_y}{\pi}z'\right\}\right]' + \frac{\Omega_x}{\pi}\frac{(1+\nu_b)\,\alpha^2}{r^2}\,, \end{aligned} \quad (2.4)$$

$$\begin{aligned} &\psi'' + \frac{(A\,r/\alpha)'}{A\,r/\alpha}\psi' - \left[4\left(\frac{r'}{r}\right)^2 - \frac{\{(1-\nu_s)\,A\,r'/\alpha\}'}{A\,r/\alpha} + 2\left(\frac{z'}{r}\right)^2\right]\psi + \frac{z'}{A\,r/\alpha}\phi = \\ &= -\frac{1}{A\,r/\alpha}\left[\frac{\Omega_x r}{\pi}\left(\frac{z}{r}\right)' + \frac{U_y}{\pi}\frac{r'}{r}\right] + \frac{1}{A\,r/\alpha}\left[\frac{\nu_s A}{r\alpha}\left\{\frac{P_x}{\pi}(r\,r' + z\,z') - \frac{T_y}{\pi}z'\right\}\right]' + \\ &+ \frac{P_x}{\pi}\frac{(1-\nu_s)\,\alpha^2}{r^2} - \frac{(A\,r^2\,\hat{p}_\theta)'}{A\,r/\alpha} + (1-\nu_s)\,\alpha\,r'\,\hat{p}_\theta\,, \end{aligned} \quad (2.5)$$

where

$$\left.\begin{aligned} P_x &= \pi\int^\xi (z'\,\hat{p}_n - r'\,\hat{p}_\xi + \alpha\,\hat{p}_\theta)\,r\,d\xi\,,\\ T_y &= \pi\int^\xi [(r\,r' + z\,z')\,\hat{p}_n + (r\,z' - z\,r')\,\hat{p}_\xi + \alpha\,z\,\hat{p}_\theta]\,r\,d\xi\,. \end{aligned}\right\} \quad (2.6)$$

Also, Ω_x and U_y are two constants of integration in the two first integrals of the compatibility equations. They have the meaning of relative edge rotation and displacement of the meridional edges of a ring shell sector (see equations (3.7)).

When $z \equiv 0$, these equations reduce to the exact equations for a flat plate. When $r =$ constant, they differ from the exact equations for a homogeneous circular cylindrical shell of constant thickness only by terms of order h^2/R_θ^2 compared to other terms in the same equations.

The left hand side of (2.4) and (2.5) is the same as that of (1. 1) except for the factor multiplied by $(z'/r)^2$ in the coefficient of the third term in each equation. These slight differences make it possible for a further reduction to a single second order equation for a complex stress function for the four shells of constant R_ξ. The method of reduction is the same as that for the corresponding axisymmetric problems. *Note* that the right hand side of (2.4) and (2.5) is no longer the same as that of (1. 1).

Corresponding simplifications in the expressions for stress resultants and couples were given in [2]. We list here instead only the expressions for strain and curvature change measures needed later:

$$\left.\begin{aligned}
e_\theta &= \frac{A}{\alpha}\left\{\psi' + (1-\nu_s)\frac{r'}{r}\psi - \frac{\nu_s P_x}{\pi}\frac{r r' + z z'}{r^2} + \frac{\nu_s T_y}{\pi}\frac{z'}{r^2} + r\alpha\hat{p}_0\right\} + \\
&\quad + \frac{2AD_S}{R_\theta\alpha}\left\{\Phi' + \left[\frac{2r'}{r} + \frac{(D_S/R_\theta)'}{D_S/R_\theta}\right]\Phi + \frac{(1+\nu_s)}{r^2}\left[\frac{\Omega_x}{\pi}(r r' + z z') - \frac{U_y}{\pi}z'\right]\right\},\\
e_\xi &= \frac{A}{\alpha}\left\{-\nu_s\psi' + (1-\nu_s)\frac{r'}{r}\psi + \frac{P_x}{\pi}\frac{r r' + z z'}{r^2} - \frac{T_y}{\pi}\frac{z'}{r^2} - \nu_s r\alpha\hat{p}_\theta\right\} + \\
&\quad + \frac{2AD_S}{R_\theta\alpha}\left\{\Phi' + \left[\frac{2r'}{r} - \frac{\nu_s(D_S/R_\theta)'}{D_S/R_\theta}\right]\Phi\right\},\\
e &= A(1+\nu_s)\frac{\psi}{r}, \qquad k_{\theta\xi} = \frac{\Phi}{r},\\
k_\xi &= -\frac{\Phi'}{\alpha} - \frac{r'\Phi}{r\alpha} + \frac{A}{R_\theta\alpha}\left\{\psi' + \left[\frac{2r'}{r} + \frac{2(A_S/R_\theta)'}{A/R_\theta}\right]\psi + \frac{P_x}{\pi}\frac{r r' + z z'}{r^2} - \frac{T_y}{\pi}\frac{z'}{r^2} - \nu_b r\alpha\hat{p}_\theta\right\},
\end{aligned}\right\} \quad (2.7)$$

where $D_S = \frac{1}{2}D(1-\nu_b)$ and $A_S = \frac{1}{2}A(1+\nu_s)$.[4]

3. Determination of Displacement Components

Associated with the sinusoidal stress distributions given by (2. 1), we have the following nonperiodic displacement field [2]:

$$\begin{aligned}(u_\xi, w, \phi_\xi) &= (U_\xi, W, \Phi_\xi)\cos\theta + \\ &+ \left(\frac{r'}{\alpha}\frac{U_y}{2\pi} + \frac{r z' - z r'}{\alpha}\frac{\Omega_x}{2\pi},\ -\frac{z'}{\alpha}\frac{U_y}{2\pi} + \frac{r' r + z' z}{\alpha}\frac{\Omega_x}{2\pi},\ -\frac{\Omega_x}{2\pi}\right)\theta\sin\theta,\end{aligned} \quad (3.1)$$

$$(u_\theta, \phi_\theta, \omega) = (U_\theta, \Phi_\theta, \Omega)\sin\theta + \left(\frac{U_y}{2\pi} - z\frac{\Omega_x}{2\pi},\ -\frac{r'}{\alpha}\frac{\Omega_x}{2\pi},\ -\frac{z'}{\alpha}\frac{\Omega_x}{2\pi}\right)\theta\cos\theta, \quad (3.2)$$

where u_ξ, u_θ and w are the middle surface translational displacement components in the meridional, circumferential and normal direction, respectively, and ϕ_θ, ϕ_ξ and ω are the rotational displacement components turning about the same directions, respectively. The &dependent quantities U_ξ, U_θ, etc. are related to the strain measures by the relations (6.4) of [2]. We list here only

$$\left.\begin{aligned}
e &= \frac{1}{2r\alpha}\left[r U_\theta' - \alpha U_\xi - r' U_\theta + r'\frac{U_y}{2\pi} + (r z' - z r')\frac{\Omega_x}{2\pi}\right], && k_\xi = \frac{\Phi_\xi'}{\alpha},\\
e_\theta &= \frac{U_\theta}{r} + \frac{r' U_\xi}{r\alpha} - \frac{z' W}{r\alpha} + \frac{U_y - z\Omega_x}{2\pi r}, && l_\theta = \frac{\Omega}{r} - \frac{\Phi_\xi}{R_\theta} - \frac{\Omega_x}{2\pi}\frac{z'}{r\alpha},\\
k_\theta &= \frac{\Phi_\theta}{r} + \frac{r'\Phi_\xi}{r\alpha} - \frac{r'}{r\alpha}\frac{\Omega_x}{2\pi}, && k_{\theta\xi} = -\frac{1}{r}\left(\Phi_\xi + \frac{r'}{\alpha}\Phi_\theta + \frac{z'\Omega}{\alpha} + \frac{\Omega_x}{2\pi}\right),
\end{aligned}\right\} \quad (3.3)$$

[4] While the simplified results presented here are valid only if $\nu_s = \nu_b$, we will continue to distinguish them for the ease of applications of the static geometric duality.

where

$$\left.\begin{aligned} \Phi_\xi &= -\frac{W'}{\alpha} + \frac{U_\xi}{R_\xi}, \qquad \Phi_\theta = \frac{W}{r} + \frac{U_\theta}{R_\theta} + \frac{z'}{r\alpha}\frac{U_y}{2\pi} - \frac{r r' + z z'}{r\alpha}\frac{\Omega_x}{2\pi}, \\ 2\,\Omega &= \frac{U_\theta'}{\alpha} + \frac{r' U_\theta}{r\alpha} + \frac{U_\xi}{r} - \frac{r'}{r\alpha}\frac{U_y}{2\pi} + \frac{z r' - r z'}{\alpha}\frac{\Omega_x}{2\pi}. \end{aligned}\right\} \tag{3.4}$$

We noted in [2] that equations (3.3) satisfy two algebraic compatibility conditions:

$$r' k_\theta + \alpha k_{\theta\xi} + z' l_\theta = -\frac{\alpha}{r}\frac{\Omega_x}{\pi}, \qquad -z' k_\theta + r' l_\theta - \frac{\alpha}{r} e_\theta - \frac{r'}{r} e = -\frac{\alpha}{r^2}\frac{U_y - z\,\Omega_x}{\pi} \tag{3.5}$$

and that the portion of the non-periodic displacement field explicitly associated with Ω_x and U_y can be written in the vectorial form

$$\boldsymbol{\phi} = \frac{\Omega_x}{2\pi}\theta\,\boldsymbol{i}_x, \qquad \boldsymbol{u} = \frac{U_y}{2\pi}\theta\,\boldsymbol{i}_y + \frac{\Omega_x}{2\pi}\theta\,\boldsymbol{i}_x \wedge \boldsymbol{r} \tag{3.6}$$

where $\boldsymbol{r}$ is the position vector of a point on the middle surface of the shell. With

$$[\boldsymbol{\phi}]_0^{2\pi} = \Omega_x \boldsymbol{i}_x, \qquad [\boldsymbol{u}]_0^{2\pi} = U_y \boldsymbol{i}_y + \Omega_x \boldsymbol{i}_x \wedge \boldsymbol{r} \tag{3.7}$$

we see that Ω_x and U_y have the meaning of relative edge rotation and displacement, respectively, for a shell slit along a meridian.

With the strain and curvature change measures already determined in section (2), it remains to obtain UE, UO, etc. by solving the system of first order differential equations (3.3) and (3.4) (see (6.4) of [2] for the complete system). This was not carried out in either [2] or [I]. We give the results here in a form convenient for our purpose:

$$\left.\begin{aligned} \Phi_\xi &= \int^\xi k_\xi\,\alpha\,d\xi, \qquad \Phi_\theta = r\,k_\theta - \frac{r'}{\alpha}\left(\Phi_\xi - \frac{\Omega_x}{2\pi}\right), \\ \Omega &= r\,l_\theta - \frac{z'}{\alpha}\left(\Phi_\xi - \frac{\Omega_x}{2\pi}\right), \\ U_\theta &= r\,e_\theta - z\,\Phi_\xi - \int^\xi (r' e_\xi - z\,\alpha\,k_\xi)\,d\xi - \frac{U_y}{2\pi} + \frac{z\,\Omega_x}{2\pi}, \\ U_\xi &= r^2 l_\theta - r\,e - \frac{r z'}{\alpha}\Phi_\xi - \frac{r'}{\alpha}\left(U_\theta - \frac{U_y}{2\pi}\right) + \frac{\Omega_x}{\pi}\frac{z' r - r' z}{\alpha}, \\ W &= r^2 k_\theta - \frac{z z' + r r'}{\alpha}\left(\Phi_\xi - \frac{\Omega_x}{\pi}\right) + \frac{z'}{\alpha}\left[r\,e_\theta - \int^\xi (r' e_\xi - z\,\alpha\,k_\xi)\,d\xi - \frac{U_y}{\pi}\right]. \end{aligned}\right| \tag{3.8}$$

4. The Side Force and Tilting Moment Problem for a Shell Frustum

For a shell of revolution complete in the circumferential direction, a class of problems of interest is the following. Consider a shell frustum free of surface loads, clamped at the edge $\xi = \xi_0$ to a rigid floor and clamped at the edge $\xi = \xi_i$ to a moveable rigid plug. The plug is subject to a side force P_x in the x-direction and/or a tilting moment T_y turning about the y-direction to produce a lateral displacement δ_x and a rotation β_y. Of interest are the relations between δ_x and β_y on the one hand and P_x and T_y on the other hand, as well as the stress distributions in the shell.

The solution of this problem for the special case of a spherical shell was obtained in [4,5] and an asymptotic solution for the dome-type shells of revolution was obtained in [6]. It is evident from these investigations that all stress, strain and displacement measures for this class of problems are sinusoidal in 8 with a period 2π and that the formulation of sections (2) and (3) (with $U_y = \Omega_x$ = o) is therefore applicable.

For a shell with no surface loads and complete in the circumferential direction, we have $\hat{p}_\xi = \hat{p}_\theta = \hat{p}_n = \Omega_x = U_y = 0$ in the differential equations (2.4) and (2.5) and the auxiliary equations (2.7) and (3.8).

The displacement conditions at the edges may be written as

$$\xi = \xi_0: \quad \Phi_\xi = u_\theta = u_\xi = w = 0\,, \tag{4.1}$$

$$\left.\begin{aligned} \xi = \xi_i: \quad & \Phi_\xi = \beta_y \cos\theta\,, & u_\theta &= -(\delta_x + z\,\beta_y)\sin\theta\,, \\ & u_\xi = \frac{r'\,\delta_x + (r' z - z' r)\beta_y}{\alpha}\cos\theta\,, & w &= -\frac{z'\,\delta_x + (r r' + z z')\,\beta_y}{\alpha}\cos\theta\,. \end{aligned}\right\} \tag{4.2}$$

Since the displacement components are integrals of the strain and curvature change measures, the four displacement boundary conditions at each edge would in general give rise to four conditions involving integrals of the unknowns ϕ and ψ. Having the displacement components in the form of (3.8, we now see that, with the indefinite integral sign $\int^{\xi}$ replaced by $\int_{\xi_0}^{\xi}$ throughout (3.8) the conditions (4.1) and (4.2) are equivalent to two conditions $e_\theta = 0$ and $l_\theta - e/r = 0$ (which, along with the second integrated compatibility equation of (3.9, imply $k_\theta = 0$) at each of the two edges of the shell and two integrated conditions

$$-\int_{\xi_i}^{\xi_0} k_\xi\,\alpha\,d\xi = \beta_y\,, \tag{4.3}$$

$$-\int_{\xi_i}^{\xi_0} (r'\,e_\xi - z\,\alpha\,k_\xi)\,d\xi = \delta_x\,. \tag{4.4}$$

In view of the first integrated compatibility equation of (3.9, we may (and will) use instead

$$\xi = \xi_i, \xi_0: \qquad k_{\theta\xi} - e/R_\theta = e_\theta = 0 \tag{4.5}$$

as our edge conditions in the subsequent development. With the help of (2.7), these conditions become conditions involving the values of ϕ, ϕ', ψ and ψ' at the edges.

The boundary value problem defined by the differential equations (2.4) and (2.5) (with $U_y = \Omega_x = \hat{p}_\theta = 0$) and the boundary conditions (4.5) determines ϕ and ψ in terms of the constants P_x and T_y. To bring out the dependence of ϕ and ψ on these two parameters, we write

$$\phi = P_x\,\phi_P + T_y\,\phi_T\,, \qquad \psi = P_x\,\psi_P + T_y\,\psi_T\,, \tag{4.6}$$

where ϕ_P, etc. are independent of P_x and T_y. Thus, if the applied force and moment are known, ϕ and ψ, and therefore the stress distributions, are completely determined without the explicit solution of the displacement components.

The two integrated conditions (4.3) and (4.4) then relate P_x and T_y to the rigid body displacement δ_x and rotation β_y of the plug. These conditions show that an accurate expression for k_ξ may be crucial for an accurate solution of our problem. Therefore, terms involving ψ in the expression for k_ξ in (2.7) should not be omitted even though their contribution to the stress level of the shell is always of the order h/R.

Upon expressing k_ξ and e_ξ in terms of ψ and ϕ and carrying out the integration with respect to ξ, we get two linear flexibility relations

$$\delta_x = P_x\,B_{\delta P} + T_y\,B_{\delta T}\,, \qquad \beta_y = P_x\,B_{\beta P} + T_y\,B_{\beta T}\,. \tag{4.7}$$

Given P_x and T_y, these two relations determine δ_x and β_y. Conversely, given δ_x and β_y, they serve to determine the necessary force and moment to produce the rigid body displacement and rotation of the rigid plug.

We may now, if we wish, execute the integration in (3.8) $\left(\text{with } U_y = \Omega_x = 0 \text{ and } \int^{\xi} \text{ replaced by } -\int_{\xi}^{\xi_0}\right)$ to get the distribution of the displacement components.

5. An Efficient Method for the Evaluation of the Influence Coefficients

To obtain the flexibility coefficients $B_{\delta P}, B_{\delta T} = B_{\beta P}$ and $B_{\beta T}$, a straightforward procedure would be to substitute the solution ϕ and ψ of the two point boundary value problem into (4.3) and (4.4), via the auxiliary equations (2.7), and then carry out the necessary integration. In general, the solution of the two point boundary problem, exact or asymptotic, is quite complicated; there is little hope for an evaluation of the relevant integrals to obtain analytical expressions for the flexibility coefficients in terms of the shell parameters.

However, it can be shown with the help of the differential equations and boundary conditions for ϕ and ψ that, except for terms of order $D\,A\,R_0^2/l^4$, the flexibility coefficients actually depend only on the value of ψ at the two edges of the shell, so that no integration of combinations of ϕ and ψ is necessary. Moreover, for a sufficiently thin shell, we only need to know the value of the particular solution of ψ at ξ_i and ξ_0. Such a particular solution can be obtained by a perturbation procedure, the leading term of which is the membrane solution for most shells. In this section, we will use the case of a circular cylindrical shell to illustrate how we can bypass the evaluation of integrals of ϕ and ψ to get the flexibility coefficients. The same technique is applicable to other shells of revolution.

With $r = r_0$ and $z = 2\,\xi, 0 \leq \xi \leq 1$, and in the absence of surface loads, the two differential equations (2.4) and (2.5) for a homogeneous shell of constant thickness and complete in the circumferential direction become

$$\phi'' - 2\frac{l^2}{r_0^2}\phi - \frac{l^2}{r_0 D}\psi = \frac{l^2}{r_0 D}\frac{P_x}{\pi}, \qquad \psi'' - 2\frac{l^2}{r_0^2}\psi + \frac{l^2}{r_0 A}\phi = \frac{l^2}{r_0^2}\frac{P_x}{\pi}, \tag{5.1}$$

where P_x is a constant. These equations can be combined into a single equation for $X = \phi + i\sqrt{A/D}\,\psi$:

$$X'' + 2\,i\,\mu^2\left(1 + 2\,i\frac{\sqrt{D\,A}}{r_0}\right)X = \frac{P_x l^2}{\pi D r_0}\left(1 + i\frac{\sqrt{D\,A}}{r_0}\right), \qquad 2\,\mu^2 = \frac{l^2}{\sqrt{D\,A\,r_0^2}}. \tag{5.2}$$

To the degree of approximation inherent in our shell theory, we have as an exact solution of (5.1)[5]

$$\left.\begin{aligned} \phi &= \sqrt{\frac{A}{D}}\{e^{\mu\xi}(c_1\cos\mu\,\xi + c_2\sin\mu\,\xi) + e^{-\mu\xi}(c_3\cos\mu\,\xi + c_4\sin\mu\,\xi)\}, \\ \psi &= e^{\mu\xi}(c_2\cos\mu\,\xi - c_1\sin\mu\,\xi) - e^{-\mu\xi}(c_4\cos\mu\,\xi - c_3\sin\mu\,\xi) - \frac{P_x}{\pi}. \end{aligned}\right\} \tag{5.3}$$

The four real constants of integrations, $c_1 - c_4$, are determined by the vanishing displacement conditions (4.5) at each of the two edges of the shell. For the present problem, these conditions take the form:

$$\xi = 0,1: \qquad \phi + \frac{A\,(1+\nu)}{r_0}\psi = 0, \qquad \psi' - \frac{D\,(1-\nu)}{r_0}\phi' = \nu\left(\frac{P_x l^2}{\pi r_0^2}\xi - \frac{T_y l}{\pi r_0^2}\right). \tag{5.4}$$

We will not give the full expressions for the constants of integration here, but will merely note that, for $\mu \gg 1$, we have

$$(e^{\mu} c_1, e^{\mu} c_2, c_3, c_4) = \frac{1}{\mu}\,O\left(\frac{P_x l^2}{\pi r_0^2}, \frac{T_y l}{\pi r_0^2}\right). \tag{5.5}$$

The overall load deformation relations (4.3) and (4.4) become

$$\beta_y = -\int_0^1 k_\xi\, l\, d\xi = \int_0^1\left[\phi' + \frac{A}{r_0}\left(\psi' + \frac{P_x l^2\xi}{\pi r_0^2} - \frac{T_y l}{\pi r_0^2}\right)\right]d\xi, \tag{5.6}$$

$$\delta_x = \int_0^1 \xi\, k_\xi\, l^2\, d\xi = -l\int_0^1\left[\xi\,\phi' + \frac{A}{r_0}\left(\xi\,\psi' + \frac{P_x l^2\xi^2}{\pi r_0^2} - \frac{T_y l\,\xi}{\pi r_0^2}\right)\right]d\xi. \tag{5.7}$$

[5] There is actually a nonvanishing particular solution for ϕ, but its contribution to the final solution is of the order of h/r_0.

The expression for β_y can be integrated immediately to give

$$\beta_y = \left[\phi + \frac{A}{r_0}\psi + \frac{P_x A\, l^2 \xi^2}{2\pi r_0^3} - \frac{T_y A\, l\, \xi}{\pi r_0^3}\right]_0^1 = \left[-\frac{\nu A}{r_0}\psi + \frac{P_x A\, l^2 \xi^2}{2\pi r_0^3} - \frac{T_y A\, l\, \xi}{\pi r_0^3}\right]_0^1, \tag{5.8}$$

where we have used the first boundary condition of (5.4) to eliminate ϕ. Upon integration by parts, the expression for δ_x becomes

$$\delta_x = \left[\frac{\nu A\, l\, \xi}{r_0}\psi - \frac{P_x A\, l^3 \xi^3}{3\pi r_0^3} + \frac{T_y A\, l^2 \xi^2}{2\pi r_0^3}\right]_0^1 + l\int_0^1 \left(\phi + \frac{A}{r_0}\psi\right) d\xi\,. \tag{5.9}$$

It would appear that we should now substitute the expressions for ϕ and ψ from (5.3) into the last term of the above relation and carry out the integration. But we can bypass this step as follows. We first use the second differential equation in (5.1) to eliminate ϕ from under the integral sign and then integrate to get

$$\int_0^1 \left(\phi + \frac{A}{r_0}\psi\right) d\xi = \left[-\frac{r_0 A}{l^2}\psi' + \frac{P_x A\, \xi}{\pi r_0}\right]_0^1 + \int_0^1 \frac{3A}{r_0}\psi\, d\xi\,. \tag{5.10}$$

We now use the first differential equation of (5.1) to express ψ under the integral sign in terms of ϕ and then integrate again to get

$$\int_0^1 \left(\phi + \frac{A}{r_0}\psi\right) d\xi = -\frac{r_0 A}{l^2}\left[\frac{(2+\nu)\, P_x\, l^2\, \xi}{\pi r_0^2} - \frac{\nu\, T_y\, l}{\pi r_0^2} - \underline{\frac{D\,(2+\nu)}{r_0}\phi'}\right]_0^1 - \underline{\int_0^1 \frac{6DA}{r_0^2}\phi\, d\xi}\,, \tag{5.11}$$

where the second boundary condition in (5.4) was used to simplify the right hand side. The underlined terms should be omitted in view of the other terms in the equation. Altogether, we have

$$\delta_x = \left[\frac{\nu A\, l}{r_0}\xi\,\psi - \frac{P_x A\, l^3}{\pi r_0^3}\left\{\frac{\xi^3}{3} + \frac{(2+\nu)\, r_0^2}{l^2}\xi\right\} + \frac{T_y A\, l^2}{2\pi r_0^3}\xi^2\right]_0^1. \tag{5.12}$$

With (5.8) and (5.12), we have that the ***exact*** influence coefficients for a circular cylindrical shell depend only on the value of ψ at the edges.

For a sufficiently thin shell so that $\mu \gg 1$, we have from (5.3) and (5.5) that the portion of ψ multiplied by the c_i's is $o(\mu^{-1})$ compared to the particular solution and can therefore be omitted[6]. In that case, (5.8) and (5.12) become

$$\beta_y \sim \frac{P_x A\, l^2}{2\pi r_0^3} - \frac{T_y A\, l}{\pi r_0^3}\,, \qquad \delta_x \sim -\frac{P_x A\, l^3}{\pi r_0^3}\left[\frac{1}{3} + 2\,(1+\nu)\frac{r_0^2}{l^2}\right] + \frac{T_y A\, l^2}{2\pi r_0^3}\,. \tag{5.13}$$

6. The Side Force and Tilting Moment Problem for a Uniform Spherical Shell

With $r = a \sin \xi$, $z = a \cos \xi$, $0 \le \xi_i \le \xi \le \xi_0 \le \pi$, and with constant A, D and ν, the governing differential equations for the side force and tilting moment problem can be combined into a single second order equation

$$X'' + \cot \xi\, X' - [4 \csc^2 \xi - n\,(n+1)]\, X =$$
$$= -\frac{P_x a}{\pi D}\csc^2 \xi \left\{\left[1 + \underline{\frac{\nu D A\,(1-\nu)}{a^2}}\right] - i\,\frac{(1-\nu)\sqrt{DA}}{a}\sqrt{1 - \underline{\frac{\nu^2 D A}{a^2}}}\right\} + \frac{T_y}{\pi D}\cot \xi \csc \xi\,, \tag{6.1}$$

where

$$n\,(n+1) = 1 - i\frac{a}{\sqrt{DA}}\sqrt{1 - \underline{\frac{\nu^2 D A}{a^2}}}\,, \qquad X = \phi + i\,\psi\sqrt{\frac{A}{D}}\left[\sqrt{1 - \underline{\frac{\nu^2 D A}{a^2}}} + i\,\nu\sqrt{\frac{DA}{a^2}}\right]. \tag{6.2}$$

[6] If $(l/r_0)^2$ is $o(\mu)$ or larger&hen the entire contribution of ψ to δ_x is negligible.

centrated forces and moments at the two poles, the conditions (64) are to be replaced by the requirement that

$$r\,e_\theta = A\left\{\sin\xi\,\psi' + (1-\nu)\cos\xi\,\psi - \frac{\nu\,T_y}{\pi\,a} + \frac{D}{a}(1-\nu)(\sin\xi\,\Phi' + 2\cos\xi\,\Phi)\right\}$$

and

$$r\left(k_{\theta\xi} - \frac{e}{a}\right) = \Phi - \frac{1}{a}A\,(1+\nu)\,\psi\,, \qquad (6.9)$$

be finite as $\xi \to 0, \pi$. Evidently, these conditions require that X be bounded at the poles.

In view of the limiting behavior of the Legendre functions near $\xi = 0$ and π [g], the finiteness condition on X determines the complex constants to be

$$B_1 = -\frac{P_x}{4}(\csc n\pi + \cot n\pi)\left(1 + i\nu\frac{\sqrt{DA}}{a}\right) - \frac{T_y}{4a}(\csc n\pi - \cot n\pi)\left(1 - i\frac{\sqrt{DA}}{a}\right),$$
$$B_2 = \frac{P_x}{2\pi}\left(1 + i\nu\frac{\sqrt{DA}}{a}\right) - \frac{T_y}{2\pi a}\left(1 - i\frac{\sqrt{DA}}{a}\right). \qquad (6.10)$$

Away from the two poles, we have from the asymptotic behavior of P_n^2 and Q_n^2 as $|n| \to \infty$ [g] that, when $(a/h)^{1/2} \gg 1$, the quantity $[Q_n^2(\cos\xi) - \frac{1}{2}\pi(\cot n\pi - \csc n\pi)P_n^2(\cos\xi)]$ is of the order

$$n\sqrt{\frac{n}{\sin\xi}} \times \begin{cases} e^{-\operatorname{Im}(n)\,\xi} & \varepsilon < \xi \le \frac{\pi}{2}, \\ e^{-\operatorname{Im}(n)\,(\pi-\xi)} & \frac{\pi}{2} \le \xi < \pi - \varepsilon, \end{cases} \qquad (6.11)$$

where $1 \gg (h/a)^{1/2}, 0$. The contribution from the P_n^2 and Q_n^2 terms in X is therefore negligible at a distance large compared to $(h/a)^{1/2}$ away from the poles and we have

$$\psi \sim -\frac{P_x}{\pi}\csc^2\xi + \frac{T_y}{\pi a}\cot\xi\csc\xi\,, \qquad \Phi \sim \frac{(1+\nu)\,T_y\,A}{\pi\,a^2}\cot\xi\csc\xi \qquad (6.12)$$

It is not difficult to verify that in this case the bending stresses, $6\,M/h^2$, are $O(h/a)$ compared to the direct stresses, N/h.

In a sufficiently small neighborhood of the pole $\xi = 0$, we have

$$X \approx -\frac{a}{12\,D}\left\{\frac{P_x}{\pi}\left[1 - i\,(1-\nu)\frac{\sqrt{DA}}{a}\right] - \frac{T_y}{\pi a}\left[1 - i\,9\frac{\sqrt{DA}}{a}\right]\right\}\left[1 + o\left(\sin^2\frac{\xi}{2}\right)\right] \qquad (6.13)$$

and correspondingly

$$\Phi \approx \frac{1}{12\,\pi\,D}(P_x\,a - T_y)\left[1 + o\left(\sin^2\frac{\xi}{2}\right)\right],$$
$$\psi \approx \frac{1}{12\,\pi\,a}[(1-\nu)P_x\,a - 9\,T_y]\left[1 + o\left(\sin^2\frac{\xi}{2}\right)\right]. \qquad (6.14)$$

From these, we obtain the following behavior of the stress resultants and couples near $\xi = 0$:

$$n_\xi \approx -\frac{(1+\nu)\,P_x\,a - 5\,T_y}{12\,\pi\,a^2\sin\xi}, \qquad n_\theta \approx \frac{P_x\,a - T_y}{12\,\pi\,a^2\sin\xi}, \qquad n_{\xi\theta} \approx n_{\theta\xi} \approx \frac{(1-\nu)\,P_x\,a - 9\,T_y}{12\,\pi\,a^2\sin\xi},$$
$$q_\xi \approx \frac{7\,(P_x\,a - T_y)}{6\,\pi\,a^2\sin^2\xi}, \qquad q_\theta \approx -\frac{P_x\,a - T_y}{6\,\pi\,a^2\sin^2\xi},$$
$$m_\xi \approx m_\theta \approx -\frac{1+\nu}{12\,\pi\,a\sin\xi}(P_x\,a - T_y)\,, \qquad m \approx -\frac{1-\nu}{12\,\pi\,a\sin\xi}(P_x\,a - T_y)\,. \qquad (6.15)$$

Similar results can be obtained near $\xi = \pi$. Together these results show that near the points of application of the concentrated forces and moments, the direct stresses are small, of relative order h/a, compared to the bending stresses. They also show that the singularity in q_ξ and q_θ is of higher order than that of the stress couples, very similar to the situation in the problem of axial point forces at the poles [3, 10]. However, it should be noted that theoretical results for concentrated loads are applicable only at a distance from the poles not less than several times the thickness h [10].

7. A Uniform Conical Shell Frustum

For a conical shell frustum with a semi-vertex angle equal to $\frac{1}{2}\pi - \beta$ $(0 < \beta < \frac{1}{2}\pi)$, we have $z = l s \xi$ and $r = 2 c \xi$ where $s = \sin\beta$ and $c = \cos\beta$. With constant A, D and ν and with $\hat{p}_\xi = \hat{p}_\theta = \hat{p}_n = 0$, equations (2.4) and (2.5) can be combined into a single equation

$$X'' + \frac{1}{\xi} X' - \left[\frac{4 + 2 t^2}{\xi^2} - \frac{i t l}{\sqrt{D A}} \frac{1}{\xi}\right] X = i \sqrt{\frac{A}{D}} \frac{P_x}{\pi c^2} \frac{1 - \nu}{\xi^2} + \frac{T_y}{\pi D c} \frac{1}{\xi^2} \left[1 + \frac{i t \nu \sqrt{D A}}{l \xi}\right], \quad (7.1)$$

for the complex function $X = \Phi + i\sqrt{A/D}\,\psi$, where $t = \tan\beta$. The two linearly independent solutions of the homogeneous equation are Bessel functions which we will write as

$$X = Z_{4\gamma}\left(2\mu\sqrt{i\xi}\right) = \begin{cases} B_1 J_{4\gamma}\left(2\mu\sqrt{i\xi}\right) + B_2 J_{-4\gamma}\left(2\mu\sqrt{i\xi}\right) & (\gamma \neq \text{integer}), \\ B_1 J_{4\gamma}\left(2\mu\sqrt{i\xi}\right) + B_2 Y_{4\gamma}\left(2\mu\sqrt{i\xi}\right) & (\gamma = \text{integer}) \end{cases} \quad (7.2)$$

where

$$\gamma = 1 + \frac{1}{2} t^2, \qquad \mu = \frac{t l}{\sqrt{D A}} \quad (7.3)$$

and where B_1 and B_2 are two complex constants of integration.

The behavior of the relevant Bessel function allows us to replace γ by unity for $t^2 = o(h/l)$. When $t^2 = O(1)$ or larger, so that $\mu \gg 1$, the leading term of the asymptotic expansion of the Bessel function $J_{4\gamma}$ differs from that of J_4 only by a constant factor. The same is true for the asymptotic expansion of $J_{-4\gamma}$ (or $Y_{4\gamma}$) and Y_4. The Bessel functions J_4 and Y_4 are the solutions obtained in [11] using Chernin's equations with several unessential terms omitted. The present formulation has the small advantage that we get a single second order equation for X without having to neglect any terms involving the complex conjugate of X.

When $t = O(1)$ so that the shell is neither nearly a flat plate nor nearly a cylindrical shell, we have to the degree of approximation inherent in shell theory

$$\psi_p = -\frac{T_y}{\pi l s \xi}, \qquad \Phi_p = \frac{P_x A}{\pi l s c} \frac{1 - \nu}{\xi} - \frac{T_y A}{\pi l^2 s^2 c} \frac{2 + c^2 - \nu s^2}{\xi^2} \quad (7.4)$$

as particular solutions for ψ and Φ.

For the determination of the two complex (or four real) constants of integration, we have from (4.5)

$$\xi = \xi_i, \xi_0: \quad \begin{cases} \Phi + \frac{s A (1 + \nu)}{r} \psi = 0, \\ \psi' + \frac{1 - \nu}{\xi} \psi - \frac{s D (1 - \nu)}{r} \left(\Phi' + \frac{1}{\xi} \Phi\right) = \frac{\nu}{r^2} \left(\frac{P_x l^2 \xi}{\pi} - \frac{T_y l s}{\pi}\right), \end{cases} \quad (7.5)$$

The flexibility relations are to be obtained from the integral relations (4.3) and (4.4). For $|D A R_\theta^2 / l^4| \ll 1$, we have by the technique outlined in the last two sections,

$$\left.\begin{aligned} \beta_y &= \left[-\frac{\nu A}{s r}\psi + \frac{T_y A (1 + 2 c^2)}{2 \pi s^2 c r^2} - \frac{P_x A}{\pi s c^2 r}\right]_{\xi_i}^{\xi_0}, \\ \delta_x &= \left[\frac{\nu A}{c}\psi - \frac{T_y A (1 - \nu c^2)}{\pi s c^2 r} - \frac{P_x A}{\pi c^3} \ln \xi\right]_{\xi_i}^{\xi_0}. \end{aligned}\right\} \quad (7.6)$$

When $\mu \gg 1$, we have, up to terms of order $1/\mu$,

$$\beta_y \sim \left[\frac{T_y A \{1 + 2 (1 + \nu) c^2\}}{2 \pi s^2 c^3 l^2} \frac{1}{\xi^2} - \frac{P_x A}{\pi s c^3 l} \frac{1}{\xi}\right]_{\xi_i}^{\xi_0}, \qquad \delta_x \sim -\left[\frac{T_y A}{\pi s c^3 l} \frac{1}{\xi} + \frac{P_x A}{\pi c^3} \ln \xi\right]_{\xi_i}^{\xi_0}. \quad (7.7)$$

8. A Static Geometric Duality

With $\hat{p}_\xi = \hat{p}_\theta = \hat{p}_n = 0$, the differential equations (2.4) and (2.5) (with $U_y = \Omega_x = 0$), the boundary conditions (4.5) and the two integral relations (4.3) and (4.4) are the static geometric duals of the differential equations (2.5) and (2.4) of the present paper (with $P_x = T_y = 0$), the boundary conditions (7.2) and the two integral relations (7.5) of [2]. That is: if we replace all the quantities appearing in the first set of equations by their static geometric duals in accordance with the following table, we will get the corresponding equations of the second and vice versa :

ϕ	D	ν_b	U_y	Ω_x	F_y	M_x	m	m_ξ	m_θ	n_θ	$n_{\xi\theta}$
ψ	$-A$	$-\nu_s$	T_y	P_x	β_y	δ_x	$-e$	e_θ	e_ξ	$-k_\xi$	$k_{\theta\xi}$

In this sense, we have that the side force and tilting moment problem for a shell frustum is the static geometric analogue of the asymmetric bending and twisting of ring shell sectors [2].

With this complete duality between the two physical problems, the solution of one of them can be obtained without another set of independent calculations from the solution of the other simply by replacing all quantities in the solution of the latter by their static geometric duals as demonstrated in [12]. For example, we can immediately write down the asymptotic expressions

$$\left.\begin{aligned} &C_{M\Omega} \sim \frac{D\,l^3}{\pi r_0^3}\left[\frac{1}{3} + 2\,(1-\nu)\frac{r_0^2}{l^2}\right], \\ &C_{MU} = ₵\ L\ ?\ - \frac{D\,l^2}{2\pi r_0^3}, \qquad C_{FU} \sim \frac{D\,l}{\pi r_0^3}, \end{aligned}\right\} \text{(cylindrical shell)} \tag{8.1}$$

$$\left.\begin{aligned} &C_{M\Omega} \sim a^2 C_{FU} \sim -\frac{D}{2\pi}(1-\nu)\left[(2\csc^2\xi + 1)\cot\xi\csc\xi - \ln\left|\tan\frac{\xi}{2}\right|\right]_{\xi_i}^{\xi_0}, \\ &C_{MU} = C_{F\Omega} \sim \frac{D}{\pi a}(1-\nu)\,[\csc^4\xi]_{\xi_i}^{\xi_0}, \end{aligned}\right\} \text{(spherical shell)} \tag{8.2}$$

$$\left.\begin{aligned} &C_{M\Omega} \sim \frac{D}{\pi c^3}[\ln\xi]_{\xi_i}^{\xi_0}, \qquad C_{MU} = C_{F\Omega} \sim \frac{D}{\pi s c^3 l}\left[\frac{1}{\xi}\right]_{\xi_i}^{\xi_0}, \\ &C_{FU} \sim -\frac{D}{2\pi s^2 c^3 l^2}[1 + 2\,(1-\nu)\,c^2]\left[\frac{1}{\xi^2}\right]_{\xi_i}^{\xi_0}, \end{aligned}\right\} \text{(conical shell)} \tag{8.3}$$

for the stiffness coefficients $C_{M\Omega}$, etc., in the stiffness relations

$$M_x = C_{M\Omega}\,\Omega_x + C_{MU}\,U_y, \qquad F_y = C_{F\Omega}\,\Omega_x + C_{FU}\,U_y \tag{8.4}$$

for asymmetric bending and twisting of ring shell sectors [2].

Moreover, a computer program developed to give a more complete solution for one problem can also be used (without any modification) to solve the other. The procedure is simply to use for inputs the values of their dual quantities and to interpret the outputs as the results for the dual quantities as demonstrated in [13, 14] for problems involving helicoidal shells.

The reformulation of the conventional displacement boundary conditions in terms of strain and curvature change measures is evidently a crucial step in establishing the above duality. The possibility of reformulating general displacement boundary conditions as the static geometric duals of the stress boundary conditions seems to have been first noted in [15].

9. Appendix

With the help of the first integrals of the differential equations of equilibrium and compatibility which are known to exist for the particular class of stress and strain distributions of interest here (see equation (2.1)) I twas shown in [2] that all but two of the shell equations are satisfied if we express the ξ-dependent portion of the stress and strain measures, $n_{\xi\theta}$, $n_{\theta\xi}$, etc., in terms of a stress function ψ and a strain function Φ by the relations

$$n_{\xi\theta} = \frac{\psi}{r}, \qquad n_{\theta\xi} = \frac{1-\varepsilon_0^2}{1+\varepsilon_0^2}\,\frac{\psi}{r} - \frac{2\,D_S\,\varrho}{1+\varepsilon_0^2}\,\frac{\Phi}{r},$$

$$k_{\theta\xi} = \frac{\Phi}{r}, \qquad k_{\xi\theta} = \frac{1-\varepsilon_0^2}{1+\varepsilon_0^2}\,\frac{\Phi}{r} + \frac{2\,A_S\,\varrho}{1+\varepsilon_0^2}\,\frac{\psi}{r},$$

$$\begin{bmatrix} q_\xi \\ m_\xi \\ m_\theta \\ \hdashline l_\theta \\ e_\theta \\ e_\xi \end{bmatrix} = \frac{1}{(1+\varepsilon_4^2)} \times \left[\begin{array}{ccc:ccc} (1+\varepsilon_3^2)\dfrac{z'}{\alpha^2} & -\dfrac{r'}{\alpha^2} & 0 & \dfrac{\nu_b D r'^2}{r\alpha^3} & \dfrac{\nu_b D r' z'}{r\alpha^3} & -\dfrac{D r'}{r\alpha} \\ -\dfrac{\varepsilon_4^2 r' R_\theta}{\alpha^2} & \dfrac{\varepsilon_4^2 r}{\alpha} & 0 & \dfrac{\nu_b D r'}{\alpha^2} & \dfrac{\nu_b D z'}{\alpha^2} & -D \\ 0 & 0 & \varepsilon_4^2 R_\theta & \dfrac{D r'}{\alpha^2} & \dfrac{D z'}{\alpha^2} & -\nu_b D \\ \hdashline \dfrac{\nu_s A r'^2}{r\alpha^3} & \dfrac{\nu_s A r' z'}{r\alpha^3} & \dfrac{A r'}{r\alpha} & (1+\varepsilon_3^2)\dfrac{z'}{\alpha^2} & -\dfrac{r'}{\alpha^2} & 0 \\ \dfrac{\nu_s A r'}{\alpha^2} & \dfrac{\nu_s A z'}{\alpha^2} & A & -\dfrac{\varepsilon_4^2 r' R_\theta}{\alpha^2} & \dfrac{\varepsilon_4^2 r}{\alpha} & 0 \\ -\dfrac{A r'}{\alpha^2} & -\dfrac{A z'}{\alpha^2} & -\nu_s A & 0 & 0 & \varepsilon_4^2 R_\theta \end{array}\right] \times$$

$$\times \left[\begin{array}{c} -\dfrac{\alpha}{r}\left(\dfrac{P_x}{\pi} + \psi\right) \\ \dfrac{2 D_S r'}{r^2(1+\varepsilon_0^2)}(\Phi + A_S\,\varrho\,\psi) + \dfrac{\alpha}{r^2}\left(\dfrac{T_y}{\pi} - \dfrac{z\,P_x}{\pi}\right) \\ \dfrac{1}{r\alpha}\left\{\left[\dfrac{1+\varepsilon_1^2}{1+\varepsilon_0^2}\,r\,\psi + \dfrac{2 D_S}{R_\theta(1+\varepsilon_0^2)}\,r\,\Phi\right]' + r^2\alpha\,\hat{p}_\theta\right\} \\ \hdashline -\dfrac{\alpha}{r}\left(\dfrac{\Omega_x}{\pi} + \Phi\right) \\ -\dfrac{2 A_S r'}{r^2(1+\varepsilon_0^2)}(\psi - D_S\,\varrho\,\Phi) + \dfrac{\alpha}{r^2}\left(\dfrac{U_y}{\pi} - \dfrac{z\,\Omega_x}{\pi}\right) \\ \dfrac{1}{r\alpha}\left[\dfrac{1+\varepsilon_1^2}{1+\varepsilon_0^2}\,r\,\Phi - \dfrac{2 A_S}{R_\theta(1+\varepsilon_0^2)}\,r\,\psi\right]' \end{array}\right],$$

$$e_\xi = A\,(n_\xi - \nu_s\,n_\theta), \qquad e_\theta = A\,(n_\theta - \nu_s\,n_\xi), \qquad e = A_S\,(n_{\xi\theta} + n_{\theta\xi}),$$

$$m_\theta = D\,(k_\theta + \nu_b\,k_\xi), \qquad m_\xi = D\,(k_\xi + \nu_b\,k_\theta), \qquad m = D_S\,(k_{\xi\theta} + k_{\theta\xi}),$$

$$r\,\alpha\,q_\theta = (r\,m)' - \alpha\,m_\theta + r'\,m, \qquad r\,\alpha\,l_\xi = (r\,e)' + \alpha\,e_\xi + r'\,e,$$

where

$$\alpha = \sqrt{r'^2 + z'^2}, \qquad \frac{1}{R_\theta} = -\frac{z'}{r\,\alpha}, \qquad \frac{1}{R_\xi} = \frac{r''\,z' - r'\,z''}{\alpha^3}, \qquad \varrho = \frac{1}{R_\theta} - \frac{1}{R_\xi},$$

$$\varepsilon_0^2 = D_S\,A_S\,\varrho^2, \qquad \varepsilon_1^2 = D_S\,A_S\,\varrho\left(\frac{3}{R_\theta} - \frac{1}{R_\xi}\right), \qquad \varepsilon_3^2 = \frac{4\,D_S\,A_S}{r^2}, \qquad \varepsilon_4^2 = \frac{4\,D_S\,A_S}{R_\theta^2}.$$

The remaining two (one equilibrium and one compatibility) equations

$$(r\,m_\xi)' + \alpha\,m - r'\,m_\theta - r\,\alpha\,q_\xi = 0, \qquad (r\,e_\theta)' - \alpha\,e - r'\,e_\xi - r\,\alpha\,l_\theta = 0$$

give two simultaneous second order differential equations for Φ and ψ of the form (2.2). In [2], these equations were simplified by simply omitting terms of order h/R and h^2/l^2, leaving us with (1.1).

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Received December 6, 1971

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