On the Equations of Linear Shallow Shell Theory

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The paper presents a formulation of the two-dimensional theory of shallow shells, including the effects of transverse shear deformation and of moments turning about the normal to the middle surface. The present formulation includes, as it must, Marguerre's theory. At the same time it is consistent with recent formulations of general linear shell theory, in particular in regard to the preservation of the static-geometric duality. Various reductions of the equation of the theory are considered. Of particular significance and effectiveness among these are reductions for the special cases of (1) shells without moments about the middle surface normal, (where an earlier result of Naghdi is extended) (2) the shell without transverse shear deformability, (the static-geometrical dual of case (1)). As an application of the general equations an explicit solution is obtained for the problem of stretching, twisting and bending of pretwisted rectangular plates.

1. Introduction

We are concerned in what follows with the equations of the linear theory of shallow shells, upon inclusion in the theory of the effects of transverse shear deformability and of couple stress stress couples, and in such a way that account is taken of the concept of the static geometric duality. Specifically, we consider the effect of introducing the Marguerre stress-function and displacement variables into the generalized theory, and the problem of obtaining differential equations for these variables. We show that a special case of our results extends previous results concerning the effect of transverse shear deformation. We also show that an important simplification is possible for the problem of the shallow spherical shell, whereby the solution of this problem is expressed entirely in terms of solutions of the equation $\nabla^2\chi = \mu^2\chi$. In addition, we obtain explicit solutions of the problems of stretching, pure bending and St. Vernant torsion of a shallow hyperbolic paraboloidal shell.

2. Basic equations

The equations of the linear theory of shallow shells are here taken in the form of equilibrium equations, compatibility equations and stress strain relations (assuming isotropy) as follows:

$$N_{11,1} + N_{21,2} + p_1 = 0, \qquad N_{12,1} + N_{22,2} + p_2 = 0,$$
$$Q_{1,1} + Q_{2,2} + z_{,11}N_{11} + z_{,12}(N_{12} + N_{21}) + z_{,22}N_{22} + p_n = 0,$$
$$M_{11,1} + M_{21,2} - Q_1 - z_{,12}P_1 - z_{,22}P_2 + q_1 = 0,$$
$$M_{12,1} + M_{22,2} - Q_2 + z_{,11}P_1 + z_{,12}P_2 + q_2 = 0,$$
$$P_{1,1} + P_{2,2} + N_{12} - N_{21} + q_n = 0. \qquad \text{(I)}$$

$$-\kappa_{22,1} + \kappa_{12,2} = 0, \qquad \kappa_{21,1} - \kappa_{11,2} = 0,$$
$$\lambda_{2,1} - \lambda_{1,2} - z_{,11}\kappa_{22} + z_{,12}(\kappa_{12} + \kappa_{21}) - z_{,22}\kappa_{11} = 0,$$
$$\varepsilon_{22,1} - \varepsilon_{12,2} - \lambda_2 - z_{,12}\gamma_2 + z_{,22}\gamma_1 = 0,$$
$$-\varepsilon_{21,1} + \varepsilon_{11,2} + \lambda_1 + z_{,11}\gamma_2 - z_{,12}\gamma_1 = 0,$$
$$\gamma_{2,1} - \gamma_{1,2} + \kappa_{21} - \kappa_{12} = 0. \qquad \text{(II)}$$

$$\varepsilon_{11} = A(N_{11} - \nu_N N_{22}), \qquad \varepsilon_{22} = A(N_{22} - \nu_N N_{11}),$$
$$\varepsilon_{12} = A_S[N_{12} - \alpha_N(N_{12} - N_{21})], \qquad \varepsilon_{21} = A_S[N_{21} - \alpha_N(N_{21} - N_{12})],$$
$$M_{11} = D(\kappa_{11} + \nu_M\kappa_{22}), \qquad M_{22} = D(\kappa_{22} + \nu_M\kappa_{11}),$$
$$M_{12} = D_S[\kappa_{12} + \alpha_M(\kappa_{12} - \kappa_{21})], \qquad M_{21} = D_S[\kappa_{21} + \alpha_M(\kappa_{21} - \kappa_{12})],$$
$$\gamma_i = A_Q Q_i, \qquad P_i = D_P\lambda_i. \qquad \text{(III)}$$

In this the N_{ij} and Q_i are stress resultants, the M_{ij} and P_i are stress couples and the $\varepsilon_{ij}, \gamma_i, \kappa_{ij}$ and λ_i are the corresponding strain resultants and couples.* The middle surface of the shell is given in the form $z = z(x_1, x_2)$ and the assumed isotropy requires that $A_S = (1 + \nu_N)A$ and $D_S = (1 - \nu_M)D$. We note, in particular, the presence of the parameters α_N and α_M in (III) of which the special values $\alpha_N = -\alpha_M = \frac{1}{2}$ and $\alpha_N = \alpha_M = 0$ have previously been considered. We also note that when $A_Q = 0$, and $D_P = 0$, (and $q_n = 0$), the terms with α_N and α_M in (III) cancel out. Equations (I) to (III) reduce to the equations of shallow shell theory without transverse shear deformability and without couple stress stress couples upon setting $A_Q = 0$ and $D_P = 0$.

Concerning the form of equations (I) and (II), we have that these equations are the simplest specialization of the corresponding equations of general linear shell theory, upon introduction of the assumption of shallowness, in such a way that the specialization produces the first three and the last of equations (I) while at the same time preserving the static-geometric duality property of the equations of the general theory.

* We note that the compatibility equations (II) may be thought of as the consequence of strain displacement relations

$$\varepsilon_{11} = u_{1,1} - z_{,11}w, \qquad \varepsilon_{12} = u_{2,1} - z_{,12}w - \omega, \qquad \ldots, \qquad \gamma_1 = w_{,1} + \phi_1, \qquad \ldots,$$
$$\kappa_{11} = \phi_{1,1}, \qquad \kappa_{12} = \phi_{2,1}, \qquad \ldots, \qquad \lambda_1 = \omega_{,1} + z_{,12}\phi_1 - z_{,11}\phi_2, \qquad \lambda_2 = \omega_{,2} + z_{,22}\phi_1 - z_{,12}\phi_2.$$

3. Derivation of Marguerre's equations

We assume for simplicity's sake that $p_i = q_i = q_n = 0$ and set in (III)

$$A_Q = 0, \qquad D_P = 0, \tag{1}$$

and therewith

$$P_i = 0, \qquad \gamma_i = 0, \tag{2}$$

in (I) and (II).

We satisfy the first two and the last of the equilibrium equations (I) identically by setting in terms of a stress function F,

$$N_{11} = F_{,22}, \qquad N_{22} = F_{,11}, \qquad N_{12} = N_{21} = -F_{,12}. \tag{3}$$

Inspection of equations (II) indicates that the static geometric duals to the relations (3) are, in terms of a strain function w,

$$\kappa_{11} = -w_{,11}, \qquad \kappa_{22} = -w_{,22}, \qquad \kappa_{12} = \kappa_{21} = -w_{,12}. \tag{4}$$

We write the remaining equilibrium and compatibility equations with the help of appropriate stress strain relations as

$$Q_{1,1} + Q_{2,2} + z_{,11}F_{,22} - 2z_{,12}F_{,12} + z_{,22}F_{,11} + p_n = 0, \tag{5}$$

$$Q_1 = -(Dw_{,11})_{,1} - (\nu_M Dw_{,22})_{,1} - [(1 - \nu_M)Dw_{,12}]_{,2},$$
$$Q_2 = -(Dw_{,22})_{,2} - (\nu_M Dw_{,11})_{,2} - [(1 - \nu_M)Dw_{,12}]_{,1}, \tag{6}$$

and

$$\lambda_{2,1} - \lambda_{1,2} + z_{,11}w_{,22} - 2z_{,12}w_{,12} + z_{,22}w_{,11} = 0, \tag{7}$$

$$\lambda_2 = (AF_{,11})_{,1} - (\nu_N AF_{,22})_{,1} + [(1 + \nu_N)AF_{,12}]_{,2},$$
$$-\lambda_1 = (AF_{,22})_{,2} - (\nu_N AF_{,11})_{,2} + [(1 + \nu_N)AF_{,12}]_{,1}. \tag{8}$$

Introduction of (6) into (5) and of (8) into (7) results in two simultaneous fourth order equations for w and F which are generally designated as Marguerre's equations [1].

We note in particular the case of constant D, ν_M, A and ν_N. For this case equations (6) and (8) reduce to

$$Q_i = -D(\nabla^2 w)_{,i}, \qquad \lambda_2 = A(\nabla^2 F)_{,1}, \qquad \lambda_1 = -A(\nabla^2 F)_{,2}, \tag{9}$$

and equations (5) and (6) become

$$D\nabla^4 w - LF = p_n, \qquad A\nabla^4 F + Lw = 0, \tag{10}$$

where $L \equiv z_{,11}(\ )_{,22} - 2z_{,12}(\ )_{,12} + z_{,22}(\ )_{,11}$.

4. Generalized Marguerre equations

We will again assume $p_i = q_i = q_n = 0$. We now first satisfy the first two equilibrium equations in (I) by means of *two* stress functions F_1 and F_2, by setting $N_{11} = F_{1,2}$, $N_{21} = -F_{1,1}$, $N_{12} = -F_{2,2}$, $N_{22} = F_{2,1}$. The last equilibrium equation, with $P_i = D_P\lambda_i$, then becomes $-F_{2,2} + F_{1,1} + (D_P\lambda_1)_{,1} + (D_P\lambda_2)_{,2} = 0$. This is satisfied by setting, in terms of a function F, $F_1 = F_{,2} - D_P\lambda_1$ and $F_2 = F_{,1} + D_P\lambda_2$. Altogether, the first two and the last equation in (I) are satisfied by setting, in generalization of equations (3),

$$\begin{aligned} N_{11} &= F_{,22} - (D_P\lambda_1)_{,2}, \qquad & N_{22} &= F_{,11} + (D_P\lambda_2)_{,1}, \\ -N_{12} &= F_{,12} + (D_P\lambda_2)_{,2}, \qquad & -N_{21} &= F_{,12} - (D_P\lambda_1)_{,1}. \end{aligned} \tag{11}$$

Analogously, we satisfy the first two and the last of the compatibility equations (II), with $\gamma_i = A_Q Q_i$, by setting in generalization of equations (4)

$$\begin{aligned} \kappa_{22} &= -w_{,22} + (A_Q Q_2)_{,2}, \qquad & \kappa_{11} &= -w_{,11} + (A_Q Q_1)_{,1}, \\ \kappa_{21} &= -w_{,12} + (A_Q Q_1)_{,2}, \qquad & \kappa_{12} &= -w_{,12} + (A_Q Q_2)_{,1}. \end{aligned} \tag{12}$$

Equations (11) and (12) serve to reduce the remaining six equilibrium and compatibility equations to a system of six equations for the six unknowns F, w, λ_i, Q_i, upon writing these six equations with the help of equations (III) in the form

$$\begin{gathered} Q_{1,1} + Q_{2,2} + z_{,11}N_{11} + z_{,12}(N_{12} + N_{21}) + z_{,22}N_{22} + p_n = 0, \\ \lambda_{2,1} - \lambda_{1,2} - z_{,11}\kappa_{22} + z_{,12}(\kappa_{21} + \kappa_{12}) - z_{,22}\kappa_{11} = 0, \end{gathered} \tag{13}$$

$$\begin{aligned} Q_1 &= \{D(\kappa_{11} + \nu_M\kappa_{22})\}_{,1} + \{(1 - \nu_M)D[(1 + \alpha_M)\kappa_{21} - \alpha_M\kappa_{12}]\}_{,2} \\ &\qquad - D_P(z_{,12}\lambda_1 + z_{,22}\lambda_2), \\ Q_2 &= \{(1 - \nu_M)D[(1 + \alpha_M)\kappa_{12} - \alpha_M\kappa_{21}]\}_{,1} + \{D(\kappa_{22} + \nu_M\kappa_{11})\}_{,2} \\ &\qquad + D_P(z_{,11}\lambda_1 + z_{,12}\lambda_2), \end{aligned} \tag{14}$$

$$\begin{aligned} \lambda_2 &= \{A(N_{22} - \nu_N N_{11})\}_{,1} - \{(1 + \nu_N)A[(1 - \alpha_N)N_{12} + \alpha_N N_{21}]\}_{,2} \\ &\qquad - A_Q(z_{,12}Q_2 - z_{,22}Q_1), \\ \lambda_1 &= \{(1 + \nu_N)A[(1 - \alpha_N)N_{21} + \alpha_N N_{12}]\}_{,1} - \{A(N_{11} - \nu_N N_{22})\}_{,2} \\ &\qquad - A_Q(z_{,11}Q_2 - z_{,12}Q_1). \end{aligned} \tag{15}$$

Consideration will be limited from here on to the case of constant D, ν_M, α_M, A, ν_N and α_N. For this case, equations (13) to (15), with the help of (11) and (12), may be written in the form

$$\begin{aligned} Q_{1,1} + Q_{2,2} &+ z_{,11}F_{,22} - 2z_{,12}F_{,12} + z_{,22}F_{,11} \\ &- D_P[z_{,11}\lambda_{1,2} + z_{,12}(\lambda_{2,2} - \lambda_{1,1}) - z_{,22}\lambda_{2,1}] + p_n = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} \lambda_{2,1} - \lambda_{1,2} &+ z_{,11}w_{,22} - 2z_{,12}w_{,12} + z_{,22}w_{,11} \\ &- A_Q[z_{,11}Q_{2,2} - z_{,12}(Q_{1,2} + Q_{2,1}) + z_{,22}Q_{1,1}] = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} Q_1 = -D(\nabla^2 w)_{,1} &+ A_Q D\{\nabla^2 Q_1 - [\alpha_M - \nu_M(1 + \alpha_M)](Q_{2,1} - Q_{1,2})_{,2}\} \\ &- D_P[z_{,12}\lambda_1 + z_{,22}\lambda_2], \end{aligned} \tag{18}$$

$$\begin{aligned} Q_2 = -D(\nabla^2 w)_{,2} &+ A_Q D\{\nabla^2 Q_2 + [\alpha_M - \nu_M(1 + \alpha_M)](Q_{2,1} - Q_{1,2})_{,1}\} \\ &+ D_P[z_{,11}\lambda_1 + z_{,12}\lambda_2], \end{aligned} \tag{19}$$

$$\lambda_2 = A(\nabla^2 F)_{,1} + D_P A\{\nabla^2 \lambda_2 - [\alpha_N - \nu_N(1 - \alpha_N)](\lambda_{1,1} + \lambda_{2,2})_{,2}\} - A_Q[z_{,12}Q_2 - z_{,22}Q_1], \tag{20}$$

$$-\lambda_1 = A(\nabla^2 F)_{,2} - D_P A\{\nabla^2 \lambda_1 - [\alpha_N - \nu_N(1 - \alpha_N)](\lambda_{1,1} + \lambda_{2,2})_{,1}\} - A_Q[z_{,12}Q_1 - z_{,11}Q_2]. \tag{21}$$

When $A_Q = D_P = 0$ then introduction of (18) to (21) into (16) and (17), leads again to Marguerre's two simultaneous equations for w and F. Reductions of the general case with $A_Q \neq 0$ *and* $D_P \neq 0$ are found to be very much less simple and will not be considered further in this account. There are, however, some cases of intermediate generality and of independent interest which will be considered in what follows.

5. The case $D_P = 0$

Equations (18) and (19) now give

$$-D\nabla^4 w = (1 - A_Q D\nabla^2)(Q_{1,1} + Q_{2,2}). \tag{22}$$

Therewith equation (16), with $D_P = 0$, gives as one equation for w and F

$$-D\nabla^4 w + (1 - A_Q D\nabla^2)(z_{,11}F_{,22} - 2z_{,12}F_{,12} + z_{,22}F_{,11}) + (1 - A_Q D\nabla^2)p_n = 0. \tag{23}$$

A second equation for w and F follows upon introducing λ_2 and λ_1 from (20) and (21), with $D_P = 0$, into (17). We obtain, *since all terms with A_Q in the resultant equation cancel out*,

$$A\nabla^4 F + z_{,11}w_{,22} - 2z_{,12}w_{,12} + z_{,22}w_{,11} = 0. \tag{24}$$

Equations (23) and (24) together are an eighth order system, the same as for the case $D_P = A_Q = 0$, and have been obtained earlier by Naghdi [2].

In Naghdi's work it is proposed to take equations (23) and (24) together with equations equivalent to our (18) and (19) (with $D_P = 0$) as the governing differential equations. However, this system is of the twelfth order while, for $D_P = 0$, it should be of the tenth order. Accordingly a further reduction is needed. This reduction is as follows. Equations (18) and (19), with $D_P = 0$, imply the second order equation

$$Q_{1,2} - Q_{2,1} = (1 + \alpha_M)(1 - \nu_M)A_Q D\nabla^2(Q_{1,2} - Q_{2,1}). \tag{25}$$

We now have in (23) to (25) a tenth order system for the *three* dependent variables w, F and $Q_{1,2} - Q_{2,1}$. All other dependent variables may be expressed as combinations of derivatives of these three. In particular, it follows from (18), (19), (16) and (25), suitably differentiating (16) in order to eliminate $\nabla^2 Q_1$ and $\nabla^2 Q_2$ from (18) and (19), that Q_1 and Q_2 are given in the form

$$Q_1 = -D(\nabla^2 w)_{,1} - A_Q D[(LF + p_n)_{,1} - (1 + \alpha_M)(1 - \nu_M)(Q_{1,2} - Q_{2,1})_{,2}],$$

$$Q_2 = -D(\nabla^2 w)_{,2} - A_Q D[(LF + p_n)_{,2} + (1 + \alpha_M)(1 + \nu_M)(Q_{1,2} - Q_{2,1})_{,1}]. \tag{26}$$

Having (26), the remaining stress and strain resultants and couples follow from (20), (21), (11), (12) and the stress strain relations (III).

6. The case $A_Q = 0$

We conclude directly, making use of the static geometric duality that the problem may be reduced to equations of the form

$$A\nabla^4 F + (1 - D_P A\nabla^2)(z_{,11}w_{,22} - 2z_{,12}w_{,12} + z_{,22}w_{,11}) = 0, \tag{27}$$

$$-D\nabla^4 w + z_{,11}F_{,22} - 2z_{,12}F_{,12} + z_{,22}F_{,11} + p_n = 0. \tag{28}$$

and

$$\lambda_{2,2} + \lambda_{1,1} = (1 - \alpha_N)(1 + \nu_N)D_P A\nabla^2(\lambda_{2,2} + \lambda_{1,1}), \tag{29}$$

with all stress and strain quantities obtainable as linear combinations of derivatives of the three dependent variables F, w and $\lambda_{2,2} + \lambda_{1,1}$.

7. The flat plate

Setting $z = 0$ we have from (16) and (17) $Q_{1,1} + Q_{2,2} + p_n = 0$ and $\lambda_{2,1} - \lambda_{1,2} = 0$. From (18) and (19), and (20) and (21), respectively follows further

$$D\nabla^4 w = (1 - A_Q D\nabla^2)p_n, \qquad A\nabla^4 F = 0. \tag{30, 31}$$

In addition to this we have equation (25) for $Q_{1,2} - Q_{2,1}$ and equation (29) for $\lambda_{2,2} + \lambda_{1,1}$, now without restrictions on A_Q and D_P. All other quantities are expressed in terms of w, F, $Q_{1,2} - Q_{2,1}$, and $\lambda_{2,2} + \lambda_{1,1}$.

The contents of equations (25) and (30) are equivalent to earlier results for the effect of transverse shear deformation in plate theory [3], where the problem, for $p_n = 0$, is further reduced by satisfying the homogeneous equation (16) identically by setting $Q_1 = \partial\chi/\partial x_2$, $Q_2 = -\partial\chi/\partial x_1$.

The corresponding problem of generalized plane stress for F and the λ_i, in accordance with (29) and (31), has previously been considered by Schäfer [5], including recognition of the fact that this problem is the static-geometric dual of the transverse bending problem with transverse shear deformation included.

8. Uniform shallow spherical shell

Setting $z = \frac{1}{2}k(x_1^2 + x_2^2)$ the problem of the spherical shell as described by equations (16) to (21) may be reduced as follows.

We define two auxiliary quantities X and Y by

$$Q_{1,1} + Q_{2,2} = X, \qquad \lambda_{2,1} - \lambda_{1,2} = Y. \tag{32}$$

Equations (18) to (21) are then seen to imply the relations

$$X = -D\nabla^4 w + A_Q D\nabla^2 X - kD_P Y, \qquad Y = A\nabla^4 F + D_P A\nabla^2 Y + kA_Q X. \tag{33}$$

At the same time, equations (16) and (17), with $p_n = 0$ for simplicity, become

$$X + kD_P Y + k\nabla^2 F = 0, \qquad Y - kA_Q X + k\nabla^2 w = 0. \tag{34}$$

From this we have further

$$\frac{X}{k} = -\frac{\nabla^2 F - kD_P\nabla^2 w}{1 + k^2 A_Q D_P}, \qquad \frac{Y}{k} = -\frac{\nabla^2 w + kA_Q\nabla^2 F}{1 + k^2 A_Q D_P}. \tag{35}$$

Introduction of (35) into (33) leads to two simultaneous equations for w and F of the form

$$D\nabla^4 w - k(1 + k^2 A_Q D_P - A_Q D\nabla^2)\nabla^2 F = 0, \tag{36a}$$

$$A\nabla^4 F + k(1 + k^2 A_Q D_P - D_P A\nabla^2)\nabla^2 w = 0. \tag{36b}$$

Upon setting $A_Q = D_P = 0$ these reduce to the known form of the equations for uniform shallow spherical shells [4].

The eighth order system (36) is complemented by a fourth order system, as follows. We define quantities X^* and Y^* by writing

$$X^* = Q_{2,1} - Q_{1,2}, \qquad Y^* = \lambda_{2,2} + \lambda_{1,1}, \tag{37}$$

and obtain then from (18) to (21) the system

$$\begin{aligned} X^* &= (1 + \alpha_M)A_Q D_S \nabla^2 X^* + kD_P Y^*, \\ Y^* &= (1 - \alpha_N)D_P A_S \nabla^2 Y^* - kA_Q X^*, \end{aligned} \tag{38}$$

in generalization of equations (25) and (29) respectively.

9. Further reduction of the equations of the shallow spherical shell

Equations (36) as well as equation (38) may be solved in terms of solutions of the second order equation

$$\nabla^2 \chi = \mu^2 \chi, \tag{39}$$

for suitable values of μ.

Setting $X^* = a\chi$ and $Y^* = b\chi$ in (38) and taking account of (39) transforms (38) into two algebraic equations for the constant coefficients a and b of the form

$$\begin{aligned} a[1 - (1 + \alpha_M)\mu^2 A_Q D_S] - bkD_P &= 0, \\ akA_Q + b[1 - (1 - \alpha_N)\mu^2 D_P A_S] &= 0. \end{aligned} \tag{40}$$

The condition of vanishing determinant of (40) gives as equation for μ^2,

$$\begin{aligned} (1 - \alpha_N)(1 + \alpha_M)D_P A_Q D_S A_S \mu^4 - [(1 + \alpha_M)D_S A_Q + (1 - \alpha_N)D_P A_S]\mu^2 \\ + (1 + k^2 D_P A_Q) = 0, \end{aligned} \tag{41}$$

with solutions μ_1^2 and μ_2^2.

Accordingly, the system (38) has solutions of the form

$$X^* = a_1\chi_1 + a_2\chi_2, \qquad Y^* = b_1\chi_1 + b_2\chi_2, \tag{42}$$

where $a_i/b_i = kD_P[1 - (1 - \alpha_M)\mu_i^2 A_Q D_S]$.

Analogously, the system (36) has solutions of the form

$$\nabla^2 w = c_3\chi_3 + c_4\chi_4, \qquad \nabla^2 F = d_3\chi_3 + d_4\chi_4 \tag{43}$$

where $d_i/c_i = \mu_i^2 D/k(1 + k^2 A_Q D_P - \mu_i^2 A_Q D)$, and where μ_3^2 and μ_4^2 are the roots of

$$DA\mu^4 - k^2(DA_Q + AD_P)\mu^2 + k^2(1 + k^2 D_P A_Q) = 0. \tag{44}$$

From (43) follows further,

$$w = w_0 + c_3\mu_3^{-2}\chi_3 + c_4\mu_4^{-2}\chi_4, \qquad F = F_0 + d_3\mu_3^{-2}\chi_3 + d_4\mu_4^{-2}\chi_4, \tag{45}$$

where w_0 and F_0 are harmonic functions. Introduction of (45) into (35) then gives

$$X = a_3\chi_3 + a_4\chi_4, \qquad Y = b_3\chi_3 + b_4\chi_4, \tag{46}$$

where the a_i and b_i for $i = 3, 4$ depend suitably on the c_i and d_i.

In order to obtain the Q_i and λ_i we write, on the basis of (32) and (37)

$$\begin{aligned} \nabla^2 Q_1 &= X_{,1} - X^*_{,2}, & \nabla^2 Q_2 &= X_{,2} + X^*_{,1}, \\ \nabla^2 \lambda_1 &= Y^*_{,1} - Y_{,2}, & \nabla^2 \lambda_2 &= Y^*_{,2} + Y_{,1}. \end{aligned} \tag{47}$$

Observation in (47) of the differential equations (39) for the functions χ_i then gives

$$\begin{aligned} Q_1 &= (a_3\mu_3^{-2}\chi_3 + a_4\mu_4^{-2}\chi_4)_{,1} - (a_1\mu_1^{-2}\chi_1 + a_2\mu_2^{-2}\chi_2)_{,2}, \\ Q_2 &= (a_3\mu_3^{-2}\chi_3 + a_4\mu_4^{-2}\chi_4)_{,2} + (a_1\mu_1^{-2}\chi_1 + a_2\mu_2^{-2}\chi_2)_{,1}, \end{aligned} \tag{48}$$

with corresponding relations for λ_1 and λ_2. The solutions of the homogeneous system (47) are found to vanish upon substituting equations (48) into equations (16) to (19).

Having (48) it is readily seen that in changing from the cartesian coordinates x_1, x_2 to polar coordinates r, θ the corresponding expressions for Q_r and Q_θ are

$$\begin{aligned} Q_r &= (a_3\mu_3^{-2}\chi_3 + a_4\mu_4^{-2}\chi_4)_{,r} - r^{-1}(a_1\mu_1^{-2}\chi_1 + a_2\mu_2^{-2}\chi_2)_{,\theta}, \\ Q_\theta &= r^{-1}(a_3\mu_3^{-2}\chi_3 + a_4\mu_4^{-2}\chi_4)_{,\theta} + (a_1\mu_1^{-2}\chi_1 + a_2\mu_2^{-2}\chi_2)_{,r}, \end{aligned} \tag{49}$$

with the functions χ subject to the differential equation $\chi_{,rr} + r^{-1}\chi_{,r} + r^{-2}\chi_{,\theta\theta} = \mu^2\chi$. Analogous expressions are equally readily obtained for λ_r, λ_θ and for the resultants and couples N_{rr}, $N_{r\theta}$, $N_{\theta r}$, $N_{\theta\theta}$, M_{rr}, $M_{r\theta}$, $M_{\theta r}$, $M_{\theta\theta}$, P_r, P_θ.

10. Stretching, twisting and bending of pretwisted rectangular plates

A class of problems of some technical interest for which explicit solutions may be obtained through use of the differential equations (I), (II) and (III) is as follows. A shallow hyperbolic paraboloidal shell with middle surface equation $z = kx_1x_2$, has two opposite edges $x_2 = \pm b$, free of stress, and the other two edges $x_1 = \pm a$, are acted upon by equal and opposite axial forces N, twisting moments M_t, plate bending moments M_p and sheet bending moments M_s. It is assumed that the properties of the shell are independent of the axial coordinate x_1.

In order to solve the problem as stated, a semi-inverse procedure is adopted. With the restriction that D, A, ν_M, ν_N, α_M and α_N are independent of x_1, we determine the class of solutions of equations (I) to (III) with the property that all stress resultants are also independent of x_1. We then show that this class of solutions is such as to allow satisfaction of the boundary conditions

$$x_2 = \pm b: \qquad N_{21} = N_{22} = Q_2 = M_{21} = M_{22} = P_2 = 0, \tag{50}$$

together with the integral conditions.

$$\begin{aligned} &\int_{-b}^{b} N_{12}\,dx_2 = 0, && \int_{-b}^{b} (Q_1 + z_{,1}N_{11} + z_{,2}N_{12})\,dx_2 = 0, \\ &\int_{-b}^{b} N_{11}\,dx_2 = N, && \int_{-b}^{b} [M_{12} + z_{,1}P_1 - x_2(Q_1 + z_{,1}N_{11})]\,dx_2 = -M_t, \end{aligned} \tag{51}$$

$$\int_{-b}^{b} (M_{11} - z_{,2}P_1 + zN_{11})\, dx_2 = M_p, \int_{-b}^{b} (P_1 - x_2 N_{11})\, dx_2 = M_s, \tag{51}$$

for $x_1 = \pm a$.

Without surface load terms, with N_{ij} and Q_i independent of x_1, and with $z = kx_1x_2$, the differential equations (I) reduce to

$$\begin{gathered} N_{21,2} = 0, \qquad N_{22,2} = 0, \qquad Q_{2,2} + k(N_{12} + N_{21}) = 0, \\ M_{11,1} + M_{21,2} - Q_1 - kP_1 = 0, \qquad M_{12,1} + M_{22,2} - Q_2 + kP_2 = 0, \\ P_{1,1} + P_{2,2} + N_{12} - N_{21} = 0. \end{gathered} \tag{52}$$

At the same time, considering that as a consequence of the assumptions made, the ε_{ij} and γ_i are also independent of x_1, equations (II) become

$$\begin{gathered} -\kappa_{22,1} + \kappa_{12,2} = 0, \qquad \kappa_{21,1} - \kappa_{11,2} = 0, \\ \lambda_{2,1} - \lambda_{1,2} + k(\kappa_{12} + \kappa_{21}) = 0, \qquad \varepsilon_{12,2} + \lambda_2 + k\gamma_2 = 0, \\ \varepsilon_{11,2} + \lambda_1 - k\gamma_1 = 0, \qquad \gamma_{1,2} + \kappa_{12} - \kappa_{21} = 0. \end{gathered} \tag{53}$$

Equations (52) and (53), jointly with the boundary conditions (50), are reduced step by step as follows.

The last four compatibility equations imply that λ_i, κ_{12} and κ_{21} are independent of x_1. The first two compatibility equations in turn imply that κ_{11} is independent of x_2 and κ_{22} is linear in x_1.

The stress strain relations then give P_i, M_{12} and M_{21} independent of x_1. Therewith, terms involving M_{12} and P_1 disappear from the last two equations in (52).

First integrals of the simplified equilibrium equations together with the boundary conditions (50) give

$$N_{21} = 0, \qquad N_{22} = 0, \qquad M_{22} = 0, \qquad Q_2 - kP_2 = 0. \tag{54}$$

The result that $M_{22} = 0$ throughout the shell, together with the results that κ_{11} is independent of x_2, κ_{22} is linear in x_1, κ_{12} is independent of x_1 and $\kappa_{12,2} = \kappa_{22,1}$ completely determines κ_{11}, κ_{22} and κ_{12} in terms of constants of integration C_1, C_3 and C_5 in the form

$$\kappa_{11} = C_1 + C_5x_1, \qquad \kappa_{22} = -\nu_M\kappa_{11}, \qquad \kappa_{12} = -(C_3 + \nu_M C_5 x_2). \tag{55}$$

The third, fifth and sixth reduced compatibility equations can be integrated to give, in terms of two additional constants of integration C_2 and C_4,

$$\begin{aligned} \lambda_1 - k\gamma_1 &= 2[C_4 - k(C_3x_2 + \tfrac{1}{2}\nu_M C_5 x_2^2)] \\ \varepsilon_{11} &= C_2 - 2C_4x_2 + k(C_3x_2^2 + \tfrac{1}{3}\nu_M C_5 x_2^3) \end{aligned} \tag{56}$$

The foregoing steps leave four first order differential equations

$$\begin{gathered} \varepsilon'_{12} + \lambda_2 + k\gamma_2 = 0, \qquad \gamma'_1 - \kappa_{21} = C_3 + \nu_M C_5 x_2, \\ P'_2 + N_{12} = 0, \qquad M'_{21} - Q_1 - kP_1 = -D(1 - \nu_M^2)C_5, \end{gathered} \tag{57}$$

where primes indicate differentiation with respect to x_2, and where P_i and M_{21} are independent of x_1. The associated boundary conditions are $P_2(\pm b) = M_{21}(\pm b) = 0$.

Equations (57) together with the appropriate stress strain relations separate into two distinct sets of two simultaneous first order equations. One of these sets, for P_2, N_{12} and Q_2, is homogeneous including the associated boundary conditions. Accordingly, we conclude immediately

$$P_2 = N_{12} = Q_2 = \lambda_2 = \gamma_2 = \varepsilon_{12} = \varepsilon_{21} = 0. \tag{58}$$

The remaining two first order differential equations may be written in the form*

$$\begin{aligned} M'_{21} - (1 + k^2 D_P A_Q)Q_1 &= 2kD_P C_4 - 2k^2 D_P(C_3 x_2 + \tfrac{1}{2}\nu_M C_5 x_2^2) \\ &\quad - D(1 - \nu_M^2)C_5, \\ (1 - \alpha)D_S(A_Q Q_1)' - M_{21} &= D_S(C_3 + \nu_M C_5 x_2). \end{aligned} \tag{59}$$

The associated boundary conditions are $M_{21}(\pm b) = 0$.

Besides M_{21} and Q_1, we are left with the following four non-vanishing couples and resultants

$$\begin{aligned} M_{11} &= D(1 - \nu_M^2)(C_1 + C_5 x_1), \\ (1 - \alpha)M_{12} &= (2\alpha - 1)D_S(C_3 + \nu_M C_5 x_2) + \alpha M_{21}, \\ P_1 &= D_P(kA_Q Q_1 + 2C_4 - 2kC_3 x_2 - \nu_M k C_5 x_2^2), \\ N_{11} &= \frac{1}{A}(C_2 - 2C_4 x_2) + \frac{k}{A}(C_3 x_2^2 + \tfrac{1}{3}\nu_M C_5 x_2^3). \end{aligned} \tag{60}$$

Determination of the five constants of integration is effected through use of the boundary conditions (51) for $x_1 = \pm a$. In view of the fact that $N_{12} = 0$, and with $z = kx_1 x_2$, these conditions may be written in the form

$$\begin{gathered} \int_{-b}^{b} N_{11}\, dx_2 = N, \qquad \int_{-b}^{b} (Q_1 + kx_2 N_{11})\, dx_2 = 0, \\ \int_{-b}^{b} [M_{12} - x_2 Q_1 + kx_2(P_1 - x_2 N_{11})]\, dx_2 = -M_t, \\ \int_{-b}^{b} [M_{11} - kx_1(P_1 - x_2 N_{11})]\, dx_2 = M_p, \qquad \int_{-b}^{b} (P_1 - x_2 N_{11})\, dx_2 = M_s. \end{gathered} \tag{61}$$

The two constants C_1 and C_5 are determined directly by means of the last two equations in (61). Upon introduction of (60) these reduce to

$$C_1 = \int_{-b}^{b} D(1 - \nu_M^2)\, dx_2 = M_p, \qquad C_5 = \int_{-b}^{b} D(1 - \nu_M^2)\, dx_2 = kM_s, \tag{62}$$

indicating that the values of these two constants are unaffected by transverse shear deformation and the existence of couple-stress stress-couples.

The remaining three constants C_2, C_3, C_4 follow upon introducing equations (60) in terms of the solution of (59) into the first three conditions in (61).

* For simplicity, we will take $-\alpha_M = \alpha_N = \alpha$ in what follows.

Further discussion will be limited to the case of constant D, A, ν_M, ν_N and α. For this case we obtain from (59) and (60) the following expressions for the resultant Q_1 and the couples M_{12}, M_{21} and P_1:

$$Q_1 = \frac{bC_3}{\Delta}\left\{\frac{\mu D_S}{b^2\Delta}[1 + (2\alpha - 1)k^2 D_P A_Q]\frac{\sinh(\mu x_2/b)}{\cosh\mu} + 2k^2 D_P\left(\frac{x_2}{b}\right)\right\} - \frac{2kD_P C_4}{\Delta}$$
$$+\frac{C_5}{\Delta}\left\{\frac{\mu\nu_M D_S}{\Delta}[1 + (2\alpha - 1)k^2 D_P A_Q]\frac{\cosh(\mu x_2/b)}{\sinh\mu}\right.$$
$$\left. + D_S\left[1 + \frac{2(1-\alpha)\nu_M k^2 D_P A_Q}{\Delta}\right] + \nu_M\lambda^2 D_P\left(\frac{x^2}{b}\right)^2\right\},$$

$$M_{21} = -\frac{D_S[1 + (2\alpha - 1)k^2 D_P A_Q]}{\Delta}\left\{C_3\left[1 - \frac{\cosh(\mu x_2/b)}{\cosh\mu}\right]\right.$$
$$\left. + \nu_M b C_5\left[\frac{x_2}{b} - \frac{\sinh(\mu x_2/b)}{\sinh\mu}\right]\right\}, \tag{63}$$

$$M_{12} = -\frac{D_S C_3}{\Delta}\left\{[1 - (2\alpha - 1)k^2 D_P A_Q] - \frac{\alpha}{1-\alpha}[1 + (2\alpha - 1)k^2 D_P A_Q]\frac{\cosh(\mu x_2/b)}{\cosh\mu}\right\}$$
$$-\frac{\nu_M b D_S C_5}{\Delta}\left\{[1 - (2\alpha - 1)k^2 D_P A_Q]\frac{x_2}{b}\right.$$
$$\left. - \frac{\alpha}{1-\alpha}[1 + (2\alpha - 1)k^2 D_P A_Q]\frac{\sinh(\mu x_2/b)}{\sinh\mu}\right\},$$

$$P_1 = \frac{D_P}{\Delta}\left\{\lambda C_3\left[\frac{1 + (2\alpha - 1)k^2 D_P A_Q}{(1-\alpha)\mu}\frac{\sinh(\mu x_2/b)}{\cosh\mu} - 2\frac{x_2}{b}\right] + 2C_4\right.$$
$$+ \lambda b C_5\left[\frac{\nu_M + (2\alpha - 1)\nu_M k^2 D_P A_Q}{(1-\alpha)\mu}\frac{\cosh(\mu x_2/b)}{\sinh\mu}\right.$$
$$\left.\left. + \frac{D_S A_Q}{b^2}\left(1 + \frac{2\nu_M k^2 D_P A_Q}{\Delta}\right) - \nu_M\left(\frac{x_2}{b}\right)^2\right]\right\}.$$

In this

$$\lambda = kb, \qquad \Delta = 1 + k^2 D_P A_Q, \qquad \mu^2 = \frac{b^2\Delta}{(1-\alpha)D_S A_Q}. \tag{64}$$

We note that for large values of μ, the contributions of the terms involving hyperbolic functions are in the nature of *edge effects*. The edge effect disappears (i.e., the width of the edge zone vanishes) for a shell which is unable to undergo transverse shear deformations.

We next use the second condition in (61) to express C_4 in terms of C_5

$$C_4 = \frac{kb^2 C_5}{10[1 + (3D_P A/b^2\Delta)]}\left\{\nu_M + \frac{5\nu_M D_P A}{b^2\Delta} + \frac{15DA(1-\nu_M^2)}{k^2 b^4\Delta}\right\}, \tag{65}$$

and write the two conditions involving N and M_t as two equations for C_2 and C_3 in the form

$$C_2 + \tfrac{1}{3}\lambda b C_3 = \frac{NA}{2b},$$

$$\frac{1}{3}\lambda C_2 + bC_3\left\{\frac{\lambda^2}{5} + \frac{4\lambda^2 D_P A}{3b^2\Delta} + \frac{D_S A}{(1-\alpha)b^2}\left[(1-2\alpha) + \frac{\langle 1 + (2\alpha - 1)k^2 D_P A_Q\rangle^2}{\Delta}\left(1 - \frac{\tanh\mu}{\mu}\right)\right]\right\} = \frac{M_t A}{2b^2}. \tag{66}$$

Upon solving equations (66) for C_2 and C_3, we get

$$C_3 = \frac{A}{6b^3\Delta_0}(3M_t - \lambda b N),$$

$$C_2 = \frac{NA}{2b} + \frac{\lambda A(\lambda b N - 3M_t)}{18b^2\Delta_0} = \frac{NA}{2b}\left(1 + \frac{\lambda^2}{9\Delta_0}\right) - \frac{\lambda M_t A}{6b^2\Delta_0}, \tag{67}$$

with

$$\Delta_0 = \frac{4\lambda^2}{45}\left(1 + \frac{15 D_P A}{b^2\Delta}\right) + \frac{D_S A}{(1-\alpha)b^2}\left\{(1-2\alpha) + \frac{[1 + (2\alpha - 1)k^2 D_P A_Q]^2}{\Delta^2}\left(1 - \frac{\tanh\mu}{\mu}\right)\right\}. \tag{68}$$

Furthermore, equations (62) give, with D and ν_M as constants,

$$C_1 = \frac{M_p}{2bD(1-\nu_M^2)}, \qquad C_5 = \frac{kM_s}{2bD(1-\nu_M^2)}, \tag{69}$$

and then equation (65) becomes

$$C_4 = \frac{M_s}{20bD(1-\nu_M^2)[1 + (3D_P A/b^2\Delta)]}\left\{\nu_M\lambda^2\left(1 + \frac{5D_P A}{b^2\Delta}\right) + \frac{15DA(1-\nu_M^2)}{b^2\Delta}\right\}. \tag{70}$$

11. Torsion of a flat plate

Setting $M_p = M_s = N = k = 0$, the results obtained in the last section reduce to

$$M_{21} = -\frac{M_t}{4b}\left[\frac{1 - \dfrac{\cosh(\mu_0 x_2/b)}{\cosh\mu_0}}{1 - \dfrac{\tanh\mu_0}{2(1-\alpha)\mu_0}}\right], \qquad M_{12} = -\frac{M_t}{4b}\left[\frac{1 - \dfrac{\alpha}{1-\alpha}\dfrac{\cosh(\mu_0 x_2/b)}{\cosh\mu_0}}{1 - \dfrac{\tanh\mu_0}{2(1-\alpha)\mu_0}}\right], \tag{71}$$

$$Q_1 = \frac{\mu_0 M_t}{4b^2}\left[\frac{\dfrac{\sinh(\mu_0 x_2/b)}{\cosh\mu_0}}{1 - \dfrac{\tanh\mu_0}{2(1-\alpha)\mu_0}}\right], \qquad \mu_0^2 = \frac{b^2}{(1-\alpha)D_S A_Q}, \tag{72}$$

while all the remaining stress resultants and couples vanish identically.

For $\alpha = \frac{1}{2}$, we have

$$M_{12} = M_{21} = -\frac{M_t}{4b}\left[\frac{1 - \dfrac{\cosh(\mu_1 x_2/b)}{\cosh \mu_1}}{1 - \dfrac{\tanh \mu_1}{\mu_1}}\right], \qquad Q_1 = \frac{\mu_1 M_t}{4b^2}\left[\frac{\dfrac{\sinh(\mu_1 x_2/b)}{\cosh \mu_1}}{1 - \dfrac{\tanh \mu_1}{\mu_1}}\right], \tag{73}$$

with $\mu_1^2 = 2b^2/D_S A_Q$, which agrees with previous results in [3].

On the other hand, for $\alpha = 0$, we have

$$M_{12} = -\frac{M_t}{4b}\left[\frac{1}{1 - \dfrac{\tanh(\mu_1/\sqrt{2})}{\mu_1/\sqrt{2}}}\right], \qquad M_{21} = -\frac{M_t}{4b}\left[\frac{1 - \dfrac{\cosh(\mu_1 x_2/\sqrt{2}b)}{\cosh(\mu_1/\sqrt{2})}}{1 - \dfrac{\tanh(\mu_1/\sqrt{2})}{\mu_1/\sqrt{2}}}\right],$$

$$Q_1 = \frac{\mu_1 M_t}{4\sqrt{2}b^2}\left[\frac{\dfrac{\sinh(\mu_1 x_2/\sqrt{2}b)}{\cosh(\mu_1/\sqrt{2})}}{1 - \dfrac{\tanh(\mu_1/\sqrt{2})}{\mu_1/\sqrt{2}}}\right]. \tag{74}$$

For values of A_Q small enough to make $\mu_0 \ll 1$, the difference between (73) and (74) is significant only in a narrow region near the edges $x_2 = \pm b$, indicating that in a theory in which the influence of transverse shear deformation is confined to narrow edge zones the results of shell theory are insensitive, in essence, to the value of α. For values of A_Q for which μ_0 is not small compared to unity, that is for cases in which transverse shear deformation is a first order effect, this difference is appreciable throughout the plate, indicating the importance of assuming for these cases appropriate values of α in the stress strain relations (III), either on the basis of three-dimensional considerations or on the basis of experiments.

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