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Rotor Blade Response to Random Loads: A Direct Time-Domain Approach

F. Y. M. WAN*
MIT, Cambridge, Mass.

AND

C. LAKSHMIKANTHAM†
Army Materials and Mechanics Research Center, Watertown, Mass.

The response of a rotor blade subject to random loads is treated within the framework of time-varying linear dynamical systems with nonstationary random forcing functions. The statistically meaningful characteristics of the response, namely the covariance and the correlation, are directly obtained from a pair of ordinary matrix differential equations in the time domain. Extensive results for different models of a rigidly flapping blade subject to quasi-stationary random excitations have been obtained by this direct time-domain approach to demonstrate its simplicity and efficiency.

Introduction

THE design of a helicopter rotor blade presents a greater challenge to structural engineers than that of a fixed airfoil such as a wing or a tail of an airplane. The aerodynamics of a rotor blade is more complex. The flowfield relative to the blade, generated by the vehicle's forward speed and the blade rotation, produces effects in a multibladed rotor such as reverse flows, compressibility at the advancing blade, and the possibility of a stall at the retreating blade.[1] Moreover, the combined flowfield yields certain damping and spring force effects in the aerodynamic forces acting on the blade which are periodic in time. Therefore, any realistic study of the response problem for rotor blades in forward flight will involve a set of coupled equations of motion with time-dependent coefficients. These equations of motion have been formulated and investigated for the response of the blade to deterministic aerodynamic loads and the associated stability criteria (e.g., Ref. 2).

As the result of high-intensity rotor-generated turbulence as well as other mechanisms, the rotor blade is also subject to severe random aerodynamic loads under certain flight conditions. When we began our work, the only papers pertaining to the response of rotor blades to random loads were those of Gaonkar and Hohenemser[3–5] who considered the single-degree-of-freedom case of the small amplitude flapping response of a rigid blade, hinged or elastically restrained at the root.‡

In Ref. 3 dealing with a small-to-moderate advance ratio ($\mu \leqq 1$) (i.e., the ratio of the vehicle forward speed to the blade-tip rotating speed), all μ^2 terms and reverse flow effects were neglected in the aerodynamic parameters. The random load in this case was chosen to be that resulting from the random change of the collective pitch. We will show in this paper that, informative though they may be, the conclusions based on such a simplified model and on their particular perturbation solution should not be taken as definitive.

The method of analysis for the high advance ratio case ($\mu > 1$) investigated in Ref. 4 is based on the mixed time-frequency domain analysis suggested in Ref. 6. A frequency decomposition of the quasi-stationary excitation characterizing the effect of atmospheric turbulence was first carried out, and the response to individual frequencies was then obtained by integrating the relevant differential equations numerically with the nonanalytic coefficients of the differential equations replaced by their 16-term Fourier series. Finally, the responses to individual frequencies were superposed to obtain statistical information such as the variance of the response. In this last step, only an approximation of the power spectral density of the stationary inflow ratio (the random part of the excitation) was used. The same method was employed to study the effect of a nonstationary inflow ratio in Ref. 5.

As the really meaningful measures for the nonstationary output of a time-varying dynamical system are the covariance and correlation functions, we propose here a direct method in the time domain to obtain these quantities. Our particular method is attractive in that the relevant algorithm is simple, efficient, and exact (in that no approximation other than a Runge-Kutta integration scheme is involved). Moreover, it can be easily adapted to more elaborate blade models which include the coupling of flapping, torsion, lead-lag, and blade flexibility. As a demonstration, the method is used in this paper to obtain extensive numerical results for the rigid blade model developed in Ref. 7 and used in Refs. 4 and 5. A brief description of this model will be given in the next section.

Flapping Motion of a Rigid Rotor Blade

In the model of rotor blades developed in Ref. 7 and used in Ref. 4, the blade is treated as a rigid beam rotating with angular speed Ω. The beam is elastically restrained at one end and free at the other end. Upon setting the spring constant of the restraining spring equal to zero, we have a hinged blade as a

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Index categories: Structural Dynamic Analysis; VTOL Vibration; Rotary Wing Aerodynamics.

* Associate Professor, Department of Mathematics.

† Research Mechanical Engineer, Theoretical and Applied Mechanics Lab. Member AIAA.

‡ After the paper had been presented, G. H. Gaonkar advised the authors that he too had used the direct-time domain approach for rotor blade problems in his recent paper in *Journal of Sound and Vibration,* Vol. 18, Nov. 1971, pp. 381–389. A referee also brought to their attention a recent AHS Preprint (No. 512, May 1971) on a multiblade analysis[10] by the method used in Refs. 4 and 5.

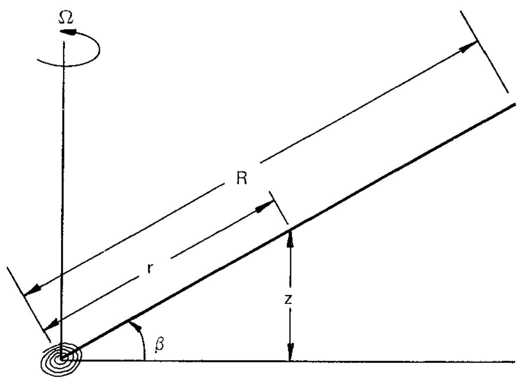

Fig. 1 Model of rigid flapping blade.

special case. The beam is assumed to be capable only of a rigid flapping motion in the z direction (Fig. 1). For small amplitude motions, the transverse displacement z at a point of distance r from the root of the blade along the blade axis is given by

$$z(r,t) = \beta(t)r \tag{1}$$

where $\beta(t)$ is the flapping angle between the blade axis and the horizon (Fig. 1) and t is a dimensionless time variable with t/Ω being the real time.

The flowfield relative to the blade consists of the vertical velocity component U_z and horizontal component U_h. If R is the blade length and V is the constant forward velocity of the vehicle, then we have

$$U_h = \Omega(r+\mu R\sin t), \quad U_z = \Omega[\dot{z}+\mu R z'\cos t]+R\Omega\lambda \tag{2}$$

where dots and primes indicate differentiation with respect to t and r, respectively, $\mu = V/\Omega R$, and λ is the dimensionless random part of U_z, the inflow ratio. It is assumed for simplicity that U_h does not have any random fluctuation.

In terms of U_z and U_h, we can obtain the aerodynamic lift acting on the blade by a linearized two-dimensional aerodynamic theory, taking into account the effect of reverse flow. A differential equation for the flapping angle can then be obtained by a moment equilibrium consideration. This differential equation of motion can be written as[7]

$$\ddot{\beta}+(\gamma/2)C(t)\dot{\beta}+[P^2+(\gamma/2)K(t)]\beta = (\gamma/2)B(t)u \tag{3}$$

Equation (3) is of the same form as the blade equation in Refs. 3 and 4. In Ref. 3, the random input is taken as the change in the angle of attack through a pitch angle θ. In Ref. 4, the random input is the inflow ratio λ. Hence, when we compare our results with those of Ref. 3, it will be understood that u stands for θ, the random change in the collective blade pitch, and $B(t) = B_\theta(t)$, the corresponding envelope function. On the other hand, to compare with results of Ref. 4, we should take $u = \lambda$ and $B(t) = B_\lambda(t)$. Also in Eq. 3, γ is the Lock number ($2 \leqq \gamma \leqq 16$, in practice) and $P^2 = 1+{\omega_\beta}^2/\Omega^2$ with ω_β as a measure of the elastic restraint at the root end ($\omega_\beta = 0$ for a hinged blade). The coefficients $C(t)$, $K(t)$, and $B(t)$ have been obtained in Ref. 7 and are listed in Appendix A to emphasize the fact that these coefficients are continuous, periodic with a period of 2π, but non-analytic functions of t. Equation (3) is supplemented by some initial conditions which, because of the linearity of our problem, may be taken as $\beta(0) = \dot{\beta}(0) = 0$.

A Direct Time-Domain Approach

When $u(t)$ is a random function with known statistics, a solution of the above initial value problem consists of obtaining moment functions (or joint probability density functions) of $\beta(t)$ of all order. In practice, we often settle for the first and second order statistics (which are all we need if u is gaussian). Ensemble averaging both sides of Eq. (3) and the initial conditions, we see immediately that $\langle\beta(t)\rangle$, the expected (or mean) value of β, is governed by the same initial value problem except for u replaced by its expected value. In particular, β will be of zero mean if u is.

The determination of the covariance and correlation functions is less straightforward. It can be shown[8] that for a general first-order linear system

$$\dot{\underline{z}} = A(t)\underline{z}+G(t)\underline{v} \tag{4}$$

with $\underline{z}(0) = 0$ and with $\underline{v}$ being a zero mean vector white noise process, i.e.

$$R_{vv}(t+\tau,\, t) = \langle \underline{v}(t+\tau)\underline{v}^T(t)\rangle = Q\delta(\tau) \tag{5}$$

where Q is a non-negative constant matrix and $(\ )^T$ is the transpose of $(\ )$, the covariance matrix $D(t) \equiv \langle \underline{z}(t)\underline{z}^T(t)\rangle$ is governed by the initial value problem

$$\dot{D} = AD+DA^T+GQG^T, \quad D(0) = 0 \tag{6}$$

and the correlation matrix $R(t+\tau, t) \equiv \langle \underline{z}(t+\tau)\underline{z}^T(t)\rangle$, $\tau > 0$ is governed by the initial value problem

$$dR/d\tau = A(t+\tau)R, \quad R(t,t) = D(t) \tag{7}$$

For completeness, a derivation of these results is sketched in Appendix B. Note that $R(t+\tau, t)$ for $\tau < 0$ is given by the symmetry condition $R(t_1, t_2) = R^T(t_2, t_1)$.

For an excitation $\underline{v}(t)$ which is not white noise, the problem can be reduced to one for a white noise excitation by associating the correlated input $\underline{v}(t)$ with the response of a supplementary linear dynamical system to white noise. By this, we mean that $\underline{v}(t)$ and the (steady state) output of the (fictitious) supplementary system have the same first and second-order statistics. The augmented linear system (the original plus the supplementary) is again one with white noise excitation if we treat $\underline{z}$ and $\underline{v}$ both as components of an unknown vector.

To illustrate, suppose that u in Eq. (3) is a zero mean stationary random process with a correlation function of the form

$$R_{uu}(t+\tau,\, t) = \sigma^2 e^{-|\tau|} \tag{8}$$

Note that even if u is stationary, the response $\beta(t)$ will be non-stationary due in part to the envelope function $B(t)$ of the stationary input u and in part to the time-varying system parameters $C(t)$ and $K(t)$. For the purpose of relating u to a zero mean white noise excitation $w(t)$, consider the supplementary dynamical system characterized by

$$\dot{u}+\alpha u = (2\alpha)^{1/2}\sigma w(t), \quad u(0) = 0 \tag{9}$$

where $R_{ww}(t+\tau, t) = \delta(\tau)$. It is not difficult to see from the impulse response representation of the solution for $t > 0$

$$u(t) = (2\alpha)^{1/2}\sigma \int_0^t e^{-\alpha(t-x)}w(x)\,dx \tag{10}$$

that u is of zero mean and is with a steady-state correlation function as given by Eq. (8). Upon introducing $z_1 = \beta(t)$, $z_2 = \dot{\beta}(t)$, and $z_3 = u(t)$, we may write the differential Eqs. (3) and (9) as the following linear first-order system:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -[P^2+\frac{\gamma}{2}K(t)] & -\frac{\gamma}{2}C(t) & \frac{\gamma}{2}B(t) \\ 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ [2\alpha\sigma^2 w]^{1/2} \end{pmatrix} \tag{11}$$

The system Eq. (11) is a special case of the following general first-order system Eq. (4) with $Q = I$ and

$$G(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & [2\alpha\sigma^2]^{1/2} \end{pmatrix}, \quad \bar{v} = \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix} \tag{12}$$

Our problem now becomes one of the response of a time-varying system excited by white noise. Equations (6) and (7) are therefore applicable for the determination of the second-order statistics.

The advantage of the above direct-time domain method is that it allows us to obtain all the second-order statistics of the non-stationary response with only two different but very similar sets of calculations in the time domain. All the necessary calculations can be relegated to the computer, since numerical techniques for the solution of initial value problems in ordinary differential equations are well developed and very reliable. Moreover, the time-varying nonanalytic coefficients $C(t)$ and $K(t)$ and the envelope function $B(t)$ themselves, rather than their approximate Fourier series representation,[4] can (and will) be used in our method of solution. Though it is known in the field of control,[8] this method does not seem to have been used for problems in random vibration (see preceding footnote).

For a typical set of parameters (say, $\mu = 1.0$, $\gamma = 4.0$, and $\alpha = 0.5$), an "exact" solution of Eq. (6) for the rotor blade problem over eight cycles by a fourth-order Runge-Kutta scheme with a step size $\Delta t = \pi/40 \simeq 0.08$ consumes less than 8 sec of machine time on a CDC 6400. An "approximate" solution of the covariance matrix over 3+ cycles by the mixed time-frequency method with a step size $\Delta t = 0.2$ requires about 3 min on an IBM 360/50.[4] § We label the solution of Ref. 4 "approximate" only to indicate the fact that, instead of the actual expressions, truncated Fourier series representation of $C(t)$, $K(t)$, and $B(t)$ and a truncated canonical series representation of the power spectral density of the input were used.

Numerical Results for Blade Flapping

The method described in the preceding section has been applied to study the flapping motion of rotor blades in response to an exponentially correlated random $u(t)$. The solution of the two relevant initial value problems was obtained by a fourth-order Runge-Kutta scheme, and then independently checked by fourth-order predictor-corrector scheme. Once the computer programs have been written, the solution for different combinations of the structural and aerodynamic parameters can be generated with very little effort. Since only a few seconds of computer time are needed for a given combination, we were able to study in some detail the effect of the parameters μ, γ, P, α, and TLF (see Appendix A) on the response statistics.

The objectives of our study are 1) to investigate the range of applicability of the simplified model and the perturbation solution scheme of Ref. 3, 2) to study the merit of the mixed time-frequency approach used in Ref. 4, and 3) to obtain additional information on the rotor blade behavior not given elsewhere. In connection with item 2, we found that our results generally support those given in Ref. 4. A preference for the direct-time domain method described herein or for the mixed time-frequency method used in Ref. 4 reflects only a concern for computing time and/or cost or an interest in the quasi-frequency composition of the response statistics. Therefore, numerical results for a random inflow excitation $\lambda(t)$ will not be included here.

To study the range of μ for which the simplified model and the perturbation solution scheme are adequate, we will fix, as in Ref. 3, $\gamma = 4$, $\alpha = 0.5$, $P = 1$ (a hinged blade), and TLF = 1, and take u to be that due a random change of the collective pitch so that $B(t) = B_\theta(t)$.

In Fig. 2, we compare the periodic mean-square response $\langle \beta^2(t) \rangle$ of the Sissingh model (S) computed over one period for values of $\mu = 0.3$, 0.6, and 1.0 by our direct method with that of the model used by Gaonkar and Hohenemser (G+H). The latter is Sissingh's model with all μ^2 terms and reverse flow omitted. Our exact result for the case $\mu = 0.3$ ¶ for the G+H model agrees extremely well with the two-term perturbation solution obtained in Ref. 3. On the other hand, we see from Fig. 2 that there is already a 20% discrepancy in the peak value of $\langle \beta^2 \rangle$ between the exact solution of the G+H model and that of the S model. We see also that the result for the S model and the result for the same model with the effect of reverse flow omitted (S′) are almost the same, so that the effect of reverse flow

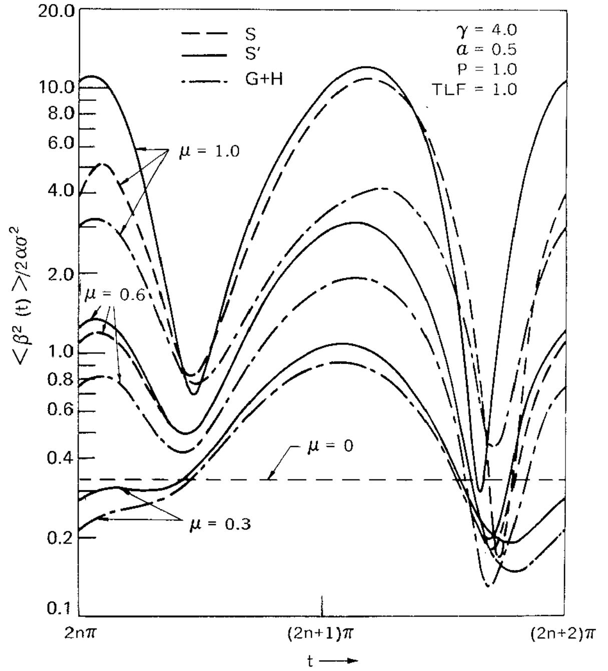

Fig. 2 Mean square of response $\beta(t)$ for various blade models and μ values.

is negligible for $\mu = 0.3$. The discrepancy in $\langle \beta^2 \rangle_{max}$ between the exact G+H and the S solutions is over 25% of the more accurate S solution for $\mu = 0.6$ and over 60% for $\mu = 1.0$. The effect of reverse flow is still unimportant for $\mu = 0.6$, but becomes very important at $\mu = 1.0$.

Beyond a comparison with the results of Ref. 3, we show in Figs. 3 and 4 how the periodic mean square velocity $\langle \dot{\beta}^2(t) \rangle$ and the covariance $\langle \beta(t)\dot{\beta}(t) \rangle$ vary with time over one period for $\mu = 0.3$, 0.6, and 1.0 for the problem of random change of collective pitch. Note that these results, not given in Ref. 3, are

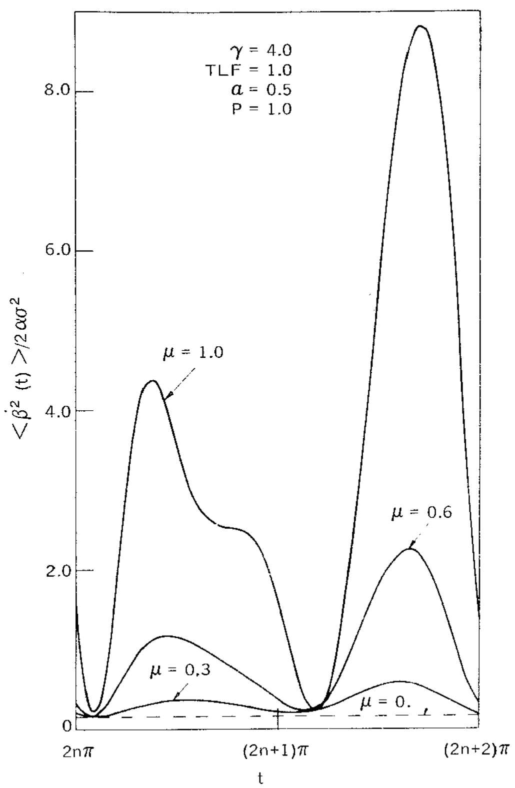

Fig. 3 Variance of $\dot{\beta}$ for various μ values.

§ The expected rate of threshold crossing for seven threshold values was also calculated. But this should consume only a very small portion of the 3 min.

¶ Note that $\mu = 0.3$ is the only case for which meaningful comparison can be made with the perturbation solution given in Ref. 3.

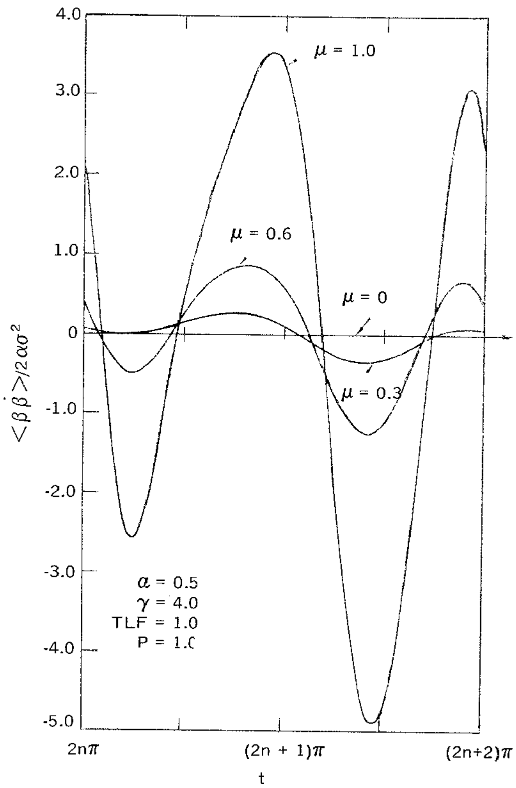

Fig. 4 Cross variance of $\beta(t)$ and $\dot{\beta}(t)$ for various μ values.

simply different elements of the covariance matrix $D(t)$. The three quantities $\langle\beta^2\rangle$, $\langle\dot{\beta}^2\rangle$, and $\langle\beta\dot{\beta}\rangle$ are needed to compute the threshold crossing expectation.[4]

Figure 5 shows the peak values of $\langle\beta^2\rangle$ and $\langle\dot{\beta}^2\rangle$ as functions of μ for $\lambda = 4$, 8, and 16. It is interesting that, within the framework of a linear theory, these peak values increase nearly exponentially with μ for a fixed γ.

Figure 6 gives a typical plot of the autocorrelation function $\langle\beta(t)\beta(t+T)\rangle$ which is the upper left-hand corner element of the correlation matrix $R(t, t+T)$. The correlation becomes significantly weaker for time intervals beyond one revolution. From similar results for other combinations of γ, α, and μ not presented herein, we found the relaxation time of the response

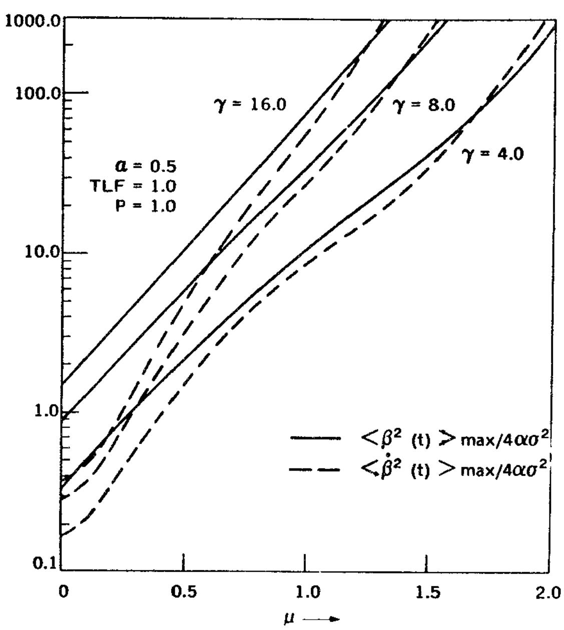

Fig. 5 Influence of lock number on the peak mean square response.

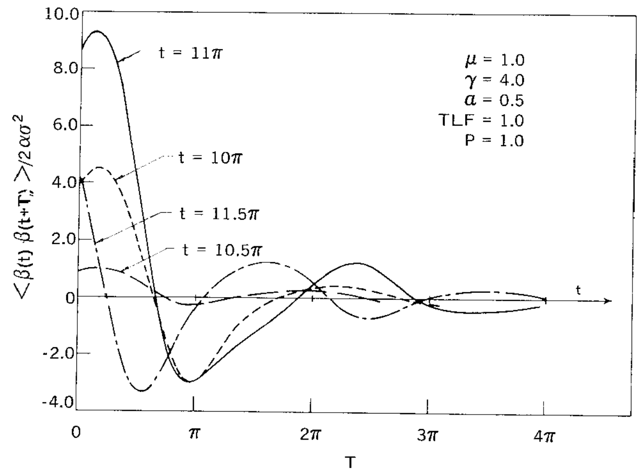

Fig. 6 Steady-state autocorrelation of $\beta(t)$.

correlation to be the larger of $8/\gamma$ and $1/\alpha$. More extensive numerical results for D and R for other combinations of the structural and aerodynamic parameters are reported in Ref. 9.

Concluding Remarks

In the preceding sections we have used a simple, efficient, and exact time-domain method of control theory for the determination of all first and second-order statistics of the response of a time-varying vibratory system to nonstationary excitations. As an illustration, the method was applied to the problem of the flapping response of a single rigid rotor blade to a stationary, exponentially correlated excitation modulated by an envelope function. With very little effort, we obtained more extensive and more accurate results than those available in the literature in the small-to-moderate advance ratio range. For the high advance ratio case, we have confirmed the results of Ref. 4 and obtained additional second-order statistics in the process, though not all the results obtained were presented herein. We feel that the method will be even more attractive than the usual mixed frequency-time analysis proposed in Refs. 4 and 6 for more elaborate blade models.

Appendix A

The coefficients $C(t)$, $K(t)$, and $B(t)$ in the equation of motion have been obtained in Ref. 7 and are listed in Table 1 to show their nonanalytic feature

Table 1 Nonanalytic coefficient occurring in Eq. (3)

Coefficient	Region
	$I\ (0 \leqq t < \pi)$
$C(t)$	$\frac{1}{4}\mathrm{TLF}^4+\frac{1}{3}\mathrm{TLF}^3\mu\sin t = C_1(t)$
$K(t)$	$\frac{1}{3}\mathrm{TLF}^3\mu\cos t+\frac{1}{4}\mathrm{TLF}^2\mu^2\sin 2t \equiv K_1(t)$
$B_\lambda(t)$	$\frac{1}{3}\mathrm{TLF}^3+\frac{1}{2}\mathrm{TLF}^2\mu\sin t \equiv B_{\lambda 1}(t)$
$B_\theta(t)$	$\frac{1}{4}\mathrm{TLF}^4+\frac{2}{3}\mathrm{TLF}^3\mu\sin t+\frac{1}{4}\mathrm{TLF}^2\mu^2 \times (1-\cos 2t) \equiv B_{\theta 1}(t)$
	$II\ (\pi \leqq t < \pi+\varepsilon,\ 2\pi-\varepsilon \leqq t < 2\pi)$
$C(t)$	$C_1(t)+(\mu^4/48)(3-4\cos 2t+\cos 4t)$
$K(t)$	$K_1(t)-(\mu^4/24)(2\sin 2t-\sin 4t)$
$B_\lambda(t)$	$B_{\lambda 1}(t)-(\mu^3/12)(3\sin t-\sin 3t)$
$B_\theta(t)$	$B_{\theta 1}(t)-(\mu^4/48)(3-4\cos 2t+\cos 4t)$
	$III\ (\pi+\varepsilon \leqq t < 2\pi-\varepsilon)$
$C(t)$	$-C_1(t)$
$K(t)$	$-K_1(t)$
$B_\lambda(t)$	$-B_{\lambda 1}(t)$
$B_\theta(t)$	$-B_{\theta 1}(t)$

where, in terms of the tip loss factor TLF and the advance ratio μ, $\sin \varepsilon = \mathrm{TLF}/\mu$, and where Region III is applicable only when TLF $< \mu$. Also, $B_\lambda(t)$ and $B_\theta(t)$ are the envelope functions for the inflow ratio and collective pitch, respectively.

Appendix B

We obtain the equation for the covariance matrix $D(t)$ by forming the combination $\dot{z}(t)z^T(t)+z(t)\dot{z}^T(t)$ and using Eq. (4) to get

$$\dot{D}(t) = \langle \bar{z}(t)\bar{z}^T(t)\rangle^{\cdot} = A(t)\langle \bar{z}(t)\bar{z}^T(t)\rangle + \langle \bar{z}(t)\bar{z}^T(t)\rangle A^T(t) + G(t)\langle \bar{v}(t)\bar{z}^T(t)\rangle + \langle \bar{z}(t)\bar{v}^T(t)\rangle G^T(t) \tag{B1}$$

In order to eliminate the covariance $\langle \bar{v}(t)\bar{z}^T(t)\rangle$ and its transpose, we note that the solution of the linear Eq. (4) may be written in terms of the impulse response matrix Z in the form of a superposition integral

$$\bar{z}(t) = \int_{-\infty}^{t} Z(t,\tau)G(\tau)\bar{v}(\tau)\,d\tau \tag{B2}$$

where an element $z_{ij}(t,\tau)$ of Z is the ith component of the vector $\underline{z}$ due to only a unit impulse excitation at the jth component of $\bar{v}$ applied at $t=\tau$. From Eq. (9) we have, with $Z(t,t)=I$ and Eq. (5)

$$\langle \bar{z}(t)\bar{v}^T(t)\rangle = \int_{-\infty}^{t} Z(t,\tau)G(\tau)\langle \bar{v}(\tau)\bar{v}^T(t)\rangle\,d\tau = (1/2)G(t)Q \tag{B3}$$

and similarly

$$\langle \bar{v}(t)\bar{z}^T(t)\rangle = (1/2)QG^T(t) \tag{B4}$$

Therefore, we may write Eq. (B2) as

$$\dot{D} = AD + DA^T + GQG^T \tag{B5}$$

Equation (B5), along with a suitable initial condition, determines the covariance matrix $D(t)$ completely.**

Having $D(t)$, we can use it to determine the correlation matrix. For all $t_2 > t_1$, we get from Eq. (4)

$$(d/dt_2)\langle \bar{z}(t_2)\bar{z}^T(t_1)\rangle = A(t_2)\langle \bar{z}(t_2)\bar{z}^T(t_1)\rangle + G(t_2)\langle \bar{v}(t_2)\bar{z}^T(t_1)\rangle \tag{B6}$$

** A referee pointed out that this derivation of Eq. (B5) is the same as that given by Van Trees in *Detection, Estimation and Modulation Theory*, Wiley, New York, 1968.

But from Eq. (B2), we have $\langle \bar{v}(t_2)\bar{z}^T(t_1)\rangle = 0$ for $t_2 > t_1$. Therefore,

$$(d/dt_2)R(t_1,t_2) = A(t_2)R(t_1,t_2), \qquad (t_2 > t_1) \tag{B7}$$

Equation (B7) is supplemented by the initial condition $R(t_1,t_1) = D(t_1)$. Together, they determine the correlation matrix $R(t_1,t_2)$ completely.

We could, of course, write Eq. (B7) alternatively as

$$(d/dT)R(t,t+T) = A(t+T)R(t,t+T), \quad R|_{T=0} = D(t) \tag{B8}$$

obtained from Eq. (B7) by setting $t_1 = t$ and $t_2 = t+T$.

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