Stress Functions for Nonlinear Shallow Shell Theory*

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This note obtains the stress function solution of the homogeneous equilibrium equations of the nonlinear shallow shell theory as formulated in [1].

The following is presented as a supplement to the results of the preceding paper [1]. Inspection of the differential equations of equilibrium (1a, b), (5), (33), (34) and (35) in [1], and some mathematical experimentation which makes use of the three linear compatibility equations (12a, b) and (16) in [1], indicate that the six homogeneous nonlinear equilibrium equations may be satisfied identically by means of six stress functions H_1, H_2, F, K_1, K_2 and J, in the form

$$N_{11} = K_{2,2}, \quad N_{12} = K_{1,2}, \quad N_{21} = -K_{2,1}, \quad N_{22} = -K_{1,1},$$

$$P_1 = -K_2 + F_{,2}, \quad P_2 = -K_1 - F_{,1}, \tag{1}$$

$$Q_1 = J_{,2} + (z_{,22} - \kappa_{22})K_1 + (z_{,12} - \kappa_{21})K_2,$$

$$Q_2 = -J_{,1} - (z_{,11} - \kappa_{11})K_2 - (z_{,12} - \kappa_{12})K_1, \tag{2}$$

$$M_{11} = H_{2,2} - (z_{,22} - \kappa_{22})F + \gamma_2 K_2,$$

$$M_{22} = -H_{1,1} - (z_{,11} - \kappa_{11})F - \gamma_1 K_1,$$

$$M_{12} = H_{1,2} + (z_{,12} - \kappa_{21})F + \gamma_2 K_1 - J,$$

$$M_{21} = -H_{2,1} + (z_{,12} - \kappa_{12})F - \gamma_1 K_2 + J. \tag{3}$$

Differential equations for the determination of the six stress functions follow upon introducing strain measures in terms of stress functions into the equations of compatibility. In order to express the κ_{ij} and γ_i in terms of stress functions, it is necessary to make use of the relevant stress strain relations. To illustrate, for stress strain relations of the form

$$M_{11} = D(\kappa_{11} + \nu\kappa_{22}), \quad M_{22} = \ldots, \quad M_{12} = D_S\kappa_{12}, \quad M_{21} = \ldots,$$

$$\gamma_i = A_Q Q_i, \tag{4}$$

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the six strain measures κ_{ij} and γ_i are determined in terms of stress functions upon solving a system of six simultaneous linear equations, obtained by substituting (4) into the left hand sides of (2) and (3),

$$\gamma_1 + A_Q K_1 \kappa_{22} + A_Q K_2 \kappa_{21} = A_Q(J_{,2} + z_{,22}K_1 + z_{,12}K_2),$$
$$\gamma_2 - A_Q K_2 \kappa_{11} - A_Q K_1 \kappa_{12} = -A_Q(J_{,1} + z_{,11}K_2 + z_{,12}K_1). \tag{5}$$
$$-K_2\gamma_2 + D\kappa_{11} + (\nu D - F)\kappa_{22} = H_{2,2} - z_{,22}F,$$
$$K_1\gamma_1 + (\nu D - F)\kappa_{11} + D\kappa_{22} = -H_{1,1} - z_{,11}F. \tag{6}$$
$$-K_1\gamma_2 + D_S\kappa_{12} + F\kappa_{21} = H_{1,2} + z_{,12}F - J,$$
$$K_2\gamma_1 + F\kappa_{12} + D_S\kappa_{21} = -H_{2,1} + z_{,12}F + J. \tag{7}$$

We limit ourselves here to the solution of (5) to (7) for the case $A_Q = 0$, for which (5) becomes $\gamma_i = 0$, and then from (6)

$$\kappa_{11} = \frac{DH_{2,2} + (\nu D - F)H_{1,1} + [z_{,11}(\nu D - F) - z_{,22}D]}{(1-\nu^2)D^2 + 2\nu DF - F^2},$$
$$\kappa_{22} = -\frac{DH_{1,1} + (\nu D - F)H_{2,2} - [z_{,22}(\nu D - F) - z_{,11}D]F}{(1-\nu^2)D^2 + 2\nu DF - F^2}. \tag{8}$$

In solving (7), we observe that equation (16) of [1], with $\gamma_i = 0$, implies that $\kappa_{12} = \kappa_{21}$. Therewith,

$$\kappa_{12} = \kappa_{21} = \frac{H_{1,2} - H_{2,1} + 2z_{,12}F}{2(D_S + F)}, \qquad J = \frac{H_{1,2} + H_{2,1}}{2}. \tag{9}$$

With (8) and (9) the stress strain relations (4) give as expressions for stress couples

$$\frac{M_{12}}{D_S} = \frac{M_{21}}{D_S} = \frac{2z_{,12}F + H_{1,2} - H_{2,1}}{2(D_S + F)},$$
$$\frac{M_{11}}{D} = \frac{[(1-\nu^2)D + \nu F](H_{2,2} - z_{,22}F) - F(H_{1,1} + z_{,11}F)}{(1-\nu^2)D^2 + 2\nu DF - F^2},$$
$$\frac{M_{22}}{D} = -\frac{[(1-\nu^2)D + \nu F](H_{1,1} + z_{,11}F) - F(H_{2,2} - z_{,22}F)}{(1-\nu^2)D^2 + 2\nu DF - F^2}, \tag{10}$$

and the equations (2) give as expressions for the transverse shear resultants

$$Q_1 = J_{,2} + K_2\frac{2z_{,12}D_S + H_{2,1} - H_{1,2}}{2(D_S + F)} + K_1\frac{z_{,22}D[(1-\nu^2)D + \nu F] + z_{,11}DF + DH_{1,1} + (\nu D - F)H_{2,2}}{(1-\nu^2)D^2 + 2\nu DF - F^2},$$

$$Q_2 = -J_{,1} - K_1\frac{2z_{,12}D_S + H_{2,1} - H_{1,2}}{2(D_S + F)} - K_2\frac{z_{,11}D[(1-\nu^2)D + \nu F] + z_{,22}DF + DH_{2,2} - (\nu D - F)H_{1,1}}{(1-\nu^2)D^2 + 2\nu D - F^2}. \tag{11}$$

Reference

1. E. REISSNER, "On the equations of nonlinear shallow shell theory", *Studies in Applied Mathematics* **48**, 2, 171–175, (1969).

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