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TWO VARIATIONAL THEOREMS FOR THIN SHELLS[1]

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The purpose of the present note is to establish two variational theorems which are particularly appropriate for the stress function formulation of boundary value problems in the linear theory of thin elastic shells. These principles are for strains and stress functions and for strains, stresses and stress functions and may be thought of as the static-geometric analogues of principles stated by Reissner for stresses and displacements [1] and for stresses, strains and displacements [3].

With reference to a set of general orthogonal middle surface coordinates ξ_1 and ξ_2, the elastostatic behavior of a thin shell is here described by strain resultant and couple vectors $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$, and stress resultant and couple vectors $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{M}_1$ and $\mathbf{M}_2$. The four strain measure vectors are subject to two vector compatibility equations [2]

$$\begin{aligned} (\alpha_2\boldsymbol{\kappa}_2)' - (\alpha_1\boldsymbol{\kappa}_1)^{\bullet} &= 0 \\ (\alpha_2\boldsymbol{\varepsilon}_2)' - (\alpha_1\boldsymbol{\varepsilon}_1)^{\bullet} + \mathbf{r}' \times (\alpha_2\boldsymbol{\kappa}_2) - \mathbf{r}^{\bullet} \times (\alpha_1\boldsymbol{\kappa}_1) &= 0 \end{aligned} \qquad (1)$$

where α_1 and α_2 are coefficients of the first fundamental form and primes and dots indicate differentiations with respect to ξ_1 and ξ_2 respectively.

The four stress measure vectors are subject to two vector equilibrium equations. Assuming the absence of distributed surface loads, these equations are satisfied identically by the two stress function vectors $\mathbf{F}$ and $\mathbf{H}$ in the form [2]

$$\begin{aligned} \alpha_2\mathbf{N}_1 &= \mathbf{F}^{\bullet}, \qquad & \alpha_2\mathbf{M}_1 &= \mathbf{H}^{\bullet} + \mathbf{r}^{\bullet} \times \mathbf{F} \\ \alpha_1\mathbf{N}_2 &= -\mathbf{F}', \qquad & \alpha_1\mathbf{M}_2 &= -\mathbf{H}' - \mathbf{r}' \times \mathbf{F} \end{aligned} \qquad (2)$$

With component representations of stress and strain measure vectors of the form

$$\begin{aligned} \boldsymbol{\varepsilon}_j &= \epsilon_{j1}\mathbf{t}_1 + \epsilon_{j2}\mathbf{t}_2 + \gamma_j\mathbf{n}, \qquad & \boldsymbol{\kappa}_j &= -\kappa_{j2}\mathbf{t}_1 + \kappa_{j1}\mathbf{t}_2 + \beta_j\mathbf{n} \\ \mathbf{N}_j &= N_{j1}\mathbf{t}_1 + N_{j2}\mathbf{t}_2 + Q_j\mathbf{n}, \qquad & \mathbf{M}_j &= -M_{j2}\mathbf{t}_1 + M_{j1}\mathbf{t}_2 + P_j\mathbf{n} \end{aligned} \qquad (3)$$

equations (1) and (2) are supplemented by twelve stress strain relations [1, 2]

$$N_{ij} = \frac{\partial A}{\partial \epsilon_{ij}}, \qquad M_{ij} = \frac{\partial A}{\partial \kappa_{ij}}, \qquad Q_j = \frac{\partial A}{\partial \gamma_j}, \qquad P_j = \frac{\partial A}{\partial \beta_j} \qquad (4)$$

which may be written in vector form as

$$\mathbf{N}_j = \frac{\partial A}{\partial \boldsymbol{\varepsilon}_j}, \qquad \mathbf{M}_j = \frac{\partial A}{\partial \boldsymbol{\kappa}_j} \qquad (5)$$

where

$$\frac{\partial A}{\partial \boldsymbol{\varepsilon}_j} = \frac{\partial A}{\partial \epsilon_{j1}}\mathbf{t}_1 + \frac{\partial A}{\partial \epsilon_{j2}}\mathbf{t}_2 + \frac{\partial A}{\partial \gamma_j}\mathbf{n}, \qquad \frac{\partial A}{\partial \boldsymbol{\kappa}_j} = -\frac{\partial A}{\partial \kappa_{j2}}\mathbf{t}_1 + \frac{\partial A}{\partial \kappa_{j1}}\mathbf{t}_2 + \frac{\partial A}{\partial \beta_j}\mathbf{n} \qquad (6)$$

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Examples of the determination of the stress potential A for certain known classes of stress strain relations may be found in [4, 5].

We consider a shell with a simply connected middle surface S whose boundary curve C has no corner. Let

$$I_{ss} = \iint_S (\mathbf{N}_1 \cdot \boldsymbol{\varepsilon}_1 + \mathbf{N}_2 \cdot \boldsymbol{\varepsilon}_2 + \mathbf{M}_1 \cdot \boldsymbol{\kappa}_1 + \mathbf{M}_2 \cdot \boldsymbol{\kappa}_2 - A)\alpha_1 \alpha_2 \, d\xi_1 \, d\xi_2 + \int_{C_d} (\bar{\boldsymbol{\kappa}}_s \cdot \mathbf{H} + \bar{\boldsymbol{\varepsilon}}_s \cdot \mathbf{F}) \, ds + \int_{C_s} [\boldsymbol{\kappa}_s \cdot (\mathbf{H} - \bar{\mathbf{H}}) + \boldsymbol{\varepsilon}_s \cdot (\mathbf{F} - \bar{\mathbf{F}})] \, ds \tag{7}$$

In this the subscript s denotes the direction tangent to the edge curve and the barred quantities are prescribed functions of position along the edge of the shell.

Our variational theorem is as follows. With the stress function representations (2) as equations of definition and with all strain measures and stress functions being varied independently, the variational equations $\delta I_{ss} = 0$ has as Euler differential equations the stress strain relations (5) and the compatibility equations (1), and as Euler boundary conditions, displacement boundary conditions (formulated in terms of strain measures) of the form

$$\Delta \boldsymbol{\kappa}_s \equiv \boldsymbol{\kappa}_s - \bar{\boldsymbol{\kappa}}_s = 0, \qquad \Delta \boldsymbol{\varepsilon}_s \equiv \boldsymbol{\varepsilon}_s - \bar{\boldsymbol{\varepsilon}}_s = 0 \tag{8}$$

on the portion C_d of the boundary curve, and stress boundary conditions (formulated in terms of stress functions) of the form

$$\Delta \mathbf{F} \equiv \mathbf{F} - \bar{\mathbf{F}} = 0, \qquad \Delta \mathbf{H} \equiv \mathbf{H} - \bar{\mathbf{H}} = 0 \tag{9}$$

on the portion C_s of the boundary curve.

Since $\boldsymbol{\kappa}_s$ and $\boldsymbol{\varepsilon}_s$ involve only derivatives of the translational and rotational displacement vectors $\mathbf{u}$ and $\boldsymbol{\phi}$ along the edge of the shell [2], $\bar{\boldsymbol{\kappa}}_s$ and $\bar{\boldsymbol{\varepsilon}}_s$ are known once $\bar{\mathbf{u}}$ and $\bar{\boldsymbol{\phi}}$ are prescribed. On the other hand, the physical meaning of the condition (9) can be seen from the relations

$$\begin{aligned} \int_{C_s} \mathbf{N}_\nu \, ds &= \int_{s_i}^{s_0} \frac{\partial \mathbf{F}}{\partial s} \, ds = [\mathbf{F}]_{s_i}^{s_0} \\ \int_{C_s} (\mathbf{M}_\nu + \mathbf{r} \times \mathbf{N}_\nu) \, ds &= \int_{s_i}^{s_0} \left(\frac{\partial \mathbf{H}}{\partial s} + \frac{\partial \mathbf{r}}{\partial s} \times \mathbf{F} + \mathbf{r} \times \frac{\partial \mathbf{F}}{\partial s} \right) ds \\ &= [\mathbf{H} + \mathbf{r} \times \mathbf{F}]_{s_i}^{s_0} \end{aligned} \tag{10}$$

where ν indicates the direction normal to the edge curve.

The above variational principle leads us to an alternate, more compact, form of the two contracted stress boundary conditions for inextensional bending theory of shells which were obtained in [1]. For such a theory, we have $\boldsymbol{\varepsilon}_s = 0$ and, from the compatibility equations, $\beta_s = 0$ (β_s being the normal component of $\boldsymbol{\kappa}_s$). The only variations which remain in the integral along C_s are $\delta\kappa_{ss}$ and $\delta\kappa_{s\nu}$; therewith the only two Euler stress boundary conditions are

$$\Delta H_\nu = \Delta H_s = 0 \tag{11}$$

It may be verified that they in fact imply the contracted conditions obtained in [1].*

We next state a variational principle for strains, stresses and stress functions which in addition to the compatibility equations and stress strain relations also has the stress function representations (2) as Euler equations. To this end, we take the stress strain relations in terms of a strain potential B in the form

$$\boldsymbol{\varepsilon}_j = \frac{\partial B}{\partial \mathbf{N}_j}, \qquad \boldsymbol{\kappa}_j = \frac{\partial B}{\partial \mathbf{M}_j} \tag{12}$$

and consider

$$\begin{aligned} I_{sss} = \iint_S \Big\{ & \left(\frac{\mathbf{F}^\circ}{\alpha_2} - \mathbf{N}_1\right)\cdot \boldsymbol{\varepsilon}_1 + \left(-\frac{\mathbf{F}'}{\alpha_1} - \mathbf{N}_2\right)\cdot \boldsymbol{\varepsilon}_2 + \left(\frac{\mathbf{H}^\circ + \mathbf{r}^\circ \times \mathbf{F}}{\alpha_2} - \mathbf{M}_1\right)\cdot \boldsymbol{\kappa}_1 \\ & + \left(-\frac{\mathbf{H}' + \mathbf{r}' \times \mathbf{F}}{\alpha_1} - \mathbf{M}_2\right)\cdot \boldsymbol{\kappa}_2 + B\Big\}\, \alpha_1\, \alpha_2\, d\xi_1\, d\xi_2 \\ & + \int_{C_d} (\bar{\boldsymbol{\kappa}}_s \cdot \mathbf{H} + \bar{\boldsymbol{\varepsilon}}_s \cdot \mathbf{F})\, ds + \int_{C_s} [\boldsymbol{\kappa}_s \cdot (\mathbf{H} - \bar{\mathbf{H}}) + \boldsymbol{\varepsilon}_s \cdot (\mathbf{F} - \bar{\mathbf{F}})]\, ds \end{aligned} \tag{13}$$

It is readily verified that with stress, strain and stress function vectors varied independently, the variational equation $\delta I_{sss} = 0$ gives as Euler differential equations the stress function representations (2), the stress strain relations (12) and the compatibility equations (1), and as Euler boundary conditions equations (8) and (9).

Finally, we note that, by the method of Lagrange multipliers indicated in [1], the appropriate displacement boundary conditions (formulated in terms of strain measures) for the classical theory of shells (i.e., a theory with $\mathrm{P}_i = \gamma_i = 0$) may be shown to be the four contracted conditions

$$\boxed{\Delta\kappa_{ss} + \frac{\Delta\epsilon_{s\nu}}{R_{s\nu}} = \Delta\kappa_{s\nu} - \frac{\Delta\epsilon_{s\nu}}{R_{ss}} = \Delta\epsilon_{ss} = \Delta\beta_s - \frac{\partial(\Delta\epsilon_{s\nu})}{\partial s} = 0} \tag{14}$$

and these conditions are the static geometric analogues of the Kirchhoff-Bassett contracted stress boundary conditions [1].

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* Similar considerations for the variational principle for stresses and displacements lead to $\Delta u_\nu = \Delta u_s = 0$ as the two appropriate displacement boundary conditions for a membrane theory of shells.