Int. J. Solids Structures, 1970, Vol. 6, pp. 959 to 973. Pergamon Press. Printed in Great Britain

CIRCUMFERENTIALLY SINUSOIDAL STRESS AND STRAIN IN SHELLS OF REVOLUTION†

FREDERIC Y. M. WAN

Department of Mathematics
Massachusetts Institute of Technology

Abstract—The system of equations of the linear theory of isotropic elastic shells of revolution for stress and strain sinusoidal in the polar angle of the base plane with period 2π is reduced to two simultaneous second order ordinary differential equations which are remarkably similar to the corresponding equations for symmetric bending. The reduction makes extensive use of the static geometric analogy to simplify the analysis and allows for the possibility of a non-periodic displacement field. The problem of asymmetric twisting and bending of ring shell sectors associated with the non-periodic displacement state is formulated for the first time within the framework of shell theory.

1. INTRODUCTION

WE ARE concerned here with the class of linear elastostatic problems of shells of revolution involving stress and strain distributions whose dependence on the polar angle θ of the base plane is through factors $\sin\theta$ or $\cos\theta$. A typical loading which gives rise to such stress distributions is wind load. Restricting himself to the case where the associate displacement state also has this dependence on θ, Chernin [1] showed that the complete system of shell equations for this class of problems may be reduced to a fourth order system of two simultaneous ordinary differential equations for a stress function and a displacement function. Starting with Marguerre's equations, a corresponding result for shallow shells was obtained by Lardner and Simmonds [4] via a different approach. For the special case of spherical shells, the possibility of lowering the order of the system of shell equations from eight to four was first noted in E. Schwerin's dissertation [13]. For conical shells, the reduction appears to have been first carried out by Love [5]. It should be mentioned that Novozhilov [7] has reduced the problem to the solution of one second order differential equation for a complex stress function. But, as pointed out by Chernin, his reduction requires the omission of a number of terms from the original system of shell equations.

In this paper, we present an alternate reduction to a set of two simultaneous second order differential equations. The following characteristics of our reduction are noted:

(1) Motivated by and patterned after a recent version [11] of the classical Reissner–Meissner reduction for symmetric bending of shells of revolution [6, 8, 12], our choice of the two primary stress function and strain function variables naturally suggests itself, and the reduction is straightforward.

(2) Starting with a formulation of the shell problem in terms of equilibrium, constitutive and compatibility equations as in [11], the present reduction makes use of the static geometric duality, in a manner more extensive than Chernin's [1, 3] and including the

† Preparation of this paper was supported by the Office of Naval Research under a contract with the Massachusetts Institute of Technology.

duality of the stress strain relations. This extensive use of the duality reduces the complexity of the analysis substantially and in a nontrivial way (see Appendix) and attains a maximum degree of symmetry in the final results.

(3) Allowance is made for the possibility of a nonperiodic displacement state compatible with the sinusoidal strain measures. The inclusion of non-periodic displacements enables us to study a class of dislocations problems, which does not appear to have been considered previously within the scope of shell theory.

2. DIFFERENTIAL EQUATIONS AND STATIC GEOMETRIC DUALITY

With reference to cylindrical coordinates (r, θ, z), the middle surface of a shell of revolution may be described by the parametric equations $r = r(\xi)$ and $z = z(\xi)$. Differential equations governing the linear elastostatic behavior of an isotropic elastic shell of revolution can be found in [9–11] and elsewhere.

We consider in this work stress resultants (N, Q), couples (M), strain (ε), curvature changes (κ, λ) and external surface load components (p) (and edge load components) of the form (see Fig. 1)

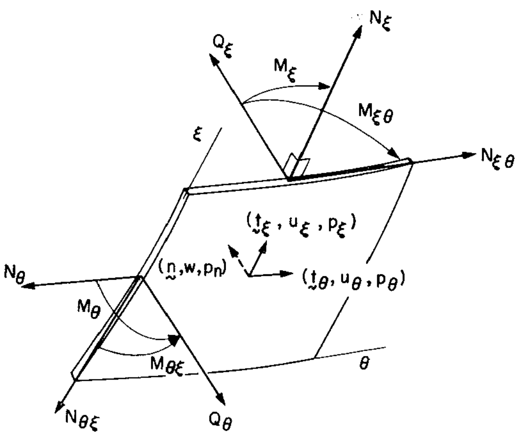

FIG. 1. An element of the shell.

$$(N_\xi, N_\theta, Q_\xi, M_\xi, M_\theta, \kappa_\theta, \kappa_\xi, \lambda_\theta, \varepsilon_\theta, \varepsilon_\xi, p_\xi, p_n) = (n_\xi, n_\theta, q_\xi, m_\xi, m_\theta, k_\theta, k_\xi, l_\theta, e_\theta, e_\xi, \hat{p}_\xi, \hat{p}_n)\cos\theta \tag{2.1}$$

$$(N_{\xi\theta}, N_{\theta\xi}, Q_\theta, M_{\xi\theta} = M_{\theta\xi}, \kappa_{\xi\theta}, \kappa_{\theta\xi}, \lambda_\xi, \varepsilon_{\xi\theta} = \varepsilon_{\theta\xi}, p_\theta) = (n_{\xi\theta}, n_{\theta\xi}, q_\theta, m, k_{\xi\theta}, k_{\theta\xi}, l_\xi, e, \hat{p}_\theta)\sin\theta.$$

We note that λ_ξ and λ_θ are the normal components of the curvature change vectors [11].

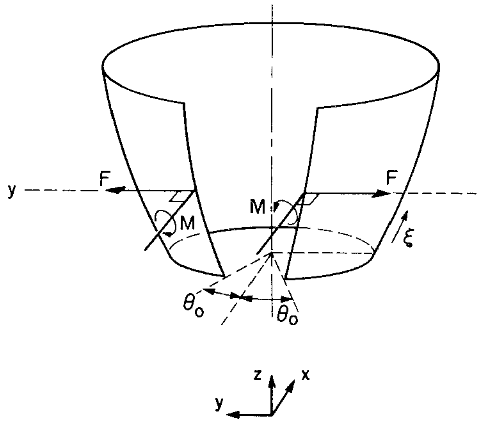

FIG. 2. Asymmetric bending and twisting of a ring sector of shell of revolution.

With (1), the system of shell equations then becomes a set of ordinary differential equations for the ξ-dependent quantities n_ξ, n_θ, etc. In particular, the equilibrium equations become [9]

$$\frac{(rn_\xi)'+\alpha n_{\theta\xi}-r'n_\theta}{r\alpha}+\frac{q_\xi}{R_\xi}+\hat{p}_\xi = 0, \qquad \frac{(rn_{\xi\theta})'-\alpha n_\theta+r'n_{\theta\xi}}{r\alpha}+\frac{q_\theta}{R_\theta}+\hat{p}_\theta = 0, \tag{2.2, 2.3}$$

$$\frac{(rq_\xi)'+\alpha q_\theta}{r\alpha}-\frac{\eta_\xi}{R_\xi}-\frac{n_\theta}{R_\theta}+\hat{p}_n = 0, \qquad \frac{(rm_\xi)'+\alpha m-r'm_\theta}{r\alpha}-q_\xi = 0, \tag{2.4, 2.5}$$

$$\frac{(rm)'-\alpha m_\theta+r'm}{r\alpha}-q_\theta = 0, \qquad n_{\xi\theta}-n_{\theta\xi}-\rho m = 0 \tag{2.6, 2.7}$$

where a prime indicates differentiation with respect to ξ and where

$$\alpha = \sqrt{[(r')^2+(z')^2]}, \qquad \frac{1}{R_\theta} = -\frac{z'}{r\alpha}, \qquad \frac{1}{R_\xi} = \frac{r''z'-r'z''}{\alpha^3}, \qquad \rho = \frac{1}{R_\theta}-\frac{1}{R_\xi} \tag{2.8}$$

and, for vanishing transverse shearing strains, the compatibility equations become

$$\frac{-(rk_\theta)'+\alpha k_{\xi\theta}-r'k_\xi}{r\alpha}+\frac{l_\theta}{R_\xi} = 0, \qquad \frac{(rk_{\theta\xi})'+\alpha k_\xi+r'k_{\xi\theta}}{r\alpha}-\frac{l_\xi}{R_\theta} = 0 \tag{2.9, 2.10}$$

$$\frac{(rl_\theta)'-\alpha l_\xi}{r\alpha}+\frac{k_\theta}{R_\xi}+\frac{k_\xi}{R_\theta} = 0, \qquad \frac{(re_\theta)'-\alpha e-r'e_\xi}{r\alpha}-l_\theta = 0 \tag{2.11, 2.12}$$

$$-\frac{(re)'+\alpha e_\xi+r'e}{r\alpha}+l_\xi = 0, \qquad k_{\theta\xi}-k_{\xi\theta}+\rho e = 0. \tag{2.13, 2.14}$$

We will take the stress–strain relations in the form

$$e_\xi = A(n_\xi-v_s n_\theta), \qquad e_\theta = A(n_\theta-v_s n_\xi), \qquad e = A_S(n_{\xi\theta}+n_{\theta\xi}) \tag{2.15}$$

$$m_\theta = D(k_\theta+v_b k_\xi), \qquad m_\xi = D(k_\xi+v_b k_\theta), \qquad m = D_S(k_{\xi\theta}+k_{\theta\xi}) \tag{2.16}$$

where $A_S = \frac{1}{2}(1+v_s)A$ and $D_S = \frac{1}{2}(1-v_b)D$. For a homogeneous shell, we have further

$$v_s = v_b = v, \qquad A = \frac{1}{Eh}, \qquad D = \frac{Eh^3}{12(1-v^2)} \tag{2.17}$$

where E, v and h are Young's modulus, Poisson's ratio and the shell thickness, respectively.

We may use (13) and (12) to eliminate l_ξ and l_θ from (9)–(11) to obtain the better known form of the compatibility equations for the conventional theory of shells [10]. We prefer the present form of six compatibility equations because it displays most simply the static geometric duality with the equilibrium equations as indicated in the following table:

n_ξ	n_θ	$n_{\xi\theta}$	$n_{\theta\xi}$	q_ξ	q_θ	m_ξ	m_θ	m
$-k_\theta$	$-k_\xi$	$k_{\theta\xi}$	$k_{\xi\theta}$	l_θ	$-l_\xi$	e_θ	e_ξ	$-e$

We note that without the explicit appearance of l_ξ and l_θ, there would have been no duals for the transverse shear stress resultants. The same duality also holds between the stress strain relations (15) and (16) if we take A and v_s to be the dual of $-D$ and $-v_b$, respectively. This concept of static geometric duality will be used extensively in the subsequent development.

The equilibrium equations (2)–(7), the compatibility equations (9)–(14) and the stress strain relations (15) and (16) are a system of eighteen equations for the eighteen unknowns on the right hand side of (1). Together with appropriately prescribed boundary conditions, they determine all measures of stress and strain. Having the strain measures, we may then, if we wish, compute the displacement components by considering the strain displacement relations as a system of first order differential equations.

3. FIRST INTEGRALS

The system of equilibrium equations (2.2)–(2.7) has two first integrals [1, 13]. One of these is obtained by forming the combination $rr'(2.2) - r\alpha\,(2.3) - rz'(2.4)$. This results in

$$\frac{rr'}{\alpha} n_\xi - \frac{rz'}{\alpha} q_\xi - r n_{\xi\theta} = \frac{P_x(\xi)}{\pi} \tag{3.1}$$

with

$$P_x(\xi) = \pi \int^\xi (z'\hat{p}_n - r'\hat{p}_\xi + \alpha\hat{p}_\theta) r \,\mathrm{d}\xi. \tag{3.2}$$

The other follows by considering the combination $r\alpha(2.5 - rr'(2.6) - r^2 r'(2.4) - r^2 z'(2.2)$ and by using equations (2.7) and (1) to get

$$\frac{r}{\alpha}(\alpha m_\xi - r'm - rr'q_\xi - rz'n_\xi) = -\frac{zP_x}{\pi} + \frac{T_y}{\pi} \tag{3.3}$$

with

$$T_y(\xi) = \pi \int^\xi [(rr' + zz')\hat{p}_n + (rz' - zr')\hat{p}_\xi + \alpha z\hat{p}_\theta] r \,\mathrm{d}\xi. \tag{3.4}$$

Overall equilibrium considerations show the quantities P_x and T_y to have the meaning of a resultant side force in the x-direction and a resultant tilting moment turning about the y-axis.

By the static geometric duality, we now know without separate derivations two first integrals of the compatibility equations in the form

$$-\left(\frac{rr'}{\alpha}k_\theta+rk_{\theta\xi}+\frac{rz'}{\alpha}l_\theta\right) = \frac{\Omega_x}{\pi} \tag{3.5}$$

$$\frac{r}{\alpha}(\alpha e_\theta+r'e-rr'l_\theta+rz'k_\theta) = -\frac{z\Omega_x}{\pi}+\frac{U_y}{\pi}. \tag{3.6}$$

In these, the quantities Ω_x and U_y are constants of integration. If the shell is slit along a meridian, these quantities will be shown in section (6) to have the meaning of relative edge rotation and displacement, i.e. dislocations of the Volterra type for shells [2].

With the four first integrals (1), (3), (5) and (6), the system of shell equations is effectively reduced from an eighth order to a fourth order system. Our next objective will be to reduce this system to two coupled second order equations of a form which is very similar to that of the Reissner–Meissner system [8, 11] for symmetric bending of shells of revolution.

4. REDUCTION TO TWO SIMULTANEOUS EQUATIONS

The reduction for the symmetric bending problem suggests that our final two differential equations will probably come from the moment equilibrium equation (2.5) and the dual compatibility equation (2.12). In deciding on two suitable primary dependent variables (a stress function and a displacement or strain function) we look at the terms entering into the moment equilibrium equation (2.5). Among these, the symmetric bending reduction suggests that we probably should not use m_ξ, m_θ and q_ξ, and therefore should not use the dual quantities e_θ, e_ξ and l_θ either, if we want to preserve the static geometric duality. This leaves us (tentatively) with m and the dual quantity e.

To express the remaining stress resultants and couples in terms of m and e, we first use the (sixth) equilibrium equation (2.7) and the last of the stress strain relations (2.15) to get

$$n_{\xi\theta} = \frac{1}{2}\left(\frac{e}{A_S}+\rho m\right), \qquad n_{\theta\xi} = \frac{1}{2}\left(\frac{e}{A_S}-\rho m\right) \tag{4.1}$$

where ρ has been defined in (1.8). The dual considerations give us

$$k_{\theta\xi} = \frac{1}{2}\left(\frac{m}{D_S}-\rho e\right), \qquad k_{\xi\theta} = \frac{1}{2}\left(\frac{m}{D_S}+\rho e\right). \tag{4.2}$$

To get m_ξ, m_θ and q_ξ [which enter into the moment equilibrium equation (2.5)] and e_θ, e_ξ and l_θ (which enter into the dual compatibility equation) in terms of m and e we observe that

(1) with n_ξ expressed in terms of e_θ and e_ξ by way of the stress–strain relations, the first integrals (3.1) and (3.3) may be written as

$$z'q_\xi-\frac{r'}{A(1-v_s^2)}(e_\xi+v_s e_\theta) = -\frac{\alpha}{2}\left(\frac{e}{A_S}+\rho m\right)-\frac{\alpha}{r}\frac{P_x}{\pi} \tag{4.3}$$

and

$$-r'q_\xi+\frac{\alpha}{r}m_\xi-\frac{z'}{A(1-v_s^2)}(e_\xi+v_s e_\theta) = \frac{r'}{r}m+\frac{\alpha}{r^2}\frac{T_y}{\pi}. \tag{4.4}$$

(2) Elimination of q_θ from (2.3) and (2.6), the use of the expression for n_θ in terms of e_θ and e_ξ along with the use of (1) give us a third equation for the quantities of interest in the form

$$\frac{\alpha}{R_\theta}m_\theta+\frac{\alpha}{A(1-v_s^2)}(e_\theta+v_s e_\xi) = \left[\frac{r}{2}\left(\frac{e}{A_S}+\rho m\right)\right]'+\frac{r'}{2}\left(\frac{e}{A_S}-\rho m\right) + \frac{(rm)'}{R_\theta}+\frac{r'm}{R_\theta}+r\alpha\hat{p}_\theta. \tag{4.5}$$

Equations (3)–(5) together with their duals may be thought of as six equations for the six unknowns, m_ξ, m_θ, q_ξ and their duals, in terms of m and e.

Equation (5) and its dual suggest that to avoid the unnecessary appearance of z''' and r''' (associated with ρ'), we should take as our stress and strain function

$$\psi = \frac{r}{2}\left(\frac{e}{A_S}+\rho m\right) = rn_{\xi\theta}, \qquad \phi = \frac{r}{2}\left(\frac{m}{D_S}-\rho e\right) = rk_{\theta\xi} \tag{4.6}$$

instead of m and e. Note that in view of (3.1), we also have $\psi = r[(r'/\alpha)n_\xi-(z'/\alpha)q_\xi-P_x/\pi r]$. Therefore, *our final choice of stress and strain functions is formally the same as that of the theory of symmetric bending* (except for the load terms) as formulated in [8, 11]†.

Upon solving for m and e from (6), we get

$$e = \frac{2A_S}{r(1+\varepsilon_0^2)}(\psi-\rho D_S\phi), \qquad m = \frac{2D_S}{r(1+\varepsilon_0^2)}(\phi+\rho A_S\psi) \tag{4.7}$$

with

$$\varepsilon_0^2 = D_S A_S \rho^2. \tag{4.8}$$

With (7), the solution of the system of equations (3)–(5) and their duals is

$$\begin{bmatrix} q_\xi \\ m_\xi \\ m_\theta \\ \hline l_\theta \\ e_\theta \\ e_\xi \end{bmatrix} = \left[\begin{array}{ccc|ccc} (1+\varepsilon_3^2)\dfrac{z'}{\alpha^2} & -\dfrac{r'}{\alpha^2} & 0 & \dfrac{v_b D r'^2}{r\alpha^3} & \dfrac{v_b D r'z'}{r\alpha^3} & -\dfrac{Dr'}{r\alpha} \\ -\dfrac{\varepsilon_4^2 r' R_\theta}{\alpha^2} & \dfrac{\varepsilon_4^2 r}{\alpha} & 0 & \dfrac{v_b D r'}{\alpha^2} & \dfrac{v_b D z'}{\alpha^2} & -D \\ 0 & 0 & \varepsilon_4^2 R_\theta & \dfrac{Dr'}{\alpha^2} & \dfrac{Dz'}{\alpha^2} & -v_b D \\ \hline \dfrac{v_s A r'^2}{r\alpha^3} & \dfrac{v_s A r'z'}{r\alpha^3} & \dfrac{Ar'}{r\alpha} & (1+\varepsilon_3^2)\dfrac{z'}{\alpha^2} & -\dfrac{r'}{\alpha^2} & 0 \\ \dfrac{v_s A r'}{\alpha^2} & \dfrac{v_s A z'}{\alpha^2} & A & -\dfrac{\varepsilon_4^2 r' R_\theta}{\alpha^2} & \dfrac{\varepsilon_4^2 r}{\alpha} & 0 \\ -\dfrac{Ar'}{\alpha^2} & -\dfrac{Az'}{\alpha^2} & -v_s A & 0 & 0 & \varepsilon_4^2 R_\theta \end{array}\right]$$

† Of course, we could have chosen $rn_{\xi\theta}$ and $rk_{\theta\xi}$ as our primary dependent variables from the very beginning. But indicating our process of arriving at them is intended to help to dispel any aura of mystery. Note that the normal is positive outward in [11] but positive inward here. This accounts for the difference in sign in front of q_ξ.

$$\times\left[\begin{array}{c} -\dfrac{\alpha}{r}\left(\dfrac{P_x}{\pi}+\psi\right) \\[2ex] \dfrac{2D_S r'(\phi+A_S\rho\psi)}{r^2(1+\varepsilon_0^2)}+\dfrac{\alpha}{r^2}\left(\dfrac{T_y}{\pi}-\dfrac{zP_x}{\pi}\right) \\[2ex] \dfrac{1}{\alpha}\left\{\left[\dfrac{1+\varepsilon_1^2}{1+\varepsilon_0^2}\psi\right]'+\dfrac{1+\varepsilon_2^2}{1+\varepsilon_0^2}\dfrac{r'}{r}\psi+\left[\dfrac{2D_S\phi}{R_\theta(1+\varepsilon_0^2)}\right]'+\dfrac{2D_S r'\phi}{rR_\theta(1+\varepsilon_0^2)}+r\alpha\hat{p}_\theta\right\} \\ \hdashline -\dfrac{\alpha}{r}\left(\dfrac{\Omega_x}{\pi}+\phi\right) \\[2ex] -\dfrac{2A_S r'(\psi-D_S\rho\phi)}{r^2(1+\varepsilon_0^2)}+\dfrac{\alpha}{r^2}\left(\dfrac{U_y}{\pi}-\dfrac{z\Omega_x}{\pi}\right) \\[2ex] \dfrac{1}{\alpha}\left\{\left[\dfrac{1+\varepsilon_1^2}{1+\varepsilon_0^2}\phi\right]'+\dfrac{1+\varepsilon_2^2}{1+\varepsilon_0^2}\dfrac{r'}{r}\phi-\left[\dfrac{2A_S\psi}{R_\theta(1+\varepsilon_0^2)}\right]'+\dfrac{2A_S r'\psi}{rR_\theta(1+\varepsilon_0^2)}\right\} \end{array}\right] \tag{4.9}$$

where

$$\varepsilon_1^2 = D_S A_S\rho\left(\frac{3}{R_\theta}-\frac{1}{R_\xi}\right), \qquad \varepsilon_2^2 = D_S A_S\left(\frac{2}{R_\theta^2}-\frac{1}{R_\xi^2}\right)$$
$$\varepsilon_3^2 = \frac{4D_S A_S}{r^2}, \qquad \varepsilon_4^2 = \frac{4D_S A_S}{R_\theta^2}. \tag{4.10}$$

For simplicity's sake, we have restricted ourselves to shells for which $\nu_s = \nu_b$ in getting (9), but will continue to distinguish the two for the ease of application of static geometric duality.

At first sight, it appears as if we would have to invert a 6×6 matrix to get (9). But by the use of the static geometric duality, we actually need only to invert two 3×3 matrices (see Appendix).

We may now substitute the expressions for $m_\xi, m_\theta, \ldots$ and l_θ given by (9) as well as the expressions for m and e from (7) into the moment equilibrium equation (2.5) and into the dual compatibility equation (2.12) to get two simultaneous second order differential equations for ψ and ϕ. We note that the resulting two equations are the exact consequences of the original shell equations and that, except for terms involving $\hat{p}_\theta$, explicitly, they are the static geometric duals of each other, with ϕ, Ω_x and U_y being the duals of ψ, P_x and T_y, respectively.

Having e_ξ and e_θ from (9), we get n_ξ and n_θ from the inverted stress strain relations obtained from the first two relations of (2.15). From (1), (6) and (7), we have further

$$n_{\xi\theta} = \frac{\psi}{r}, \qquad n_{\theta\xi} = \frac{1-\varepsilon_0^2}{1+\varepsilon_0^2}\frac{\psi}{r}-\frac{2D_S\rho}{1+\varepsilon_0^2}\frac{\phi}{r}. \tag{4.11, 4.12}$$

Finally, q_θ may be obtained from the moment equilibrium equation (2.6).

5. AN ACCURATE SET OF SIMPLIFIED EQUATIONS

Upon carrying out the substitution of (4.9) and (4.7) into (2.5) and (2.12), we obtain two simultaneous equations which are exact consequences of the original shell equations. These exact results are rather complicated. Fortunately, the bulk of the complexity is due to terms which are $O(DA/R^2)$ or $O(DA/r_0^2)$ where R and r_0 are representative magnitudes of the radii or curvature and of the radial dimension of the shell, respectively. We follow Chernin and consider henceforth only shells for which $O(DA/R^2)$ and $O(DA/r_0^2)$ are negligible in comparison with unity†. The two exact differential equations for ϕ and ψ can then be simplified to read

$$\phi''+\frac{(Dr/\alpha)'}{Dr/\alpha}\phi'-\left\{4\left(\frac{r'}{r}\right)^2-\frac{[(1+\nu_b)Dr'/\alpha]'}{Dr/\alpha}+(1-\nu_b)\left(\frac{z'}{r}\right)^2\right\}\phi-\frac{z'}{Dr/\alpha}\psi$$
$$=\frac{P_x}{\pi}\frac{(z/r)'}{D/\alpha}+\frac{T_y}{\pi}\frac{r'}{Dr^2/\alpha}-\frac{1}{Dr/\alpha}\left\{\frac{\nu_b D}{r\alpha}\left[\frac{\Omega_x}{\pi}(rr'+zz')-\frac{U_y}{\pi}z'\right]\right\}' \tag{5.1}$$
$$+\frac{(1+\nu_b)r'}{r^3}\left\{\frac{\Omega_x}{\pi}(rr'+zz')-\frac{U_y}{\pi}z'\right\},$$

$$\psi''+\frac{(Ar/\alpha)'}{Ar/\alpha}\psi'-\left\{4\left(\frac{r'}{r}\right)^2-\frac{[(1-\nu_s)Ar'/\alpha]'}{Ar/\alpha}+(1+\nu_s)\left(\frac{z'}{r}\right)^2\right\}\psi+\frac{z'}{Ar/\alpha}\phi$$
$$=-\frac{\Omega_x}{\pi}\frac{(z/r)'}{A/\alpha}-\frac{U_y}{\pi}\frac{r'}{Ar^2/\alpha}+\frac{1}{Ar/\alpha}\left\{\frac{\nu_s A}{r\alpha}\left[\frac{P_x}{\pi}(rr'+zz')-\frac{T_y}{\pi}z'\right]\right\}' \tag{5.2}$$
$$+\frac{(1-\nu_s)r'}{r^3}\left\{\frac{P_x}{\pi}(rr'+zz')-\frac{T_y}{\pi}z'\right\}-\frac{(Ar^2\hat{p}_\theta)'}{Ar/\alpha}+(1-\nu_s)r'\alpha\hat{p}_\theta.$$

We note the remarkable fact that the left hand sides of (1) and (2) differ only in the coefficient of the third term from those of the symmetric bending equations [11]‡.

Corresponding simplifications in the expressions for stress resultants and couples give us

$$n_{\xi\theta}=\frac{\psi}{r},\qquad n_{\theta\xi}=\frac{\psi}{r}-\underline{D(1-\nu_b)\left(\frac{1}{R_\theta}-\frac{1}{R_\xi}\right)\frac{\phi}{r}}$$

$$n_\xi=\frac{r'\psi}{r\alpha}+\frac{P_x}{\pi}\frac{rr'+zz'}{r^2\alpha}-\frac{T_y}{\pi}\frac{z'}{r^2\alpha}+\underline{\frac{D}{R_\theta\alpha}\left\{\phi'+\frac{2r'}{r}\phi+\frac{\nu_b\Omega_x}{\pi}\frac{rr'+zz'}{r^2}-\frac{\nu_b U_y}{\pi}\frac{z'}{r^2}\right\}}$$

$$n_\theta=\frac{\psi'}{\alpha}+\frac{r'\psi}{r\alpha}+r\hat{p}_\theta+\frac{D}{R_\theta\alpha}\left\{\phi'+\left[\frac{2r'}{r}+\frac{2(D_S/R_\theta)'}{D/R_\theta}\right]\phi+\frac{\Omega_x}{\pi}\frac{rr'+zz'}{r^2}-\frac{U_y}{\pi}\frac{z'}{r^2}\right\}$$

$$m=D(1-\nu_b)\left\{\frac{\phi}{r}+\underline{A_S\left(\frac{1}{R_\theta}-\frac{1}{R_\xi}\right)\frac{\psi}{r}}\right\} \tag{5.3}$$

$$m_\xi=-\frac{D}{\alpha}\left\{\phi'+(1+\nu_b)\frac{r'}{r}\phi+\frac{\nu_b\Omega_x}{\pi}\frac{rr'+zz'}{r^2}-\frac{\nu_b U_y}{\pi}\frac{z'}{r^2}\right\}$$
$$+\frac{2DA_S}{R_\theta\alpha}\left\{\psi'+\left[\frac{2r'}{r}+\frac{(A_S/R_\theta)'}{A_S/R_\theta}\right]\psi+\frac{(1-\nu_b)}{r^2}\left[\frac{P_x}{\pi}(rr'+zz')-\frac{T_y}{\pi}z'\right]\right\}$$

† Note that $R = r_0$ for cylindrical shells and that $DA = 0(h^2)$ for homogeneous and isotropic shells.

‡ For instance, the homogeneous counterpart of (1) for symmetric bending is

$$\phi''+\frac{(Dr/\alpha)'}{Dr/\alpha}\phi'-\left\{\left(\frac{r'}{r}\right)^2-\frac{(\nu_b Dr'/\alpha)'}{Dr/\alpha}\right\}\phi-\frac{z'}{Dr/\alpha}\psi=0.$$

$$m_\theta = -\frac{D}{\alpha}\left\{\nu_b\phi' + (1+\nu_b)\frac{r'}{r}\phi + \frac{\Omega_x}{\pi}\frac{rr'+zz'}{r^2} - \frac{U_y}{\pi}\frac{z'}{r^2}\right\}$$
$$+\frac{2DA_S}{R_\theta\alpha}\left\{\psi' + \left[\frac{2r'}{r} + \frac{\nu_b(A_S/R_\theta)'}{A_S/R_\theta}\right]\psi - 2(1-\nu_b)z'\hat{p}_\theta\right\}$$
$$q_\xi = -\frac{1}{\alpha}\left\{\frac{z'}{r}\psi + \frac{P_x}{\pi}\left(\frac{z}{r}\right)' + \frac{T_y}{\pi}\frac{r'}{r^2}\right\} - \frac{Dr'}{r\alpha^2}\left\{\phi' + \frac{2r'}{r}\phi + \frac{\nu_b\Omega_x}{\pi}\frac{rr'+zz'}{r^2} - \frac{\nu_b U_y}{\pi}\frac{z'}{r^2}\right\}$$
$$q_\theta = \frac{D}{r\alpha}\left\{\phi' + 2\left(\frac{r'}{r} + \frac{D'}{D}\right)\phi + \frac{\Omega_x}{\pi}\frac{rr'+zz'}{r^2} - \frac{U_y}{\pi}\frac{z'}{r^2} + \underline{O\left(\frac{A_S\psi}{R_\theta}\right)}\right\}$$

and these expressions for stress measures and strain measures satisfy all equilibrium, compatibility and constitutive equations up to terms of the order DA/R^2 and DA/r_0^2.

For the case of homogeneous shells with periodic displacement fields, equations (1)–(3) constitute a simplified theory which is essentially equivalent to that of Chernin [1]. For instance, the counterpart of equation (1) as obtained by Chernin (with negligible terms deleted) is

$$\frac{1}{R_\varphi^2}\frac{d^2\Phi}{d\varphi^2} + \frac{1}{rR_\varphi}\left[\cos\varphi - \frac{r}{R_\varphi^2}\frac{dR_\varphi}{d\varphi}\right]\frac{d\Phi}{d\varphi}$$
$$-\frac{1}{rR_\varphi}\left[(1+\nu)\sin\varphi + 2(1-\nu)\frac{R_\varphi}{r} + 2(1+\nu)\cos^2\varphi\frac{R_\varphi}{r}\right]\Phi - \frac{\sin\varphi}{Dr}\Psi = F(\varphi). \tag{5.4}$$

The displacement function Φ is related to the middle surface meridional and normal displacement components $u_\varphi = U_\varphi(\varphi)\cos\theta$ and $w = W(\varphi)\cos\theta$ by the relation

$$\Phi = -\frac{1}{R_\varphi}\left(\frac{dW}{d\varphi} - U_\varphi\right) + \frac{1}{r}(W\cos\varphi - U_\varphi\sin\varphi). \tag{5.5}$$

The stress function Ψ is the static geometric dual of Φ†. It can be verified that, except for load terms, $\Phi = -rk_{\theta\xi} + z'e/\alpha$ and that for a homogeneous shell (4) is the same (up to terms of the order h^2/R^2 and h^2/r_0^2) as (1).

Chernin's use of the azimuthal angle φ enclosed by the midsurface normal and the axis of revolution makes necessary additional transformations of the results whenever $1/R_\varphi = 0$. While these transformations are not difficult in general, it seems more satisfying to have a set of equations which are directly usable for all shells of revolution including the limiting case of a circular cylindrical shell and the limiting case of a flat plate.

An additional simplification of (3) not considered by Chernin consists of omitting all underlined terms in the expressions for the stress resultants n_ξ, $n_{\theta\xi}$ and q_θ and the twisting couple m. That this simplification does not affect the accuracy of our results is evident from the following consideration.

Suppose that the underlined terms contributed significantly to either n_ξ or $n_{\theta\xi}$. In this case, we would have

$$\psi = O\left(\frac{D\phi}{\mu R}\right) \tag{5.6}$$

where μ is a dimensionless scale length at most of order unity and max $(\sqrt{(DA)}/R,$

† How Chernin arrived at this ingenious choice of primary dependent variables is not clear to this writer.

$\sqrt{(DA)/r_0} \ll \mu$. The contributions of ψ-terms (i.e. the underlined terms) to m and q_θ are then of order $DA/\mu R^2$ compared to the contribution of the ϕ-terms and are therefore negligible. Moreover, the relative magnitude of the direct and bending stresses in this case is

$$\frac{\sigma_D}{\sigma_B} = O\left(\frac{n/h}{6m/h^2}\right) = O\left(\frac{h}{R}\right). \tag{5.7}$$

Because of this, the accuracy of the expressions for n_ξ and $n_{\theta\xi}$ for the present class of problems is of no consequence and therefore, the underlined terms in these expressions may be omitted.

The argument for omitting the underlined terms in cases where the ψ-terms contributes significantly to m and q_θ proceeds similarly. Together, they suggest that all underlined terms in (3) may be omitted without affecting the accuracy of the final results. While the same arguments suggest that we may also simplify the expressions for n_θ, m_ξ and m_θ, the fact that these quantities are also involved in the accuracy of the expressions for resultant force and moment (see Section 7) prevents us from making these simplifications. At the same time, the ϕ-terms in q_ξ definitely cannot be omitted since the magnitude of z may be small throughout the shell (z vanishes identically for the limiting case of a flat plate).

6. A CLASS OF NON-PERIODIC DISPLACEMENT FIELDS

With a view toward an interpretation of the constants Ω_x and U_y, we consider non-periodic displacement states of the form

$$\begin{aligned}(u_\xi, W, \phi_\xi) &= (U_\xi, W, \Phi_\xi)\cos\theta + (\hat{U}_\xi, \hat{W}, \hat{\Phi}_\xi)\theta\sin\theta \\ (u_\theta, \phi_\theta, \omega) &= (U_\theta, \Phi_\theta, \Omega)\sin\theta + (\hat{U}_\theta, \hat{\Phi}_\theta, \hat{\Omega})\theta\cos\theta\end{aligned} \tag{6.1}$$

where u_ξ, u_θ and w are mid-surface translational displacement components in the meridional, circumferential and normal direction, respectively, and ϕ_θ, ϕ_ξ and ω are rotational displacement components turning about these same directions, respectively. The quantities inside the parentheses on the right hand side of (1) are functions of ξ only†. Upon substituting (1) into the strain displacement relations [9, 11]

$$\begin{aligned}
\varepsilon_\xi &= \frac{u_\xi'}{\alpha} + \frac{w}{R_\xi}, & \varepsilon_{\xi\theta} &= \frac{u_\theta'}{\alpha} - \omega, & \gamma_\xi &= \phi_\xi + \frac{w'}{\alpha} - \frac{u_\xi}{R_\xi} \\
\varepsilon_\theta &= \frac{\dot{u}_\theta}{r} + \frac{r'u_\xi}{r\alpha} + \frac{w}{R_\theta}, & \varepsilon_{\theta\xi} &= \frac{\dot{u}_\xi}{r} - \frac{r'u_\theta}{r\alpha} + \omega, & \gamma_\theta &= \phi_\theta + \frac{\dot{w}}{r} - \frac{u_\theta}{R_\theta} \\
\kappa_\xi &= \frac{\phi_\xi'}{\alpha}, & \kappa_{\xi\theta} &= \frac{\phi_\theta'}{\alpha} - \frac{\omega}{R_\xi}, & \lambda_\xi &= \frac{\omega'}{\alpha} + \frac{\phi_0}{R_\xi} \\
\kappa_\theta &= \frac{\dot{\phi}_\theta}{r} + \frac{r'\phi_\xi}{r\alpha}, & \kappa_{\theta\xi} &= \frac{\dot{\phi}_\xi}{r} - \frac{r'\phi_\xi}{r\alpha} + \frac{\omega}{R_\theta}, & \lambda_\theta &= \frac{\dot{\omega}}{r} - \frac{\phi_\xi}{R_\theta}
\end{aligned} \tag{6.2}$$

(where a dot indicates differentiation with respect to the polar angle θ)

† The author is indebted to E. Reissner for the suggestion to consider the possible use of displacement fields of this form in the theory of shells of revolution.

and requiring that the strain measures vary sinusoidally as given by (2.1), we find that possible displacement fields of the form (1) are

$$(u_\xi, w, \phi_\xi) = (U_\xi, W, \Phi_\xi)\cos\theta + \left(\frac{r'}{\alpha}u_0 + \frac{zr'-rz'}{\alpha}\phi_0,\ -\frac{z'}{\alpha}u_0 - \frac{rr'+zz'}{\alpha}\phi_0,\ \phi_0\right)\theta\sin\theta$$
$$(u_\theta, \phi_\theta, \omega) = (U_\theta, \Phi_\theta, \Omega)\sin\theta + \left(u_0 + z\phi_0,\ \frac{r'}{\alpha}\phi_0,\ \frac{z'}{\alpha}\phi_0\right)\ \theta\cos\theta \tag{6.3}$$

where u_0 and ϕ_0 are arbitrary constants.

The corresponding strain measures are given by

$$\varepsilon_\xi = \left(\frac{U'_\xi}{\alpha} + \frac{W}{R_\xi}\right)\cos\theta, \qquad \varepsilon_\theta = \frac{1}{r}\left(U_\theta + \frac{r'}{\alpha}U_\xi - \frac{z'}{\alpha}W + z\phi_0 + u_0\right)\cos\theta$$

$$\varepsilon_{\xi\theta} = \left(\frac{U'_\theta}{\alpha} - \Omega\right)\sin\theta, \qquad \varepsilon_{\theta\xi} = -\frac{1}{r}\left(U_\xi + \frac{r'}{\alpha}U_\theta - r\Omega - \frac{r'}{\alpha}u_0 - \frac{zr'-rz'}{\alpha}\phi_0\right)\sin\theta$$

$$\gamma_\xi = \left(\Phi_\xi + \frac{W'}{\alpha} - \frac{U_\xi}{R_\xi}\right)\cos\theta, \qquad \gamma_\theta = \left(\Phi_\theta - \frac{W}{r} - \frac{U_\theta}{R_\theta} - \frac{z'}{r\alpha}u_0 - \frac{rr'+zz'}{r\alpha}\phi_0\right)\sin\theta$$

$$\kappa_\xi = \frac{\Phi'_\xi}{\alpha}\cos\theta, \qquad \kappa_\theta = \frac{1}{r}\left(\Phi_\theta + \frac{r'}{\alpha}\Phi_\xi + \frac{r'}{\alpha}\phi_0\right)\cos\theta, \qquad \lambda_\theta = \left(\frac{\Omega}{r} - \frac{\Phi_\xi + \phi_0}{R_\theta}\right)\cos\theta \tag{6.4}$$

$$\kappa_{\xi\theta} = \left(\frac{\Phi'_\theta}{\alpha} - \frac{\Omega}{R_\xi}\right)\sin\theta, \qquad \kappa_{\theta\xi} = -\frac{1}{r}\left(\Phi_\xi + \frac{r'}{\alpha}\Phi_\theta + \frac{z'}{\alpha}\Omega - \phi_0\right)\sin\theta, \qquad \lambda_\xi = \left(\frac{\Omega'}{\alpha} + \frac{\Phi_\theta}{R_\xi}\right)\sin\theta.$$

From these follow the relations

$$r'\mathrm{k}_\theta + \alpha\mathrm{k}_{\theta\xi} + z'l_\theta = \frac{2\alpha}{r}\phi_0 \tag{6.5}$$

and

$$z'\mathrm{k}_\theta - r'l_\theta + \frac{\alpha}{r}e_\theta + \frac{r'}{r}e = \frac{2\alpha}{r^2}(u_0 + z\phi_0). \tag{6.6}$$

A comparison of (5) and (6) with (3.5) and (3.6) leads to the identifications

$$\phi_0 = -\frac{\Omega_x}{2\pi}, \qquad u_0 = \frac{U_y}{2\pi}. \tag{6.7}$$

Inasmuch as the above non-periodic displacement state is not possible if the shell is closed in the circumferential direction, the constants Ω_x and U_y must be equal to zero for such a shell.

On the other hand, the portion of the displacement field associated with u_0 and ϕ_0 in (3) can be written in the form

$$\begin{aligned}\boldsymbol{\phi}_0 &\equiv \phi_0\theta\left\{-\frac{r'}{\alpha}\cos\theta\,\mathbf{t}_\xi+\sin\theta\,\mathbf{t}_\theta+\frac{z'}{\alpha}\cos\theta\,\mathbf{n}\right\} = -\phi_0\theta\mathbf{i}_x\\ \mathbf{u}_0 &\equiv \theta\left\{\frac{r'u_0+(zr'-rz')\phi_0}{\alpha}\sin\theta\,\mathbf{t}_\xi+(u_0+z\phi_0)\cos\theta\,\mathbf{t}_\theta-\frac{z'u_0+(rr'+zz')\phi_0}{\alpha}\sin\theta\,\mathbf{n}\right\}\\ &= u_0\theta\mathbf{i}_y\end{aligned} \tag{6.8}$$

where, in terms of the position vector $\mathbf{r}$ of a point on the middle surface, $\mathbf{t}_\xi = \mathbf{r}'/\alpha$, $\mathbf{t}_\theta = \mathbf{r}^{\cdot}/r$ and $\mathbf{n} = \mathbf{t}_\xi\times\mathbf{t}_\theta$. With

$$[\boldsymbol{\phi}_0]_0^{2\pi} = -2\pi\phi_0\mathbf{i}_x = \Omega_x\mathbf{i}_x, \qquad [\mathbf{u}_0]_0^{2\pi} = 2\pi u_0\mathbf{i}_y = U_y\mathbf{i}_y \tag{6.9}$$

we see that Ω_x and U_y have the meaning of relative circumferential edge rotation and displacement, respectively, for a shell slit along a meridian.

7. ASYMMETRIC BENDING AND TWISTING OF RING SHELL SECTORS

We now formulate the boundary value problem involving non-periodic displacement fields of the type discussed in section (6). Consider a ring shell sector bounded by $\xi = \xi_i$, $\xi = \xi_0$ $(\xi_i < \xi_0)$ and $\theta = \pm\theta_0$ $(\theta_0 < \pi)$. The shell is free of surface loads and the two parallel circular edges $\xi = \xi_i$ and $\xi = \xi_0$ are free of edge tractions. At the meridional edges $\theta = \pm\theta_0$, the shell is subject to equal and opposite resultant forces acting in directions parallel to the base plane, and to equal and opposite resultant moments turning about axes parallel to the base plane. Of interest are the relative edge displacement and rotation as well as the stress distributions produced by these forces and moments.

The equations developed in the preceding sections are directly applicable for the analysis of the special case of this problem where the forces are in directions parallel to the y-axis and the moments are turning about axes parallel to the x-axis (see Fig. 2). Results for the case where the forces are in directions parallel to the x-axis (with equilibrating moments) and the moments are turning about axes parallel to the y-axis can be obtained from the results of the preceding case by interchanging the roles of $\sin\theta$ and $\cos\theta$. The solution for the general case can be obtained by an appropriate combination of the results of these two special cases.

For the case shown in Fig. 2, overall equilibrium requires that $P_x = T_y = 0$ since the shell is free of surface loads as well as edge loads along $\xi = \xi_i$ and $\xi = \xi_0$. The governing differential equations are therefore (5.1) and (5.2) with terms involving P_x, T_y and $\hat{p}_\theta$ omitted.

The conditions of no edge tractions along the parallel edges require the stress resultants and stress couples to satisfy the homogeneous Kirchhoff–Bassett contracted conditions which, in view of (2.1), take the form $n_\xi = n_{\xi\theta}+m/R_\theta = m_\xi = q_\xi+m/r = 0$ for $\xi = \xi_i$ and ξ_0. Of the four conditions at each edge, only two are independent conditions. This is evident once we rewrite the first integrals (3.1) and (3.3) (with $P_x = T_y = 0$) in the form

$$\frac{r'}{\alpha}n_\xi-\left(n_{\xi\theta}+\frac{m}{R_\theta}\right)-\frac{z'}{\alpha}\left(q_\xi+\frac{m}{r}\right) = 0, \quad \alpha m_\xi-rz'n_\xi-rr'\left(q_\xi+\frac{m}{r}\right) = 0. \tag{7.1}$$

The differential equations and the two independent boundary conditions

$$\xi = \xi_i, \xi_o: \qquad n_{\xi\theta} + \frac{m}{R_\theta} = m_\xi = 0 \tag{7.2}$$

[expressed in terms of ψ and ϕ by way of (5.3)] determine ϕ and ψ in terms of Ω_x and U_y. We write the solution in the form

$$\phi = \Omega_x \phi_\Omega + U_y \phi_U, \qquad \psi = \Omega_x \psi_\Omega + U_y \psi_U \tag{7.3}$$

where $\phi_\Omega, \ldots, \psi_U$ do not depend on Ω_x and U_y.

Along the edges $\theta = \pm\theta_0$, we will prescribe only that overall force and moment conditions be satisfied. In view of (2.1), the six force and moment conditions take the form

$$\int_{\xi_i}^{\xi_o} \left\{ \frac{r'}{\alpha}\left(n_{\theta\xi} + \frac{m}{R_\xi}\right) - \frac{z'}{\alpha}\left(q_\theta + \frac{m'}{\alpha}\right) - n_\theta \right\} \alpha \, d\xi + \left[\frac{2z'm}{\alpha}\right]_{\xi_i}^{\xi_o} = 0$$

----, ----, ----,

$$\begin{aligned}
&\int_{\xi_i}^{\xi_o} \left\{ \left[\frac{r'}{\alpha}\left(n_{\theta\xi} + \frac{m}{R_\xi}\right) - \frac{z'}{\alpha}\left(q_\theta + \frac{m'}{\alpha}\right)\right] \sin^2\theta + n_\theta \cos^2\theta \right\} \alpha \, d\xi + \left[\frac{2z'm}{\alpha}\sin^2\theta\right]_{\xi_i}^{\xi_o} = F \\
&\int_{\xi_i}^{\xi_o} \left\{ \left[\frac{rz' - zr'}{\alpha}\left(n_{\theta\xi} + \frac{m}{R_\xi}\right) + \frac{rr' + zz'}{\alpha}\left(q_\theta + \frac{m'}{\alpha}\right)\right] \sin^2\theta - \left[\frac{r'}{\alpha} m_\theta + z n_\theta\right] \cos^2\theta \right\} \alpha \, d\xi \\
&\qquad - \left[\frac{rr' + zz'}{\alpha}(2m)\sin^2\theta\right]_{\xi_i}^{\xi_o} = M.
\end{aligned} \tag{7.4}$$

Terms outside the integral signs in these conditions represent the corner forces introduced by the assumption of no transverse shear deformation.

Through the use of the equilibrium equations (3.2)–(3.7) (with $\hat{p}_\xi = \hat{p}_\theta = \hat{p}_n = 0$) the boundary conditions (2), and the differentiation formulas $(r'/\alpha)' = z'/R_\xi$ and $(z'/\alpha)' = -r'/R_\xi$, it can be shown that the first four homogeneous conditions in (4) are satisfied identically while the last two inhomogeneous conditions become

$$F = \int_{\xi_i}^{\xi_o} n_\theta \alpha \, d\xi, \qquad M = -\int_{\xi_i}^{\xi_o} (r' m_\theta + z\alpha n_\theta) \, d\xi \tag{7.5}$$

where F and M are the applied force and moment respectively. To illustrate the nature of the analysis, we note that integration by parts and the use of the equilibrium equation (2.3) transform the first condition of (4) to read

$$\begin{aligned}
&\int_{\xi_i}^{\xi_o} \left\{ \frac{r'}{\alpha}\left(n_{\theta\xi} + \frac{m}{R_\xi}\right) - \frac{z'}{\alpha}\left(q_\theta + \frac{m'}{\alpha}\right) - n_\theta \right\} \alpha \, d\xi + \left[\frac{2z'm}{\alpha}\right]_{\xi_i}^{\xi_o} \\
&= \int_{\xi_i}^{\xi_o} \left\{ r'\left(n_{\theta\xi} + \frac{m}{R_\xi}\right) + \left(\frac{z'}{\alpha}\right)' m - \alpha n_\theta - z' q_\theta \right\} d\xi + \left[\frac{z'm}{\alpha}\right]_{\xi_i}^{\xi_o} \\
&= \int_{\xi_i}^{\xi_o} \{r' n_{\theta\xi} - \alpha n_\theta - z' q_\theta\} \, d\xi + \left[\frac{z'm}{\alpha}\right]_{\xi_i}^{\xi_o} \\
&= -\int_{\xi_i}^{\xi_o} (r n_{\xi\theta})' \, d\xi + \left[\frac{z'm}{\alpha}\right]_{\xi_i}^{\xi_o} = -\left[r\left(n_{\xi\theta} + \frac{m}{R_\theta}\right)\right]_{\xi_i}^{\xi_o}
\end{aligned} \tag{7.6}$$

where the right hand side vanishes because of the first boundary condition of (2).

Upon expressing n_θ and m_θ in terms of ϕ and ψ, we have two linear stiffness relations of the form

$$F = U_y C_{FU} + \Omega_x C_{F\Omega}, \qquad M = U_y C_{MU} + \Omega_x C_{M\Omega} \tag{7.7}$$

where the stiffness coefficients C_{FU}, etc. do not depend on U_y and Ω_x. These stiffness relations determine the force F and moment M needed to produce the edge dislocations associated with Ω_x and U_y. Alternately, given F and M, they determine Ω_x and U_y and therefore the corresponding edge dislocations.

We indicated in Section (5) that the argument which allowed us to simplify the expression for n_ξ, $n_{\theta\xi}$, q_θ and m is also applicable to the expressions for n_θ, m_ξ and m_θ. In particular, it can be shown that when the ϕ-terms contribute significantly to n_θ in (5.3), the direct stress associated with n_θ is $O(h/R)$ compared to the bending stresses and the accuracy of n_θ in this case is therefore of no consequence insofar as the accurate determination of the stress level in the structure is concerned. However, we refrained at the time from omitting the ϕ-terms in n_θ and the ψ-terms in m_ξ and m_θ anticipating that the accuracy of n_θ, m_ξ and m_θ may affect the final solution of a given problem in some other ways. It is now evident from the first equation of (5) that the accuracy of n_θ is crucial to an accurate determination of the stiffness coefficients and therefore the final solution of the present problem. On the other hand, the underlined terms in m_θ and m_ξ do not contribute significantly to (5). However, the static geometric duality suggests that these terms will be significant for an accurate determination of k_ξ, which may be important in another class of problems to be discussed in a future publication.

REFERENCES

[1] V. S. CHERNIN, On the system of differential equations of equilibrium of shells of revolution under bending loads. *Prikl. Mat. Mech.* **23**, 258–265 (1959).

[2] K. F. CHERNYKH, Relation between dislocations and concentrated loadings in the theory of shells. *Prikl. Mat. Mech.* **23**, 249–257 (1959).

[3] A. L. GOLDENVEIZER, The equations of the theory of thin shells. *Prikl. Mat. Mech.* **4**, 32–42 (1940).

[4] T. J. LARDNER and J. G. SIMMONDS, On the lateral deformation of shallow shells of revolution. *Int. J. Solids Struct.* **1**, 337–384 (1965).

[5] A. E. H. LOVE, *The Mathematical Theory of Elasticity*, 4th edition, pp. 595–600. Dover (1944).

[6] E. MEISSNER, Das Elastizitätsproblem für dünne Schalen von Ringflächen-Kugel oder Kegelform. *Z. Phys.* **14**, 343–349 (1913).

[7] V. V. NOVOZHILOV, *Thin Shell Theory*, 2nd augmented and revised edition, pp. 80–88 and 364–367. Noordhoff (1964).

[8] E. REISSNER, On the theory of thin elastic shells. *Reissner Anniversary Volume*, pp. 230–247. Ann Arbor Michigan (1949).

[9] E. REISSNER, Variational considerations for elastic beams and shells. *J. Engng Mech. Div., Am. Soc. civ. Engrs* **88**, 23–57 (1962).

[10] E. REISSNER, A note on stress functions and compatibility equations in shell theory. *Topics in Applied Mechanics* (*Schwerin Memorial Volume*), pp. 23–32. Elsevier (1965).

[11] E. REISSNER and F. Y. M. WAN, Rotationally symmetric stress and strain in shells of revolution. *Studies Appl. Math.* **48**, 1–17 (1969).

[12] H. REISSNER, *Spannungen in Kugelschalen* (*Kuppeln*) pp. 181–193. Müller-Breslau Festschrift (1912).

[13] E. SCHWERIN, Über Spannungen in symmetrisch and unsymmetrisch belasteten Kugelschalen (Kuppeln) insbesondere bei Belastung durch Winddruck. *Armierter Beton* **12**, 25–37, 54–63, 81–88 (1919).

APPENDIX

To obtain the solution (4.9), it seems that one would have to invert a 6×6 matrix. We will show here that by using the static geometric duality properties of the elements of the relevant matrix, we only need to invert two 3×3 matrices.

Denoting the coefficient matrix of the relevant matrix equation by A and its inverse by B, we partition A and B each into four 3×3 submatrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{11}^* \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{11}^* \end{bmatrix} \tag{1}$$

Since

$$BA = \begin{bmatrix} B_{11}A_{11}+B_{12}A_{12}^* & B_{11}A_{12}+B_{12}A_{11}^* \\ B_{12}^*A_{11}+B_{11}^*A_{11}^* & B_{12}^*A_{12}+B_{11}^*A_{11}^* \end{bmatrix} = I_6 \tag{2}$$

where I_n is an $n \times n$ identity matrix, we must have

$$B_{11}A_{11}+B_{12}A_{12}^* = I_3, \qquad B_{11}A_{12}+B_{12}A_{11}^* = 0 \tag{3}$$

$$B_{11}^*A_{11}^*+B_{12}^*A_{12} = I_3, \qquad B_{11}^*A_{12}^*+B_{12}^*A_{11} = 0. \tag{4}$$

From the second relation in (3) and (4), we have further

$$B_{12} = -B_{11}A_{12}(A_{11}^*)^{-1}, \qquad B_{12}^* = -B_{11}^*A_{12}^*(A_{11})^{-1}. \tag{5}$$

Upon introducing (5) into the first relation in (3) and (4), we get

$$B_{11} = [A_{11}-A_{12}(A_{11}^*)^{-1}A_{12}^*]^{-1}, \quad B_{11}^* = [A_{11}^*-A_{12}^*(A_{11})^{-1}A_{12}]^{-1}. \tag{6}$$

Having (6), we get B_{12} and B_{12}^* from (5).

Now, for our particular matrix A, A_{11}^* and A_{12}^* are the static geometric duals of A_{11} and A_{12} respectively. From (5) and (6), B_{11}^* and B_{12}^* are evidently the duals of B_{11} and B_{12}, respectively, so that we need only to do the necessary calculations for B_{11} and B_{12}. These calculations involve only the inversion of the two 3×3 matrices A_{11}^* and $[A_{11}-A_{12}(A_{11}^*)^{-1}A_{12}^*]$.

(*Received* 24 *April* 1969; *revised* 1 *October* 1969)

Абстракт—Система уравнений линейной теории изотропных упругих оболочек вращения, для синусоидальных в полярном угле основной плоскости, с периодом 2п, напряжений и деформаций, сокращается к двум совместным обыкновенным дифференниальным уравнениям второго порядка. Эти уравнения похожи на соответствующие уравнения для симметрического изгиба. В сведении экстенсивно используется статическая геометрическая аналогия, с целью упрощения анализа и дается возможность получить непериодическое поле перемещений. Даются, во первые, в рамках теории оболочек, формулы для задачи осесимметрического кручения и изгиба секторов кольцевых оболочек, связанной с непериодическим деформационным состоянием.