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On the Equations of the Linear Theory of Elastic Conical Shells

By Frederic Y. M. Wan

1. Introduction

In a recent article [19], it was shown that the equations of the linear theory of elastic shells of revolution under arbitrary loads, after a harmonic analysis in the polar angle of the base plane of the shell, can be reduced to two simultaneous fourth order ordinary differential equations for a stress function and a displacement function. The exact reduction procedure of [19] was applied to a circular cylindrical shell. Also mentioned was the possibility of a simplified procedure based on the fact that there is an inherent error in shell theory of the order h/R, where h is the shell thickness and R is a representative magnitude of the two principal radii of curvature.

In the present paper, the exact and the simplified procedure of [19] will be applied to a conical shell frustrum. The simplified procedure leads to two differential equations as well as to auxiliary equations for the calculation of stress resultants and couples which differ from those of the exact procedure only in terms which are $O(h/R)$. The differential equations obtained here are very similar in form to those of shallow shell theory [6,8] as well as to those of the Donnell type theory [15]. They do in fact reduce to the shallow shell equations when the slope angle β between a meridian and the base plane is very small. On the other hand, for $\beta = \pi/2$, they reduce to the "correct" equations for circular cylindrical shells of [16,19] while the results of [15] reduce only to the Donnell equation.

Aside from their usefulness in obtaining the solution of specific problems, our results also delimit the range of applicability of the Donnell type equations for conical shells previously obtained in [5,15].

2. Differential equations

In cylindrical coordinates (r, θ, z), the middle surface of a conical shell may be described by the parametric equations

$$r = r_0 + c\xi, \qquad z = z_0 + s\xi, \tag{1}$$

where r_0, z_0, c and s are fixed geometrical parameters and, in terms of the slope angle β, $c = \cos\beta$ and $s = \sin\beta$. Correspondingly, we have

$$\frac{1}{R_\xi} = 0, \qquad \frac{1}{R_\theta} = -\frac{s}{r}, \tag{2}$$

where R_ξ and R_θ are the two principal radii of curvature.

Upon specializing the strain displacement relations as given in [19] to the case of a conical shell with no transverse shear deformations, we have, with $\alpha = 1$, the following expressions for the mid-surface strains $\varepsilon_\xi, \varepsilon_\theta, \varepsilon_{\xi\theta} = \varepsilon_{\theta\xi}$, and curvature changes $\kappa_\xi, \kappa_\theta, \kappa_{\xi\theta}, \kappa_{\theta\xi}$:

$$\varepsilon_\xi = u', \qquad \varepsilon_\theta = \frac{v^{\cdot} + cu - sw}{r}, \qquad \varepsilon_{\xi\theta} = \varepsilon_{\theta\xi} = \frac{rv' - cv + u^{\cdot}}{2r},$$

$$\kappa_\xi = -w'', \qquad \kappa_\theta = -\frac{rcw' + w^{\cdot\cdot} + sv^{\cdot}}{r^2}, \tag{3}$$

$$\kappa_{\xi\theta} = -\left(\frac{w^{\cdot} + sv}{r}\right)', \qquad \kappa_{\theta\xi} = -\frac{2rw'^{\cdot} - 2cw^{\cdot} - csv + srv' - su}{2r^2}.$$

In (3) primes and dots indicate differentiation with respect to ξ and θ, respectively, and u, v and w are the mid-surface displacement components in the meridional, circumferential and normal (positive inward) directions respectively.

Midsurface strains and curvature changes are related to stress resultants N and stress couples M, by a system of stress strain relations which we will take to be of the form :

$$\varepsilon_\xi = A(N_\xi - \nu_s N_\theta), \qquad \varepsilon_\theta = A(N_\theta - \nu_s N_\xi), \qquad \varepsilon_{\xi\theta} = \varepsilon_{\theta\xi} = A_S(N_{\theta\xi} + N_{\xi\theta}), \tag{4}$$

$$M_\theta = D(\kappa_\theta + \nu_b \kappa_\xi), \qquad M_\xi = D(\kappa_\xi + \nu_b \kappa_\theta), \qquad M_{\theta\xi} = M_{\xi\theta} = D_S(\kappa_{\theta\xi} + \kappa_{\xi\theta}), \tag{5}$$

with $A_S = \frac{1}{2}A(1 + \nu_s)$ and $D_S = \frac{1}{2}D(1 - \nu_b)$. For an isotropic and homogeneous medium, we have

$$D = \frac{Eh^3}{12(1 - \nu^2)}, \qquad A = \frac{1}{Eh}, \qquad \nu_s = \nu_b = \nu, \tag{6}$$

where E is Young's modulus, v is Poisson's ratio and h is the shell thickness.

The stress resultants and stress couples, N and M, together with the two transverse shear resultants Q_ξ and Q_θ are subject to six differential equations of equilibrium [11]

$$(rN_\xi)' + N^{\cdot}_{\theta\xi} - cN_\theta + rp_\xi = 0, \qquad (rN_{\xi\theta})' + N^{\cdot}_\theta + cN_{\theta\xi} - sQ_\theta + rp_\theta = 0,$$

$$(rQ_\xi)' + Q^{\cdot}_\theta + sN_\theta + rp_n = 0, \qquad (rM_\xi)' + M^{\cdot}_{\theta\xi} - cM_\theta - rQ_\xi = 0, \tag{7}$$

$$(rM_{\xi\theta})' + M^{\cdot}_\theta + cM_{\theta\xi} - rQ_\theta = 0, \qquad N_{\xi\theta} - N_{\theta\xi} + \frac{s}{r}M_{\theta\xi} = 0,$$

where p_ξ, p_θ and p_n are distributed surface load intensity components in the meridional, circumferential and normal directions, respectively.

The equilibrium equations (7), with $M_{\xi\theta} = M_{\theta\xi}$, may be satisfied identically by means of three stress functions G, H and F in the form

$$M_\theta = G', \qquad M_\xi = r^{-1}(H^{\cdot} + cG - sF) + M_{\xi p},$$

$$M_{\theta\xi} = M_{\xi\theta} = -\tfrac{1}{2}r^{-1}(rH' - cH + G^{\cdot}),$$

$$N_\theta = F'' + N_{\theta p}, \qquad N_\xi = r^{-2}(F^{\cdot\cdot} + sH^{\cdot} + crF') + N_{\xi p}, \tag{8}$$

$$N_{\theta\xi} = -[r^{-1}(F^{\cdot} + sH)]' + N_p,$$

$$N_{\xi\theta} = -\tfrac{1}{2}r^{-2}(zrF'^{\cdot} - 2cF^{\cdot} - scH + srH' - sG^{\cdot}) + N_p,$$

where

$$N_{\theta p} = -srp_n, \qquad N_{\xi p} = r^{-1}\int^{\xi}\left[r^{-2}\int^{\xi}(p_\theta - sp_n^{\cdot\cdot})r^2\,d\xi - r(p_\xi + scp_n)\right]d\xi,$$
$$N_p = r^{-2}\int^{\xi}(sp_n^{\cdot} - p_\theta)r^2\,d\xi, \qquad M_{\xi p} = -c^2r^{-1}\int^{\xi}\left(\int^{\xi}p_n r\,d\xi\right)d\xi, \tag{9}$$

and

$$Q_\xi = -\tfrac{1}{2}r^{-2}(2srF' - rH'^{\cdot} - cH^{\cdot} + G^{\cdot\cdot}) - c^2r^{-1}\int^{\xi}p_n r\,d\xi,$$
$$Q_\theta = -\tfrac{1}{2}[H' + r^{-1}(cH - G^{\cdot})]'. \tag{10}$$

Equations (3), (4), (5) and (8) constitute a system of twenty simultaneous partial differential equations for the twenty unknowns, u, v, w, ε_ξ, ε_θ, $\varepsilon_{\xi\theta}$, κ_ξ, κ_θ, $\kappa_{\xi\theta}$, $\kappa_{\theta\xi}$, N_ξ, N_θ, $N_{\xi\theta}$, $N_{\theta\xi}$, M_ξ, M_θ, $M_{\xi\theta}$, G, H, and F.

3. Exact reduction to two simultaneous equations

We consider stress functions, displacement components and load distributions of the form

$$(u, w, G, F, p_\xi, p_n) = (U, W, g, f, P_\xi, P_n)\cos n\theta,$$
$$(v, H, p_\theta) = (V, h, P_\theta)\sin n\theta, \tag{1}$$

where the quantities inside the parentheses on the right are functions of ξ only. In view of the known reductions for the cases $n = 0$ and $n = 1$ [1,7,9,10,13, 14,18], we are concerned here only with the cases $n \geq 2$. Corresponding to (l), we have

$$(\varepsilon_\xi, \varepsilon_\theta, \kappa_\xi, \kappa_\theta, N_\xi, N_\theta, Q_\xi, M_\xi, M_\theta) = (e_\xi, e_\theta, k_\xi, k_\theta, n_\xi, n_\theta, q_\xi, m_\xi, m_\theta)\cos n\theta,$$
$$(\varepsilon_{\xi\theta} = \varepsilon_{\theta\xi}, \kappa_{\xi\theta}, \kappa_{\theta\xi}, N_{\xi\theta}, N_{\theta\xi}, Q_\theta, M_{\xi\theta} = M_{\theta\xi}) = (e, k_{\xi\theta}, k_{\theta\xi}, n_{\xi\theta}, n_{\theta\xi}, q_\theta, m)\sin n\theta. \tag{2}$$

With (1) and (2), the strain displacement relations (2.3)* and the stress-stress function relations (2.8) become

$$e_\xi = U', \qquad e_\theta = r^{-1}(nV + cU - sW), \qquad e = \tfrac{1}{2}r^{-1}(rV' - cV - nU), \tag{3,4,5}$$
$$k_\xi = -W'', \qquad k_\theta = -r^{-2}(crW' - n^2W + snV), \tag{6,7}$$
$$k_{\xi\theta} = r^{-2}(nrW' - ncW - srV' + scV), \tag{8}$$
$$k_{\theta\xi} = \tfrac{1}{2}r^{-2}(2nrW' - 2ncW - srV' + scV - snU), \tag{9}$$

and

$$m_\theta = g', \qquad m_\xi = r^{-1}(nh + cg - sf) + m_{\xi p}, \tag{10, 11}$$
$$m = -\tfrac{1}{2}r^{-1}(rh' - ch - ng), \tag{12}$$
$$ne = f'' + n_{\theta p}, \qquad n_\xi = r^{-2}(rcf' - n^2f + snh) + n_{\xi p}, \tag{13, 14}$$
$$n_{\theta\xi} = r^{-2}(nrf' - ncf - srh' + sch) + n_p, \tag{15}$$
$$n_{\xi\theta} = \tfrac{1}{2}r^{-2}(2nrf' - 2ncf - srh' + sch - nsg) + n_p, \tag{16}$$

* **(2.3)** denotes equation (3) of section (2).

where

$$n_{\theta p} = -srP_n, \qquad n_{\xi p} = r^{-1}\int^{\xi}\left[r^{-2}\int^{\xi}(sn^2P_n + nP_\theta)r^2\,d\xi - r(P_\xi + scP_n)\right]d\xi,$$
$$n_p = -r^{-2}\int^{\xi}(snP_n + P_\theta)r^2\,d\xi, \qquad m_{\xi p} = -c^2r^{-1}\int^{\xi}\left(\int^{\xi}P_nr\,d\xi\right)d\xi. \tag{17}$$

The stress strain relations (2.4) and (2.5) may be written, in a moreconvenient form for our purpose, as

$$n_\xi = \frac{e_\xi + \nu_s e_\theta}{A(1-\nu_s^2)}, \qquad e_\theta = A(n_\theta - \nu_s n_\xi), \qquad e = A_S(n_{\xi\theta} + n_{\theta\xi}), \quad (18,19,20)$$

and

$$k_e = \frac{m_\theta - \nu_b m_\xi}{D(1-\nu_b^2)}, \qquad m_\xi = D(k_\xi + \nu_\theta k_\theta), \qquad m = D_S(k_{\theta\xi} + k_{\xi\theta}). \quad (21,22,23)$$

Finally, equations (2.10) for the transverse shear resultants become

$$q_\xi = -\tfrac{1}{2}r^{-2}(2srf' - nrh' - nch - n^2g) - c^2r^{-1}\int^{\xi}P_nr\,d\xi,$$
$$q_\theta = -\tfrac{1}{2}[h' + r^{-1}(ch + ng)]'. \tag{24}$$

With equations (3) through (16), the six stress strain relations (18) through (23) can be thought of as six equations for the six unknowns W, U, V, f, g and h. We will use four of these relations to eliminate U, V, g and h from the remaining two equations and get two simultaneous equations for$W$ and f. To effect this elimination, we begin by writing out the six stress strain relations in terms of the three displacement functions and the three stress functions.

Upon introducing (3), (4) and (14) into (18) and re-arranging the result, we get

$$U' = -\nu_s(cr^{-1}U + nr^{-1}V) + A(1-\nu_s^2)nsr^{-2}h$$
$$+ \nu_s sr^{-1}W + A(1-\nu_s^2)(cr^{-1}f' - n^2r^{-2}f + n_{\xi p}). \tag{25}$$

Analogously, we get from (7), (l0), (11) and (21)

$$g' = \nu_b(cr^{-1}g + nr^{-1}h) - D(1-\nu_b^2)nsr^{-2}V$$
$$- \nu_b sr^{-1}f - D(1-\nu_b^2)(cr^{-1}W' - n^2r^{-2}W) + \nu_b m_{\xi p}. \tag{26}$$

Next, we introduce (5), (15) and (16) into (20) and re-arrange the result to read

$$V' + 3sA_Sr^{-1}h' = r^{-1}(nU + cV) - A_Sr^{-2}(nsg - 3csh)$$
$$+ 4A_S(nr^{-1}f' - cnr^{-2}f + n_p). \tag{27}$$

Similarly, we have from *(12), (8), (9)* and (23)

$$h' - 3sD_Sr^{-1}V' = r^{-1}(ng + ch) + D_Sr^{-2}(nsU - 3csV)$$
$$- 4nr^{-1}D_S(W' - cr^{-1}W). \tag{28}$$

We then solve (27) and (28) for V' and h' to get

$$V' = -\frac{4nsA_S}{r^2(1+3\varepsilon^2)}g + \frac{c}{r}V + \frac{n(1-\varepsilon^2)}{r(1+3\varepsilon^2)}U + \frac{4nA_S}{r(1+3\varepsilon^2)}\left(f' - \frac{c}{r}f\right)$$

$$+ \frac{12nsD_SA_S}{r^2(1+3\varepsilon^2)}\left(W' - \frac{c}{r}W\right) + \frac{4A_S}{1+3\varepsilon^2}n_p, \tag{29}$$

$$h' = \frac{4nsD_S}{r^2(1+3\varepsilon^2)}U + \frac{c}{r}h + \frac{n(1-\varepsilon^2)}{r(1+3\varepsilon^2)}g - \frac{4nD_S}{r(1+3\varepsilon^2)}\left(W' - \frac{c}{r}W\right)$$

$$+ \frac{12nsD_SA_S}{r^2(1+3\varepsilon^2)}\left(f' - \frac{c}{r}f\right) + \frac{12D_SA_S}{r}sn_p, \tag{30}$$

where

$$\varepsilon^2 = \frac{3s^2 D_SA_S}{r^2} = \frac{3DsAs}{R_\theta^2} \tag{31}$$

Finally, the remaining two stress strain relations *(19)* and (22) may be written in the form

$$cg + nh + \nu_b nsr^{-1}DV = -Dr[W'' + \nu_b cr^{-1}W' - \nu_b n^2r^{-2}W] + sf - rm_{\xi p}, \tag{32}$$

$$cV + nV + \nu_s nsr^{-1}Ah = Ar[f'' - \nu_s cr^{-1}f' + \nu_s n^2r^{-2}f] + sW + re_{\theta p}, \tag{33}$$

where use has been made of (25) and (26) to eliminate the explicit appearance of g' and U' and where

$$e_{\theta p} = A(n_{\theta p} - \nu_s n_{\xi p}). \tag{34}$$

Our objective now is to eliminate U, V, g and h from (32) and (33) to give us two simultaneous differential equations for W and f. This can be done as follows. We differentiate (32) and (33) once with respect to ξ and use (25), (26), (29) and (30) to eliminate U', V', g' and h', assuming henceforth constant D, A, ν_s and ν_b and $\nu_s = \nu_b$ for simplicity. After using (32) and (33) to simplify the resulting expressions somewhat, we have

$$\left[n^2 - c^2 - \frac{4n^2\varepsilon^2(3-\nu_b)}{3(1-\nu_b)(1+3\varepsilon^2)}\right]rg + Dn^2s\left[2 - \nu_b - \frac{2\varepsilon^2(3-\nu_b)}{1+3\varepsilon^2}\right]U - D(1+\nu_b)nscV$$

$$= -D\left\{r^3 W''' - \left[n^2(2-\nu_b) + c^2(1+\nu_b) - \frac{2n^2\varepsilon^2(3-\nu_b)}{1+3\varepsilon^2}\right]rW'\right.$$

$$\left. + \left[3 - \frac{2\varepsilon^2(3-\nu_b)}{1+3\varepsilon^2}\right]n^2cW\right\} + \left[1 - \frac{n^2DA(1+\nu_s)(3-\nu_b)}{r^2(1+3\varepsilon^2)}\right]sr^2\left(f' - \frac{c}{r}f\right)$$

$$- r^3m'_{\xi p} - nsDA(1+\nu_s)\left[3 - \nu_b - \frac{6\nu_b\varepsilon^2}{1+3\varepsilon^2}\right]n_p, \tag{35}$$

and

$$\left[n^2 - c^2 - \frac{4n^2\varepsilon^2(3+\nu_s)}{3(1+\nu_s)(1+3\varepsilon^2)}\right] rU - An^2 s\left[2+\nu_s - \frac{2\varepsilon^2(3+\nu_s)}{1+3\varepsilon^2}\right] g + A(1-\nu_s)nsch$$

$$= A\left\{r^3 f''' - \left[n^2(2+\nu_s) + c^2(1-\nu_s) - \frac{2n^2\varepsilon^2(3+\nu_s)}{1+3\varepsilon^2}\right] rf'\right.$$

$$\left. + \left[3 - \frac{2\varepsilon^2(3+\nu_s)}{1+3\varepsilon^2}\right] n^2 cf\right\} + \left[1 - \frac{n^2 DA(1-\nu_b)(3+\nu_s)}{r^2(1+3\varepsilon^2)}\right] sr^2\left(W' - \frac{c}{r}W\right)$$

$$+ r^3 e'_{\theta p} - r^2\left\{ce_{\xi p} + 4n\left[1 - \frac{(3+\nu_s)\varepsilon^2 - 6\nu_s\varepsilon^4}{(1+\nu_s)(1+3\varepsilon^2)}\right] e_p\right\}, \tag{36}$$

where

$$e_{\xi p} = A(n_{\xi p} - \nu_s n_{\theta p}), \qquad e_p = A_S n_p. \tag{37}$$

We then solve the four equations (32), (33), (35) and (36) for the four unknowns U, V, g and h in terms of W and f and the distributed loads, writing the solutions (which are linear in W and f and their derivatives up to third order) symbolically in the form

$$\begin{aligned} U &= L_U(f''', f'', f', f; W''', W'', W', W; P_\xi, P_\theta, P_n), \\ V &= L_V(f''', f'', f', f; W''', W'', W', W; P_\xi, P_\theta, P_n), \\ g &= L_g(f''', f'', f', f; W''', W'', W', W; P_\xi, P_\theta, P_n), \\ h &= L_h(f''', f'', f', f; W''', W'', W', W; P_\xi, P_\theta, P_n). \end{aligned} \tag{38}$$

We now differentiate (35) and (36) once more with respect to ξ. Upon using equations (25), (26), (29), (30) and (38) to eliminate U, V, g, h and their first derivatives from the resulting two equations, we obtain the desired two simultaneous fourth order differential equations for W and f.

We emphasize that the above procedure is exact and has been carried out by the author. Rather than reporting the details of this reduction and the final exact results, we will in the next section discuss a simplified reduction procedure which yields a set of differential equations for W and f differing from the exact set only by terms of the order *(,/m/R).* Whenever $\sqrt{DA} = O(h)$, as is certainly the case for an isotropic and homogeneous medium, the results of the simplified procedure are as accurate as those of the exact procedure considering the inherent error of order h/R in the constitutive equations of this theory*.

4. A simplified reduction

Assuming $\sqrt{DA} = O(h)$ and omitting terms of the order $\varepsilon^2 = O(h^2/R^2)$ compared to unity in equations (3.25), (3.29), (3.26) and (3.30) we have

$$U' = -\nu_s r^{-1}(cU + nV) + nsr^{-2}(1-\nu_s^2)Ah + \nu_s sr^{-1}W$$
$$+ A(1-\nu_s^2)(cr^{-1}f' - n^2 r^{-2} f - n_{\xi p}), \tag{1}$$

* To include the limiting case of a circular cylindrical shell, we will take $h/R = O(h/r_0)$ where r_0 is the minimum radius of the conical shell.

$$V' = -4nsr^{-2}A_S g + r^{-1}(nU + cV) + 4nr^{-1}A_S(f' - cr^{-1}f) + 12snr^{-2}D_S A_S(W' - cr^{-1}W) + 4A_S n_p, \tag{2}$$

$$g' = v_p r^{-1}(cg + nh) - nsr^{-2}(1 - v_b^2)DV - v_b sr^{-1}f - D(1 - v_b^2)(cr^{-1}W' - n^2 r^{-2}W) + v_b m_{\xi p}, \tag{3}$$

$$h' = 4nsr^{-2}U + r^{-1}(ng + ch) - 4nr^{-1}D_S(W' - cr^{-1}W) + 12nsr^{-2}D_S A_S(f' - cr^{-1}f) + 12sr^{-1}D_S A_S n_p. \tag{4}$$

Omitting terms of the order ε^2 in equations (3.32), (3.33), (3.35) and (3.36), we get

$$cg + nh + v_b nsr^{-1}DV = -D(rW'' + v_b cW' - v_b n^2 r^{-1}W) + sf - rm_{\xi p}, \tag{5}$$

$$(n^2 - c^2)rg + n^2 s(2 - v_b)DU - nsc(1 + v_b)DV = -D\{r^3 W''' - r[n^2(2 - v_b) + c^2(1 + v_b)]W' + 3cn^2 W\} + sr^2(f' - cr^{-1}f) - r^3 m'_{\xi p}, \tag{6}$$

$$cU + nV + v_s nsr^{-1}Ah = A(rf'' - v_s cf' + v_s n^2 r^{-1}f) + sW + re_{\theta p}, \tag{7}$$

$$(n^2 - c^2)rU - n^2 s(2 + v_s)Ag + nsc(1 - v_s)Ah = A\{r^3 f''' - r[n^2(2 + v_s) + c^2(1 - v_s)]f' + 3cn^2 f\} + sr^2(W' - cr^{-1}W) + r^2(re'_{\theta p} - ce_{\xi p} - 4ne_p). \tag{8}$$

We solve (5), (6), (7) and (8) for U, V, g and h and then omit all terms of the order h/R compared to unity, leaving us with*

$$(n^2 - c^2)g = -Dr^2\left\{W''' - \frac{n^2 - 1}{n^2 - c^2}\left[\frac{n^2(2 - v_b) + c^2(1 + v_b)}{r^2}W' - \frac{3n^2 c}{r^3}W\right]\right\} + s(rf' - cf) - r^2 m'_{\xi p}, \tag{9}$$

$$n(n^2 - c^2)h = Dr^2\left\{cW''' - \frac{n^2 - c^2}{r}W'' - \frac{c(2n^2 + c^2)(n^2 - 1)}{r^2(n^2 - c^2)}W' + \frac{n^2(n^2 - 1)}{r^3(n^2 - c^2)}[v_b n^2 + c^2(3 - v_b)]W\right\} - s(crf' - n^2 f) + cr^2 m'_{\xi p} - r(n^2 - c^2)m_{\xi p}, \tag{10}$$

$$(n^2 - c^2)U = Ar^2\left\{f''' - \frac{n^2 - 1}{n^2 - c^2}\left[\frac{n^2(2 + v_s) + c^2(1 - v_s)}{r^2}f' - \frac{3n^2 c}{r^3}f\right]\right\} + s(rW' - cW) + (r^2 e'_{\theta p} - cre_{\xi p} - 4nre_p) - \frac{sA}{n^2 - c^2}\{[n^2(2 + v_s) + c^2(1 - v_s)]rm'_{\xi p} - c^2(n^2 - c^2)(1 - v_s)m_{\xi p}\}, \tag{11}$$

* In this step, we have made use of the well known fact that, differentiation with respect to ξ changes order of magnitude by a factor λ/r_0 at most where $\lambda = \max[n, r_0/(Rh)^{1/2}]$.

$$n(n^2 - c^2)V = -Ar^2\left\{cf''' - \frac{n^2 - c^2}{r^2}f'' - \frac{c(2n^2 + c^2)(n^2 - 1)}{r^2(n^2 - c^2)}f' + \frac{n^2(n^2 - 1)}{r^3(n^2 - c^2)}[-v_s n^2 + c^2(3 + v_s)]f\right\} s(crW' - n^2 W) + sA\left\{\frac{2n^2 + c^2}{n^2 - c^2}crm'_{\xi p} - (c^2 - v_s n^2)m_{\xi p}\right\} - [cr^2 e'_{\theta p} - (n^2 - c^2)re_{\theta p} - c^2 re_{\xi p} - 4nre_p]. \tag{12}$$

In solving the system (5) to (8), it appears that we would have to invert a 4 x 4 matrix. But by taking advantage of the static geometric duality inherent in the left hand side of these equations, the task is reduced to the inversion of two 2 x 2 submatrices (see Appendix).

Differentiating (6) and (8) once more with respect to ξ, and then using (1) to (4) and (9) to (12) to eliminate U, V, g, h and their first derivatives from the resulting two equations give us, after omitting terms of the order h/R,

$$D\left\{W^{\text{iv}} + \frac{2c}{r}W''' - \frac{2n^2 + c^2}{r^2}W'' + \frac{c(2n^2 + c^2)(n^2 - 1)}{r^3(n^2 - c^2)}W' + \frac{n^2(n^2 - 4c^2)(n^2 - 1)}{r^4(n^2 - c^2)}W\right\} - \frac{s}{r}f'' = c^2 P_n, \tag{13}$$

and

$$A\left\{f^{\text{iv}} + \frac{2c}{r}f''' - \frac{2n^2 + c^2}{r^2}f'' + \frac{c(2n^2 + c^2)(n^2 - 1)}{r^3(n^2 - c^2)}f' + \frac{n^2(n^2 - 4c^2)(n^2 - 1)}{r^4(n^2 - c^2)}f\right\} + \frac{s}{r}W'' = -\frac{A}{r}P_0, \tag{14}$$

where

$$P_0 = \left[v_s(rP_\xi)' + cP_\xi + \frac{n^2 - c^2}{r^2}\int^\xi P_\xi r\, d\xi\right] + n\left[(2 + v_s)P_\theta - \frac{3c}{r^3}\int^\xi P_\theta r^2\, d\xi - \frac{n^2 - c^2}{r^2}\int^\xi\left(\frac{1}{r^2}\int^\xi P_\theta r^2\, d\xi\right)d\xi\right] - s\left[(r^2 P_n)'' - 2n^2 P_n + \frac{cn^2(2n^2 + c^2)}{r^3(n^2 - c^2)}\int^\xi P_n r^2\, d\xi + \frac{n^2(n^2 - c^2)}{r^2}\int^\xi\left(\frac{1}{r^2}\int^\xi P_n r^2\, d\xi\right)d\xi\right]. \tag{15}$$

Equations (13) and (14) are two simultaneous differential equations for W and f and, except for terms of the order h/R, are the same as those obtained by way of the exact reduction described in the preceding section.

The two differential equations for W and f (load terms included) reduce correctly to those for a circular cylindrical shell when c = 0 and s = 1 [16, 19]*. For a shallow shell, we may replace c by unity and these equations become Marguerre's equations for a shallow conical shell [6,8]. When c = 1 and s = 0, equations (13) and (14) (load terms included) reduce correctly to the governing differential equations for transverse bending and inplane stretching, respectively, of flat plates.

5. **Simplified auxiliary equations**

Having W and f from the differential equations (4.13) and (4.14) and U, V, g and h from (4.9) to (4.12), we can now use (3.10) to (3.16) and (3.24) to express all stress resultants and couples in terms of W and f. Omitting all terms which are $O(h/R)$ compared to unity, we obtain the following equations for the stress and displacement measures

$$n_\xi = \frac{n^2-1}{n^2-c^2}\left(\frac{c}{r}f' - \frac{n^2}{r^2}f\right) + \frac{sD}{n^2-c^2}\left[cW''' - \frac{n^2-c^2}{r}W'' - \frac{c(n^2-1)(2n^2+c^2)}{r^2(n^2-c^2)}W' + \frac{n^2(n^2-1)}{n^2-c^2}\frac{\nu_b n^2 + c^2(3-\nu_b)}{r^3}W\right] + n_{\xi p} + \frac{sc}{n^2-c^2}m'_{\xi p} - \frac{s}{r}m_{\xi p}, \tag{1}$$

$$n_\theta = f'' - srP_n, \tag{2}$$

$$n_{\xi\theta} = \frac{n(n^2-1)}{n^2-c^2}\left(\frac{1}{r}f' - \frac{c}{r^2}f\right) + \frac{snD}{n^2-c^2}\left\{W''' - \frac{n^2-1}{n^2-c^2}\left[\frac{n^2+2c^2}{r^2}W' - \frac{n^2(2+\nu_b)+c^2(1-\nu_b)}{r^3}cW\right]\right\} + n_p + \frac{sn}{n^2-c^2}m'_{\xi p}, \tag{3}$$

$$n_{\theta\xi} = \frac{n(n^2-1)}{n^2-c^2}\left(\frac{1}{r}f' - \frac{c}{r^2}f\right) + \frac{snD}{n^2-c^2}\left\{W''' - \frac{n^2-1}{n^2-c^2}\left[\frac{\nu_b n^2 + c^2(3-\nu_b)}{r^2}W' - \frac{n^2(1+2\nu_b)+2c^2(1-\nu_b)}{r^3}cW\right]\right\} + n_p + \frac{sn}{n^2-c^2}m'_{\xi p}, \tag{4}$$

$$q_\xi = -D\left\{W''' + \frac{c}{r}W' + \frac{(n^2-1)(n^2+c^2)}{r^2(n^2-c^2)}W' + \frac{2n^2c}{r^3}\frac{n^2-1}{n^2-c^2}W\right\}, \tag{5}$$

* Because of the difference in the direction of the midsurface normal, w and p in [16] correspond to our $-W$ and $-P_n$.

$$q_\theta = nD\left\{W'' + \frac{n^2-1}{n^2-c^2}\left(\frac{c}{r}W' - \frac{n^2}{r^2}W\right)\right\}, \tag{6}$$

$$m_\xi = -D\left\{W'' + \nu_b\frac{n^2-1}{n^2-c^2}\left(\frac{c}{r}W' - \frac{n^2}{r^2}W\right)\right\}, \tag{7}$$

$$m_\theta = -D\left\{\nu_b W'' + \frac{n^2-1}{n^2-c^2}\left(\frac{c}{r}W' - \frac{n^2}{r^2}W\right)\right\}, \tag{8}$$

$$\mathbf{m} = n(1-\nu_b)D\left\{\frac{n^2-1}{n^2-c^2}\left(\frac{1}{r}W' - \frac{c}{r^2}W\right) - \frac{sA}{n^2-c^2}f'''\right\},$$

$$U = \frac{s}{n^2-c^2}(rW' - cW) + \frac{Ar^2}{n^2-c^2}\left\{f''' - \frac{n^2-1}{n^2-c^2}\left[\frac{n^2(2+\nu_s)+c^2(1-\nu_s)}{r^2}f' - \frac{3n^2c}{r^3}f\right]\right\} + \frac{1}{n^2-c^2}\{r^2e_{\theta p} - cre_{\xi p} - 4nre_p\}$$
$$- \frac{sA}{(n^2-c^2)^2}\{[n^2(2+\nu_s)+c^2(1-\nu_s)]rm'_{\xi p} - c^2(n^2-c^2)(1-\nu_s)m_{\xi p}\}, \tag{10}$$

$$V = -\frac{s}{n(n^2-c^2)}(crW' - n^2W)$$
$$- \frac{Ar^2}{n(n^2-c^2)}\left\{cf''' - \frac{n^2-c^2}{r}f'' - \frac{c(2n^2+c^2)(n^2-1)}{r^2(n^2-c^2)}f' + \frac{n^2(n^2-1)}{r^3(n^2-c^2)}[-\nu_s n^2 + c^2(3+\nu_s)]f\right\}$$
$$+ \frac{sA}{n(n^2-c^2)}\left\{\frac{2n^2+c^2}{n^2-c^2}crm'_{\xi p} - (c^2-\nu_s n^2)m_{\xi p}\right\}$$
$$- \frac{1}{n(n^2-c^2)}\{cr^2e'_{\theta p} - (n^2-c^2)re_{\theta p} - c^2re_{\xi p} - 4nre_p\}. \tag{11}$$

We note that the above expressions for the stress measures satisfy the differential equations of equilibrium (2.7) up to terms of the order h/R. They also reduce correctly to the corresponding expressions for flat plates, shallow conical shells [6, 8] and circular cylindrical shells [16, 19]*.

6. Comparison with Seide's equations

A Donnell type theory for asymmetrical bending of conical shell was proposed earlier by N. J. Hoff [5] and by P. Seide [15]. It was also shown in [151 that the differential equations for such a theory can be reduced to two simultaneous fourth order differential equations for a stress function and a displacement function.

* An exception is they-terms in ***m*** which is in the result of [19] but not in the result of [161. Without these terms, the equilibrium equations (2.7) in general are not satisfied in the sense stated above.

With load terms omitted, the two governing differential equations of [15] are, in our terminology,

$$D\left[W^{\text{iv}}+\frac{2c}{r}W'''-\frac{2n^2+c^2}{r^2}W''+\frac{c(2n^2+c^2)}{r^3}W'+\frac{n^2(n^2-4c^2)}{r^4}W\right]$$

$$=\frac{s}{r}f'', \tag{1}$$

$$-A\left[f^{\text{iv}}+\frac{2c}{r}f'''-\frac{2n^2+c^2}{r^2}f''+\frac{c(2n^2+c^2)}{r^3}f'+\frac{n^2(n^2-4c^2)}{r^4}f\right]$$

$$=\frac{s}{r}W''. \tag{2}$$

As pointed out in [15], they reduce to Donnell's equations for circular cylindrical shells [2] if we set $c=0$ and $s=1$. We see the equations (1) and (2) differ from the exact (within the accuracy of shell theory) equations (4.13) and (4.14) only by a factor $(n^2-1)/(n^2-c^2)=1-s^2/(n^2-c^2)$ in the first and zeroth derivative terms on the left. As such, they are correct if $s^2/(n^2-c^2)=O(\varepsilon)$ but should be adequate whenever $s^2/(n^2-c^2)\ll 1$, keeping in mind that our results are valid only for $n\geq 2$*.

The auxiliary equations for stress resultants and couples given in [15] also differ from those obtained here. In particular, Seide's auxiliary equations do not include the W-terms in the expressions for the in-plane stress resultants and do not include the f-terms in the expressions for stress couples. The auxiliary equations of [15] are known to lead to an incorrect solution for certain problems of circular cylindrical shells involving symmetric stress distribution (see [16] for references). Whether the simplified expressions would lead to an incorrect solution for problems involving unsymmetric stress distributions remains to be investigated.

7. A single complex equation and generalized hypergeometric function solutions

The form of (4.13) and (4.14) permits the usual further reduction to a fourth order complex equation for a complex function Φ.

$$\Phi^{\text{iv}}+\frac{2c}{r}\Phi'''-\frac{2n^2+c^2}{r^2}\Phi''+\frac{c(2n^2+c^2)(n^2-1)}{r^3(n^2-c^2)}\Phi'+\frac{n^2(n^2-4c^2)(n^2-1)}{r^4(n^2-c^2)}\Phi$$

$$=\frac{is}{r\sqrt{DA}}\Phi''+\frac{1}{D}\left(c^2P_n+\frac{i\sqrt{DA}}{r}P_0\right), \tag{1}$$

where

$$\Phi=W-i\sqrt{\frac{A}{D}}f. \tag{2}$$

* For $n=0$, equations (4.1) through (4.8) uncouple into four separate sets. Further reductions should be carried out independently for each set to avoid inverting a singular matrix. For $n=1$, the simplified reduction of section (4) is not applicable whenever $(n^2-c^2)=(1-c^2)=O(\varepsilon^2)$. In both cases, multi-valued stress and displacement functions should be included for a complete theory.

For c $\neq 0$, we may introduce a new independent variable

$$x = \frac{isr}{c^2\sqrt{DA}}, \qquad (\ )^{\cdot} = \frac{d(\ )}{dx}, \tag{3}$$

and rewrite the differential equation (1) with x as the independent variable. Omitting the load terms, we get

$$\Phi^{\cdot\cdot\cdot\cdot} + \frac{2}{x}\Phi^{\cdot\cdot\cdot} - \frac{c_2}{x^2}\Phi^{\cdot\cdot} + \frac{c_1}{x^3}\Phi^{\cdot} + \frac{c_0}{x^4}\Phi = \frac{1}{x}\Phi^{\cdot\cdot}, \tag{4}$$

where

$$c_2 = \frac{2n^2 + c^2}{c^2}, \qquad c_1 = \frac{2n^2 + 1}{c^2}\,\frac{n^2 - 1}{n^2 - c^2}, \qquad c_0 = \frac{n^2(n^4 - 4c^2)(n^2 - 1)}{c^4(n^2 - c^2)} \tag{5}$$

A straightforward application of Frobenius' method leads to four independent power series solutions about the regular singular point x = 0 of the form

$$\Phi_i = x^{\alpha_i}\left[1 + \sum_{k=1}^{\infty} \frac{\prod_{l=1}^{k} P(\alpha_i + l)}{\prod_{l=1}^{k} Q(\alpha_i + l + 1)} x^k\right], \tag{6}$$

where

$$P(\gamma) = \gamma(\gamma - 1), \qquad Q(\gamma) = \gamma(\gamma - 1)[\gamma^2 - 3\gamma - c^{-2}(2n^2 - c^2)] + c_1\gamma + c_0, \tag{7}$$

and where the α_i's are the four roots of $Q(\alpha) = 0$. If these roots are not distinct or if any two roots differ only by an integer, other solutions involving logarithmic terms will be needed. For a shallow shell, for instance, we take c= 1 in $Q(\gamma)$ and get as we should (see [6] and references contained therein)

$$\begin{aligned} Q(\gamma) &= \gamma(\gamma - 1)(\gamma^2 - 3\gamma - 2n^2 + 1) + (2n^2 + 1)\gamma + n^2(n^2 - 4) \\ &= (\gamma - n)(\gamma + n)(\gamma - n - 2)(\gamma + n - 2), \end{aligned} \tag{8}$$

so that two solutions involving logarithmic terms must be used.

It should be noted that the series (6) are related to the generalized hypergeometric function ${}_2F_3$. To see this relationship, we write the differential equation (4) in the form

$$[Q(\delta) - xP(\delta)]\Phi \equiv \left[\prod_{i=1}^{4}(\delta - \alpha_i) - x\delta(\delta - 1)\right]\Phi = 0 \tag{9}$$

where $\delta = x(d/dx)$ and Q and P are polynomials of degree four and two respectively, as defined in (7). If we now set $\Phi = x^{\alpha_4}\phi_4$ and make use of the relation

$$Q(\delta)(x^{\alpha_4}\phi_4) = x^{\alpha_4}Q(\delta + \alpha_4)\phi_4, \qquad P(\delta)(x^{\alpha_4}\phi_4) = x^{\alpha_4}P(\delta + \alpha_4)\phi_4, \tag{10}$$

to transform (9) into

$$x^{\alpha_4}[Q(\delta + \alpha_4) - xP(\delta + \alpha_4)]\phi_4 = 0, \tag{11}$$

or

$$\left[\delta \prod_{i=1}^{3}(\delta - \alpha_i) - x(\delta + \alpha_4)(\delta + \alpha_4 - 1)\right]\phi_4 = 0. \tag{11'}$$

The last equation is known to be satisfied by the generalized hypogeometric function [3, eq. 4.2(2)]

$$\phi_4 = {}_2F_3\begin{bmatrix} \alpha_4, & \alpha_4 - 1, & x \\ \alpha_4 - \alpha_1 + 1, & \alpha_4 - \alpha_2 + 1, & \alpha_4 - \alpha_3 + 1 \end{bmatrix} \tag{12}$$

It is now evident that, except for the factor x^{α_i}, the series (6) is a generalized hypergeometric function ${}_2F_3$ with suitable parameters readily deduced from the typical one given by (12). The asymptotic and other properties of such functions are given in [3].

Since the irregular singular point at infinity is the only other singularity of the differential equation (4), the series solution (6) converges for the entire complex x-plane with the origin deleted. However, the convergence of these series will in general be slow for moderate values of n since

$$x = i\mu^2(r/r_0), \tag{13}$$

and

$$\mu^2 = \frac{sr_0}{c^2\sqrt{DA}}, \tag{14}$$

is usually large compared to unity for a shell. On the other hand, the fact that μ^2 is usually much greater than unity suggests that an asymptotic solution suffices for most purposes. A direct asymptotic solution of the differential equation proceeds exactly as in the case of a shallow conical shell [6, 121.

8. Remarks

Since n^2 is effectively $\partial^2(\,)/\partial\theta^2$ for shells of revolution, the two differential equations (4.13) and (4.14) for f and Wsuggest that a reduction of the equations of the linear theory of elastic shells to two simultaneous fourth order partial differential equations for F and w is not possible except for special cases (e.g. cylindrical, spherical and shallow shells; see references given in [19]). It would appear that for general shells of revolution, the pair of equations for F and w will be fourth order in ξ and sixth order in θ. Analogously, one may expect that for noncircular cylindrical shells, a procedure similar to our reduction with the role of ξ and θ interchanged would reduce the shell equations to two equations for F and w which are fourth order in θ and sixth order in ξ. For an arbitrary shell, the corresponding two equations appear to be sixth order partial differential equations. A reduction to a single sixth order partial differential equation for the complex function $\phi = w + i\lambda F$ (or equivalently two coupled sixth order equations for w and F) has been discussed by A. L. Goldenveiser [4]. The relevant complex equation remains a sixth order equation for general shells of revolution and reduces only to a fifth order ordinary differential equation in ξ for solutions whose dependence on θ is of the form $e^{in\theta}$.

On the other hand, it may still be possible to reduce the shell equations to two simultaneous fourth order partial differential equations for some other choice of two dependent variables. A recent work by J. G. Simmonds illustrates this possibility for catenoidal and helicoidal shells [17].

* Professor Simmonds brought Goldenveiser's work to the author's attention after he had read a preprint of [19].

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Appendix

One of the steps in the reduction described in Section (4) was to obtain from equations (5) through (8) the expression for U V, g and h in terms of W, f and their derivatives. Writing the four equations as

$$\left[\begin{array}{cc|cc} c & 1 & 0 & v_b s D r^{-1} \\ r(n^2-c^2) & 0 & n^2 s(2-v_b)D & -sc(1+v_b)D \\ \hline 0 & v_s s A r^{-1} & c & 1 \\ -n^2 s(2+v_s)A & sc(1-v_s)A & r(n^2-c^2) & 0 \end{array}\right] \left[\begin{array}{c} g \\ nh \\ \hline U \\ nV \end{array}\right] = \left[\begin{array}{c} \text{right} \\ \text{hand} \\ \text{side} \end{array}\right] \tag{1}$$

this step amounts to the inversion of the 4 x 4 coefficient matrix of the left hand side of (1). We will show here that by using the static geometric duality properties of the elements of the matrix, the task can be reduced to the inversion of only 2 x 2 matrices.

Denoting the coefficient matrix by C and inverse by B, we partition C and B each into four 2 x 2 submatrices:

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{11}^* \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{11}^* \end{bmatrix}. \tag{2}$$

Since

$$BA = \begin{bmatrix} B_{11}C_{11} + B_{12}C_{12}^* & B_{11}C_{12} + B_{12}C_{11}^* \\ B_{12}^*C_{11} + B_{11}^*C_{12}^* & B_{11}^*C_{11}^* + B_{12}^*C_{12} \end{bmatrix} = I_4, \tag{3}$$

where I_n is an n x n identity matrix, we must have

$$B_{11}C_{11} + B_{12}C_{12}^* = I_2, \qquad B_{11}C_{12} + B_{12}C_{11}^* = 0, \tag{4}$$

$$B_{11}^*C_{11}^* + B_{12}^*C_{12} = I_2, \qquad B_{11}^*C_{12}^* + B_{12}^*C_{11} = 0, \tag{5}$$

from the second relation in (4) and (5), we have further

$$B_{12} = -B_{11}C_{12}(C_{11}^*)^{-1}, \qquad B_{12}^* = -B_{11}^*C_{12}^*(C_{11})^{-1}. \tag{6}$$

Upon introducing (6) into the first relations in (4) and (5), we get

$$B_{11} = [C_{11} - C_{12}(C_{11}^*)^{-1}C_{12}^*]^{-1}, \qquad B_{11}^* = [C_{11}^* - C_{12}^*(C_{11})^{-1}C_{12}]^{-1}. \tag{7}$$

Having (7), we get B_{12} and B_{12}^* from (6).

Now for our particular matrix C, C_{11}^* and C_{12}^* are the static geometric duals of C_{11} and C_{12} respectively. That is, C_{11}^* and C_{12}^* can be obtained from C_{11} and C_{12}

respectively by replacing v_b and D in the latter by $-v_s$ and $-A$. From (5) and (7), B^*_{11} and B^*_{12} are evidently the duals of B_{11} and B_{12}, respectively, so that we need only to do the necessary calculations for B_{11} and B_{12}. These calculations involve only the inversion of the two 2 x 2 matrices C^*_{11} and $C_{11}-C_{12}(C^*_{11})^{-1}C^*_{12}$.

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