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Pure Bending of Shallow Helicoidal Shells[1]

The solution of the problem of pure bending of shallow helicoidal shells is obtained in terms of elementary functions. Numerical results for the direct and bending stresses as well as for the overall stiffness coefficient of the shell are given for various pitch-to-thickness ratios. An interesting aspect of the analysis is that the displacement state of the shell is multivalued in the polar angle θ while the stress state is unsymmetric but periodic in θ. Moreover, unlike the problem of axial torsion and extension where there is no edge effect, the stress state for the present problem exhibits both an interior state and an edge zone state.

Introduction

THE PRESENT work is concerned with the linear elastostatic behavior of a thin homogeneous isotropic shallow helicoidal shell subject to equal and opposite bending moments at the axial edges, Fig. 1. The problem can be thought of as the shell-theoretic formulation of the problem of pure bending of helicoidal springs [5][2] and will be of practical importance for springs with a rectangular cross section so wide that beam theory becomes inapplicable.

It was found in [4] that the solution of the problem of axial extension and torsion of helicoidal shells requires an axisymmetric stress distribution which is associated with a multivalued displacement state. We will show that the solution to the present problem also requires a multivalued displacement state associated this time with a nonaxisymmetric stress state. We will show also that a shell subject to end bending moments exhibits an (inextensional bending) interior stress state as well as an edge zone state, while no edge effect is present in a shell under axial forces and/or torques.

The present work will be confined to shells with small pitch for which Marguerre's shallow shell equations are applicable. However, a reduction of the problem to a two-point boundary-value problem in ordinary differential equations similar to that carried out herein is also possible for nonshallow shells [6]. The exact solution for the other extreme case of a slightly pretwisted strip (i.e., a helicoidal shell with large pitch) has been obtained previously in [3].

The limiting case of a ring plate sector (a shell with zero pitch) of our problem was discussed in [2]. Our results show that when the pitch of the shell is large compared to its thickness, the behavior of the structure differs significantly from that of a flat plate and, except for edge effects, is asymptotic to an inextensional bending shell behavior.

Formulation

In cylindrical coordinates (r, θ, z), the middle surface of a helicoidal shell is given by the equation $z = a\theta$ where the constant $2\pi a$ is the pitch of the helicoid. Within the framework of a linear shallow shell theory without surface loads, the inplane stress resultants N_r, N_θ, and $N_{r\theta}$ are given in terms of a stress function F by

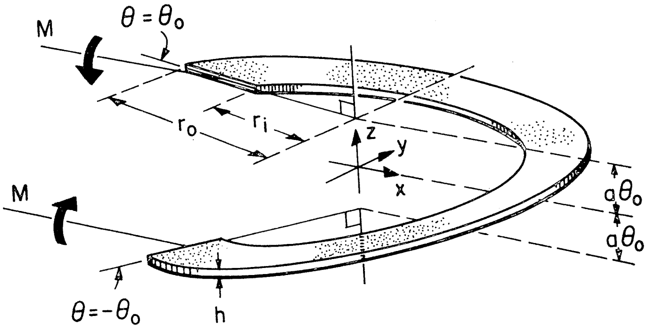

Fig. 1 Helicoidal shell subject to equal and opposite end-bending moments in the x-direction

$$N_r = r^{-1}F' + r^{-2}F^{\cdot\cdot}, \qquad N_\theta = F'', \qquad N_{r\theta} = -(r^{-1}F^{\cdot})' \quad (1)$$

where primes and dots indicate differentiation with respect to r and θ, respectively. The stress couples M_r, M_θ, and $M_{r\theta}$ and the transverse shear resultants Q_r and Q_θ are given in terms of the axial middle surface displacement component w by

$$\begin{aligned} M_r &= -D[w'' + \nu_b(r^{-1}w' + r^{-2}w^{\cdot\cdot})], \\ M_\theta &= -D[\nu_b w'' + r^{-1}w' + r^{-2}w^{\cdot\cdot}], \\ M_{r\theta} &= -D(1 - \nu_b)(r^{-1}w^{\cdot})', \\ Q_r &= -D(\nabla^2 w)', \qquad Q_\theta = -Dr^{-1}(\nabla^2 w)^{\cdot} \end{aligned} \quad (2)$$

where $\nabla^2(\quad) = (\quad)'' + r^{-1}(\quad)' + r^{-2}(\quad)^{\cdot\cdot}$, D is the bending stiffness of the shell, and ν_b the corresponding effective Poisson's ratio. The two unknown functions w and F are determined by the two coupled partial differential equations

$$D\nabla^2\nabla^2 w = 2ar^{2}(r^{-1}F^{\cdot})', \qquad A\nabla^2\nabla^2 F = -2ar^{-2}(r^{-1}w^{\cdot})' \quad (3)$$

where $1/A$ is the stretching stiffness of the shell.

In the derivation of the second equation in (3), use has been made of stress-strain relations

$$\begin{aligned} \epsilon_r = A(N_r - \nu_s N_\theta), \qquad \epsilon_\theta = A(N_\theta - \nu_s N_r), \\ \gamma_{r\theta} = 2(1 + \nu_s)AN_{r\theta} \end{aligned} \quad (4)$$

where the middle surface strain components ϵ_r, ϵ_θ, and $\gamma_{r\theta}$ are given in terms of w and the radial and circumferential displacement components, u_r and u_θ, by

$$\begin{aligned} \epsilon_r = u_r', \qquad \epsilon_\theta = r^{-1}(u_r + u_\theta^{\cdot}) + ar^{-2}w^{\cdot} \\ \gamma_{r\theta} = r^{-1}u_r^{\cdot} + r(r^{-1}u_\theta)' + ar^{-1}w' \end{aligned} \quad (5)$$

The force and moment equilibrium equations for stress resultants and couples are

$$\begin{aligned} (rN_r)' + N_{r\theta}^{\cdot} - N_\theta = 0, \qquad (rN_{r\theta})' + N_\theta^{\cdot} + N_{r\theta} = 0, \\ (rQ_r)' + Q_\theta^{\cdot} - 2ar^{-1}N_{r\theta} = 0 \end{aligned} \quad (6)$$

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[2] Numbers in brackets designate References at end of paper.

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$$(rM_r)' + M_{r\theta}^{\cdot} - M_\theta - rQ_r = 0, \qquad (6)$$
$$(rM_{r\theta})' + M_\theta^{\cdot} + M_{r\theta} - rQ_\theta = 0 \qquad \text{(Cont.)}$$

While these are satisfied identically by the given expressions for the resultants and couples provided F and w satisfy (3), we will use them on a later occasion to simplify the discussion of axial-edge conditions.

The foregoing set of differential equations is supplemented by appropriate sets of boundary conditions. In this work, we consider a shell bounded by $r = r_i$, $r = r_0$ $(0 < r_i < r_0)$, and $\theta = \pm\theta_0$. Along the radial edges ($r = r_i$ and $r = r_0$), the shell is assumed to be free of tractions. The homogeneous Kirchhoff-Bassett stress boundary conditions are:

$$N_r = N_{r\theta} = M_r = Q_r + r^{-1}M_{r\theta}^{\cdot} = 0 \qquad (7)$$

for $r = r_i$ and $r = r_0$. Along the axial edges $\theta = \pm\theta_0$, we require that the resultant force vanish and that the resultant moment be equal to $\pm M$, turning in the direction of the z-axis, Fig. 1. In terms of the stress resultants and stress couples, these requirements take the form

$$\int_{r_i}^{r_0} (N_\theta \cos\theta - N_\theta \sin\theta)dr = 0,$$
$$\int_{r_i}^{r_0} (N_{r\theta} \sin\theta + N_\theta \cos\theta)dr = 0,$$
$$\int_{r_i}^{r_0} (ar^{-1}N_\theta + Q_\theta + M'_{r\theta})dr - [2M_{r\theta}]_{r_i}^{r_0} = 0,$$
$$\int_{r_i}^{r_0} rN_\theta dr = 0, \qquad (8)$$
$$-\int_{r_i}^{r_0} [(aN_\theta + rQ_\theta + rM'_{r\theta})\cos\theta + M_\theta \sin\theta]dr + [2rM_{r\theta}\cos\theta]_{r_i}^{r_0} = 0,$$
$$\int_{r_i}^{r_0} [(aN_\theta + rQ_\theta + rM'_{r\theta})\sin\theta - M_\theta\cos\theta]dr - [2rM_{r\theta}\sin\theta]_{r_i}^{r_0} = M$$

where the nonintegrated terms represent the corner forces introduced by the assumption of negligible transverse shear deformation.

In the subsequent development, we seek a solution to the differential equations (3), which satisfies the boundary conditions (7) at the radial edges, and the overall conditions (8) at the axial edges. The inplane displacement components, u_r and u_θ, can then be computed through (5).

Reduction

Motivated by the result for a flat plate (see Appendix), we consider solutions to the differential equations (3) in the form

$$F(r, \theta) = f(r)\sin\theta, \qquad w(r, \theta) = W(r)\cos\theta + \frac{1}{2}ka\theta r\sin\theta \qquad (9)$$

where k is an arbitrary constant. The foregoing expressions for F and w reduce (3) to two fourth-order-coupled ordinary differential equations for $f(r)$ and $W(r)$:

$$D\Delta\Delta W = 2ar^{-2}(r^{-1}f)', \qquad A\Delta\Delta f = 2ar^{-2}(r^{-1}W)' \qquad (10)$$

where

$$\Delta(\ ) = (\ )'' + r^{-1}(\ )' - r^{-2}(\ ).$$

The corresponding expressions for resultants and couples are

$$N_r = n_r(r)\sin\theta = (r^{-1}f)'\sin\theta, \qquad N_\theta = n_\theta(r)\sin\theta = f''\sin\theta \qquad (11)$$

$$N_{r\theta} = n_{r\theta}(r)\cos\theta = -(r^{-1}f)'\cos\theta \qquad (11)\ \text{(Cont.)}$$
$$M_r = m_r(r)\cos\theta = -D[W'' + \nu_b(r^{-1}W)' + \nu_b kar^{-1}]\cos\theta$$
$$M_\theta = m_\theta(r)\cos\theta = -D[\nu_b W'' + (r^{-1}W)' + kar^{-1}]\cos\theta$$
$$M_{r\theta} = m_{r\theta}(r)\sin\theta = D(1-\nu_b)(r^{-1}W)'\sin\theta$$
$$Q_r = q_r(r)\cos\theta = -D[W''' + (r^{-1}W)'' - kar^{-2}]\cos\theta$$
$$Q_\theta = q_\theta(r)\sin\theta = Dr^{-1}[W'' + (r^{-1}W)' + kar^{-1}]\sin\theta$$

and the equilibrium equations (6) can now be written as

$$(rn_r)' - n_{r\theta} - n_\theta = 0, \qquad (rn_{r\theta})' + n_\theta + n_{r\theta} = 0$$
$$(rq_r)' + q_\theta - 2r^{-1}an_{r\theta} = 0$$
$$(rm_r)' + m_{r\theta} - m_\theta - rq_r = 0, \qquad (12)$$
$$(rm_{r\theta})' - m_\theta + m_{r\theta} - rq_\theta = 0$$

The boundary conditions at the radial edges (7) take on the form

$$n_r = n_{r\theta} = m_r = q_r + r^{-1}m_{r\theta} = 0 \qquad (13)$$

or

$$(r^{-1}f)' = W'' + \nu_b(r^{-1}W)' + \nu_b kar^{-1}$$
$$= W''' + (r^{-1}W)'' - r^{-1}(1-\nu_b)(r^{-1}W)' - kar^{-2} = 0 \qquad (14)$$

for $r = r_i$ and $r = r_0$. Note that these boundary conditions give only three independent conditions at each edge.

The integrated boundary conditions (8) for the axial edges can be written as

$$\int_{r_i}^{r_0} (n_{r\theta}\cos^2\theta - n_\theta\sin^2\theta)dr = 0, \qquad \int_{r_i}^{r_0} (n_{r\theta} + n_\theta)dr = 0,$$
$$\int_{r_i}^{r_0} (ar^{-1}n_\theta + q_\theta + m'_{r\theta})dr - [2m_{r\theta}]_{r_i}^{r_0} = 0,$$
$$\int_{r_i}^{r_0} rn_\theta dr = 0, \qquad (15)$$
$$\int_{r_i}^{r_0} (an_\theta + m_\theta + rq_\theta + rm'_{r\theta})dr - [2rm_{r\theta}]_{r_i}^{r_0} = 0$$
$$\int_{r_i}^{r_0} [(an_\theta + rq_\theta + rm'_{r\theta})\sin^2\theta - m_\theta\cos^2\theta]dr - [2rm_{r\theta}\sin^2\theta]_{r_i}^{r_0} = M$$

at $\theta = \pm\theta_0$. Judicious uses of the equilibrium equations (12), as well as of the boundary conditions (13) for the radial edges, show that the first five conditions of (15) are satisfied identically. For example, using the second and third equilibrium equations along with integration by parts, we get for the third condition

$$\int_{r_i}^{r_0} (ar^{-1}n_\theta + q_\theta + m'_{r\theta})dr - [2m_{r\theta}]_{r_i}^{r_0}$$
$$= \int_{r_i}^{r_0} (ar^{-1}n_\theta + q_\theta)dr - [m_{r\theta}]_{r_i}^{r_0}$$
$$= \int_{r_i}^{r_0} [-2ar^{-1}n_{r\theta} + q_\theta - an'_{r\theta}]dr - [m_{r\theta}]_{r_i}^{r_0} \qquad (16)$$
$$= -\int_{r_i}^{r_0} (rq_r)'dr - [m_{r\theta}]_{r_i}^{r_0} - [an_{r\theta}]_{r_i}^{r_0}$$
$$= -[r(q_r + r^{-1}m_{r\theta})]_{r_i}^{r_0}$$

The right-hand side vanishes because of the last boundary condition of (13). Similarly, the last condition of (15) becomes simply

$$M = -\int_{r_i}^{r_0} m_\theta dr \tag{17}$$

which, upon carrying out the integration, gives

$$\frac{M}{D} = [\nu_b W' + r^{-1}W + ka \ln(r)]_{r_i}^{r_0} \tag{18}$$

The system of ordinary differential equations for f and W (10) and the boundary conditions (14) form a two-point (inhomogeneous) boundary-value problem, notwithstanding the fact (as we shall see in the next section) that (14) comprises only three independent conditions at each edge. The boundary-value problem effectively determines f and W in terms of the parameter k; k will then be related to the applied bending moment M by the condition (18).

Solution of the Two-Point Boundary-Value Problem

The system of ordinary differential equations (10) for $f(r)$ and $W(r)$ is equidimensional. It is not difficult to verify that the solutions of this system are

$$\begin{aligned} W(r) &= C_0 r + C_1 r^{\alpha_1} + C_2 r^{\alpha_2} + C_3 r^{\alpha_3} \\ &\quad + \bar{C}_3 r^{\bar{\alpha}_3} + C_4 r^{\alpha_4} + \bar{C}_4 r^{\bar{\alpha}_4} \\ f(r) &= \sqrt{\frac{D}{A}}\,(D_0 r + C_1 r^{\alpha_1} - C_2 r^{\alpha_2} + C_3 r^{\alpha_3} \\ &\quad + \bar{C}_3 r^{\bar{\alpha}_3} - C_4 r^{\alpha_4} - \bar{C}_4 r^{\bar{\alpha}_4}) \end{aligned} \tag{19}$$

where $D_0, C_0, C_1, \ldots, \bar{C}_4$ are constants of integration, α_1 is the real root, α_3 and $\bar{\alpha}_3$ are the complex conjugate roots of the cubic equation:

$$(\alpha^2 - 1)(\alpha - 3) = \pm\frac{2a}{\sqrt{DA}} \equiv \pm 2\delta^3 \tag{20}$$

with the upper sign, and α_2, α_4, and $\bar{\alpha}_4$ are the corresponding roots for the lower sign. In terms of the parameter δ, we have

$$\left.\begin{matrix}\alpha_1 \\ \text{or } \alpha_2\end{matrix}\right\} = 1 \pm \alpha_R, \qquad \begin{pmatrix}\alpha_3 \\ \alpha_4\end{pmatrix} = \left(1 \mp \frac{\alpha_R}{2}\right) + i\alpha_I \tag{21}$$

where

$$\alpha_R = \delta\left[\left(1 + \sqrt{1 - \left(\frac{4}{3\delta^2}\right)^3}\right)^{1/3} + \left(1 - \sqrt{1 - \left(\frac{4}{3\delta^2}\right)^3}\right)^{1/3}\right]$$

$$\alpha_I = \frac{\delta\sqrt{3}}{2}\left[\left(1 + \sqrt{1 - \left(\frac{4}{3\delta^2}\right)^3}\right)^{1/3} - \left(1 - \sqrt{1 - \left(\frac{4}{3\delta^2}\right)^3}\right)^{1/3}\right]$$

To insure real-valued $W(r)$ and $f(r)$, we take D_0, C_0, C_1, and C_2 to be real and $\bar{C}_3$ and $\bar{C}_4$ to be the complex conjugates of C_3 and C_4, respectively.

For an isotropic homogeneous medium, we have

$$\delta = \left(\frac{a^2}{DA}\right)^{1/6} = O\left(\sqrt[3]{\frac{a}{h}}\right) \tag{22a}$$

For typical shell behavior, we are mainly interested in the range $\delta \gg 1$. In this range, we have

$$\alpha_R \sim \sqrt[3]{2}\,\delta, \qquad \alpha_I \sim \sqrt{3}\,\delta/\sqrt[3]{4} \tag{22b}$$

so that α_R and α_I are $O(\sqrt[3]{a/h})$.

For our purpose, it will be more convenient to write (19) in terms of a dimensionless variable $\rho = r/r_0$, in the form

$$\begin{aligned} W(r) &= \frac{kar_0\rho}{\alpha_R^2}\{c_1\rho^{\alpha_R} + c_2\rho^{-\alpha_R} \\ &\quad + \rho^{-\alpha_R/2}[c_3\cos(\alpha_I \ln\rho) + c_4\sin(\alpha_I\ln\rho)] \\ &\quad + \rho^{\alpha_R/2}[c_5\cos(\alpha_I\ln\rho) + c_6\sin(\alpha_I\ln\rho)]\} + c_7 r_0\rho \\ f(r) &= \frac{kar_0\rho}{\alpha_R^2}\sqrt{\frac{D}{A}}\{c_1\rho^{\alpha_R} - c_2\rho^{-\alpha_R} \\ &\quad + \rho^{-\alpha_R/2}[c_3\cos(\alpha_I\ln\rho) + c_4\sin(\alpha_I\ln\rho)] \\ &\quad - \rho^{\alpha_R/2}[c_5\cos(\alpha_I\ln\rho) + c_6\sin(\alpha_I\ln\rho)]\} + c_8 r_0\rho \end{aligned} \tag{23}$$

where $c_1, \ldots, c_8$ are dimensionless real constants.

Correspondingly, we have

$$\begin{aligned} n_r &= -n_{r\theta} = \frac{ka}{\alpha_R r_0\rho}\sqrt{\frac{D}{A}}\{c_1\rho^{\alpha_R} + c_2\rho^{-\alpha_R} \\ &\quad + \rho^{-\alpha_R/2}[c_3G_{r3}(\rho) + c_4G_{r4}(\rho)] \\ &\quad - \rho^{\alpha_R/2}[c_5G_{r5}(\rho) + c_6G_{r6}(r)]\} \\ n_\theta &= \frac{ka}{r_0\rho}\sqrt{\frac{D}{A}}\{c_1(1 + \alpha_R^{-1})\rho^{\alpha_R} - c_2(1 - \alpha_R^{-1})\rho^{-\alpha_R} \\ &\quad + \rho^{-\alpha_R/2}[c_3G_{\theta3}(\rho) + c_4G_{\theta4}(\rho)] - \rho^{\alpha_R/2}[c_5G_{\theta5}(\rho) + c_6G_{\theta6}(\rho)]\} \\ m_r &= -\frac{kaD}{r_0\rho}\{c_1[1 + (1 + \nu_b)\alpha_R^{-1}]\rho^{\alpha_R} \\ &\quad + c_2[1 - (1 + \nu_b)\alpha_R^{-1}]\rho^{-\alpha_R} \\ &\quad + \rho^{-\alpha_R/2}[c_3(G_{\theta3} + \nu_b\alpha_R^{-1}G_{r3}) + c_4(G_{\theta4} + \nu_b\alpha_R^{-1}G_{r4})] \\ &\quad + \rho^{\alpha_R/2}[c_5(G_{\theta5} + \nu_b\alpha_R^{-1}G_{r5}) + c_6(G_{\theta6} + \nu_b\alpha_R^{-1}G_{r6})] + \nu_b\} \\ m_\theta &= -\frac{kaD}{r_0\rho}\{c_1[\nu_b + (1 + \nu_b)\alpha_R^{-1}]\rho^{\alpha_R} \\ &\quad + c_2[\nu_b - (1 + \nu_b)\alpha_R^{-1}]\rho^{-\alpha_R} \\ &\quad + \rho^{-\alpha_R/2}[c_3(\nu_bG_{\theta3} + \alpha_R^{-1}G_{r3}) + c_4(\nu_bG_{\theta4} + \alpha_R^{-1}G_{r4})] \\ &\quad + \rho^{\alpha_R/2}[c_5(\nu_bG_{\theta5} + \alpha_R^{-1}G_{r5}) + c_6(\nu_bG_{\theta6} + \alpha_R^{-1}G_{r6})] + 1\} \\ m_{r\theta} &= \frac{ka(1 - \nu_b)D}{\alpha_R r_0\rho}\{c_1\rho^{\alpha_R} - c_2\rho^{-\alpha_R} + \rho^{-\alpha_R/2}[c_3G_{r3} + c_4G_{r4}] \\ &\quad + \rho^{\alpha_R/2}[c_5G_{r5} + c_6G_{r6}]\} \\ q_\theta &= \frac{kaD}{(r_0\rho)^2}\{c_1(1 + 2\alpha_R^{-1})\rho^{\alpha_R} + c_2(1 - 2\alpha_R^{-1})\rho^{-\alpha_R} \\ &\quad + \rho^{-\alpha_R/2}[c_3(G_{\theta3} + \alpha_R^{-1}G_{r3}) + c_4(G_{\theta4} + \alpha_R^{-1}G_{r4})] \\ &\quad + \rho^{\alpha_R/2}[c_5(G_{\theta5} + \alpha_R^{-1}G_{r5}) + c_6(G_{\theta6} + \alpha_R^{-1}G_{r6})] + 1\} \\ q_r &= -\frac{\alpha_R kaD}{(r_0\rho)^2}\{c_1(1 + \alpha_R^{-1} - 2\alpha_R^{-2})\rho^{\alpha_R} \\ &\quad - c_2(1 - \alpha_R^{-1} - 2\alpha_R^{-2})\rho^{-\alpha_R} \\ &\quad + \rho^{-\alpha_R/2}(c_3G_{q3} + c_4G_{q4}) + \rho^{\alpha_R/2}(c_5G_{q5} + c_6G_{q6}) - \alpha_R^{-1}\} \end{aligned} \tag{24}$$

where

$$\begin{aligned} (G_{r3}, G_{r4}, G_{r5}, G_{r6}) &= \cos(\alpha_I\ln\rho)\left\{-\frac{1}{2}, \frac{\alpha_I}{\alpha_R}, \frac{1}{2}, \frac{\alpha_I}{\alpha_R}\right\} \\ &\quad + \sin(\alpha_I\ln\rho)\left\{-\frac{\alpha_I}{\alpha_R}, -\frac{1}{2}, -\frac{\alpha_I}{\alpha_R}, \frac{1}{2}\right\} \\ (G_{\theta3}, G_{\theta4}, G_{\theta5}, G_{\theta6}) &= \frac{1}{4}\cos(\alpha_I\ln\rho)\left\{\left(1 - \frac{4\alpha_I^2}{\alpha_R^2} - \frac{2}{\alpha_R}\right),\right. \\ &\quad \left. -\frac{4\alpha_I}{\alpha_R}\left(1 - \frac{1}{\alpha_R}\right), \left(1 - \frac{4\alpha_I^2}{\alpha_R^2} + \frac{2}{\alpha_R}\right), \frac{4\alpha_I}{\alpha_R}\left(1 + \frac{1}{\alpha_R}\right)\right\} \end{aligned} \tag{25}$$

$$+\frac{1}{4}\sin(\alpha_I \ln\rho)\left\{\frac{4\alpha_I}{\alpha_R}\left(1-\frac{1}{\alpha_R}\right)\left(1-\frac{4\alpha_I^2}{\alpha_R^2}-\frac{2}{\alpha_R}\right) - \frac{4\alpha_I}{\alpha_R}\left(1+\frac{1}{\alpha_R}\right)\left(1-\frac{4\alpha_I^2}{\alpha_R^2}+\frac{2}{\alpha_R}\right)\right\} \qquad (25)\ (Cont.)$$

$$G_{q3} = -\frac{1}{2}\left[\left(1+\frac{2}{\alpha_R}\right)\left(G_{\theta 3}+\frac{1}{\alpha_R}G_{r3}\right) + \frac{4\alpha_I}{\alpha_R}\left(G_{\theta 4}+\frac{1}{\alpha_R}G_{r4}\right)\right]$$

$$G_{q4} = -\frac{1}{2}\left[\left(1+\frac{2}{\alpha_R}\right)\left(G_{\theta 4}+\frac{1}{\alpha_R}G_{r4}\right) - \frac{4\alpha_I}{\alpha_R}\left(G_{\theta 3}+\frac{1}{\alpha_R}G_{r3}\right)\right]$$

$$G_{q5} = \frac{1}{2}\left[\left(1-\frac{2}{\alpha_R}\right)\left(G_{\theta 5}+\frac{1}{\alpha_R}G_{r5}\right) - \frac{4\alpha_I}{\alpha_R}\left(G_{\theta 6}+\frac{1}{\alpha_R}G_{r6}\right)\right]$$

$$G_{q6} = \frac{1}{2}\left[\left(1-\frac{2}{\alpha_R}\right)\left(G_{\theta 6}+\frac{1}{\alpha_R}G_{r6}\right) + \frac{4\alpha_I}{\alpha_R}\left(G_{\theta 5}+\frac{1}{\alpha_R}G_{r5}\right)\right]$$

The six independent conditions of no stress at the outer edge $\rho = 1$ and at the inner edge $\rho = \rho_i = r_i/r_0$ now take on the form

$$c_1 + c_2 + c_3G_{r3}(1) + c_4G_{r4}(1) - c_5G_{r5}(1) - c_6G_{r6}(1) = 0$$

$$\begin{aligned} & c_1[1+(1+\nu_b)\alpha_R^{-1}] + c_2[1-(1+\nu_b)\alpha_R^{-1}] \\ & + c_3[G_{\theta 3}(1)+\nu_b\alpha_R^{-1}G_{r3}(1)] + c_4[G_{\theta 4}(1)+\nu_b\alpha_R^{-1}G_{r4}(1)] \\ & + c_5[G_{\theta 5}(1)+\nu_b\alpha_R^{-1}G_{r5}(1)] \\ & + c_6[G_{\theta 6}(1)+\nu_b\alpha_R^{-1}G_{r6}(1)] = -\nu_b \end{aligned} \qquad (25a)$$

$$\begin{aligned} & c_1[1+\alpha_R^{-1}-(3-\nu_b)\alpha_R^{-2}] \\ & - c_2[1-\alpha_R^{-1}-(3-\nu_b)\alpha_R^{-2}] \\ & + c_3[G_{q3}(1)-(1-\nu_b)\alpha_r^{-2}G_{r3}(1)] \\ & + c_4[G_{q4}(1)-(1-\nu_b)\alpha_R^{-2}G_{r4}(1)] \\ & + c_5[G_{q5}(1)-(1-\nu_b)\alpha_R^{-2}G_{r5}(1)] \\ & + c_6[G_{q6}(1)-(1-\nu_b)\alpha_R^{-2}G_{r6}(1)] = \alpha_R^{-1} \end{aligned}$$

and

$$\begin{aligned} & c_1\rho_i^{\alpha_R} + c_2\rho_i^{-\alpha_R} + \rho_i^{-\alpha_R/2}[c_3G_{r3}(\rho_i)+c_4G_{r4}(\rho_i)] \\ & - \rho_i^{\alpha_R/2}[c_5G_{r5}(\rho_i)+c_6G_{r6}(\rho_i)] = 0 \end{aligned}$$

$$\begin{aligned} & c_1[1+(1+\nu_b)\alpha_R^{-1}]\rho_i^{\alpha_R} + c_2[1-(1+\nu_b)\alpha_R^{-1}]\rho_i^{-\alpha_R} \\ & + \rho_i^{-\alpha_R/2}\{c_3[G_{\theta 3}(\rho_i)+\nu_b\alpha_R^{-1}G_{r3}(\rho_i)] \\ & + c_4[G_{\theta 4}(\rho_i)+\nu_b\alpha_R^{-1}G_{r4}(\rho_i)]\} \\ & + \rho_i^{\alpha_R/2}\{c_5[G_{\theta 5}(\rho_i)+\nu_b\alpha_R^{-1}G_{r5}(\rho_i)] \\ & + c_6[G_{\theta 6}(\rho_i)+\nu_b\alpha_R^{-1}G_{r6}(\rho_i)]\} = -\nu_b \end{aligned} \qquad (25b)$$

$$\begin{aligned} & c_1[1+\alpha_R^{-1}-(3-\nu_b)\alpha_R^{-2}]\rho_i^{\alpha_R} \\ & - c_2[1-\alpha_R^{-1}-(3-\nu_b)\alpha_R^{-2}]\rho_i^{-\alpha_R} \\ & + \rho_i^{-\alpha_R/2}\{c_3[G_{q3}(\rho_i)-(1-\nu_b)\alpha_R^{-2}G_{r3}(\rho_i)] \\ & + c_4[G_{q4}(\rho_i)-(1-\nu_b)\alpha_R^{-2}G_{r4}(\rho_i)]\} \\ & + \rho_i^{\alpha_R/2}\{c_5[G_{q5}(\rho_i)-(1-\nu_b)\alpha_R^{-2}G_{r5}(\rho_i)] \\ & c_6[G_{q6}(\rho_i)-(1-\nu_b)\alpha_R^{-2}G_{r6}(\rho_i)]\} = \alpha_R^{-1} \end{aligned}$$

These six inhomogeneous conditions determine the six dimensionless constants c_j in terms of ρ_i, δ, and ν_b.

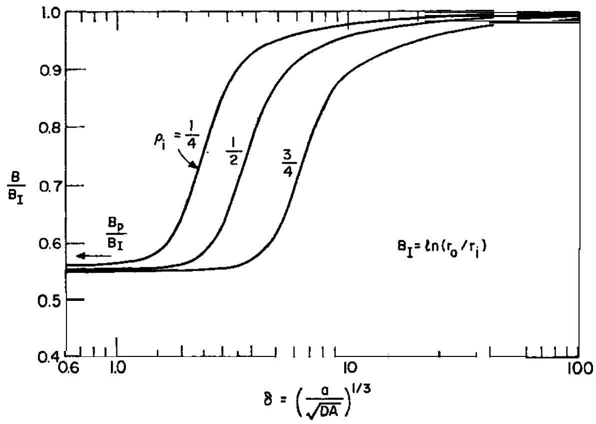

Fig. 2 Dimensionless overall bending stiffness coefficient versus δ for $\nu_b = 0.3$

Overall load- Deformation Relation and Stress Distributions

The remaining undetermined parameter k can now be related to the applied moment M by means of (18). The result is a linear relation between k and M in the form

$$M = Bk \equiv kaD\left\{-\ln\rho_i + \alpha_R^{-1}\left(\sum_{n=1}^{6} c_n\Delta_n\right)\right\} \qquad (26)$$

where

$$\begin{Bmatrix}\Delta_1\\ \Delta_2\end{Bmatrix} = \pm(1-\rho_i^{\pm\alpha_R})[\nu_b \pm (1+\nu_b)\alpha_R^{-1}]$$

$$\begin{Bmatrix}\Delta_3\\ \Delta_5\end{Bmatrix} = \frac{1}{2}\left\{\frac{2\nu_b\alpha_I}{\alpha_R}\rho_i^{\mp\alpha_R/2}\sin(\alpha_I\ln\rho_i) \mp \left[\nu_b \mp \frac{2(1+\nu_b)}{\alpha_R}\right][1-\rho_i^{\mp\alpha_R/2}\cos(\alpha_I\ln\rho_i)]\right\}$$

$$\begin{Bmatrix}\Delta_4\\ \Delta_6\end{Bmatrix} = \frac{2\nu_b\alpha_I}{\alpha_R}[1-\rho_i^{\mp\alpha_R/2}\cos(\alpha_I\ln\rho_i)] - \left[\mp\frac{\nu_b}{2} + \frac{1+\nu_b}{\alpha_R}\right]\rho_i^{\mp\alpha_R/2}\sin(\alpha_I\ln\rho_i)$$

The variation of B as a function of δ is shown in Fig. 2 for representative values of ρ_i and for $\nu_b = 0.3$. This graph shows that for $\delta \gg 1$, we have

$$B \sim -Da\ln\rho_i \equiv B_I \qquad (27)$$

Thus only the multivalued portion of the solution in (9) contributes significantly to B for $\delta \gg 1$. As a tends to zero so that δ tends to zero, Fig. 2 shows that B tends to the corresponding result for a flat plate (see Appendix):

$$B_P = -\frac{Da(1-\nu_b)}{4(3+\nu_b)}\left[(3+\nu_b)^2\ln\rho_i + \frac{(1-\nu_b)^2(1-\rho_i^2)}{1+\rho_i^2}\right] \qquad (28)$$

In order for the result to be meaningful, we keep $k_0 = ka$ finite as a tends to zero.

Turning now to the stress distributions, the expressions for the resultants and couples given by (23) suggest that for $\delta \gg 1$, there are two distinct stress states in the shell. The contribution from the term $\frac{1}{2}kar\theta\sin\theta$ in (9) to the stress couples is significant everywhere in the shell and can therefore be considered as the interior state. Inasmuch as this same term does not contribute to the inplane stress resultants, the interior state of the shell is purely an ***inextensional bending state.***

In contrast, the contribution of term associated with the constants c_j becomes insignificant at a distance $O(r_0/\delta)$ from the edges for $\delta \gg 1$. As such, they are of the nature of a boundary-

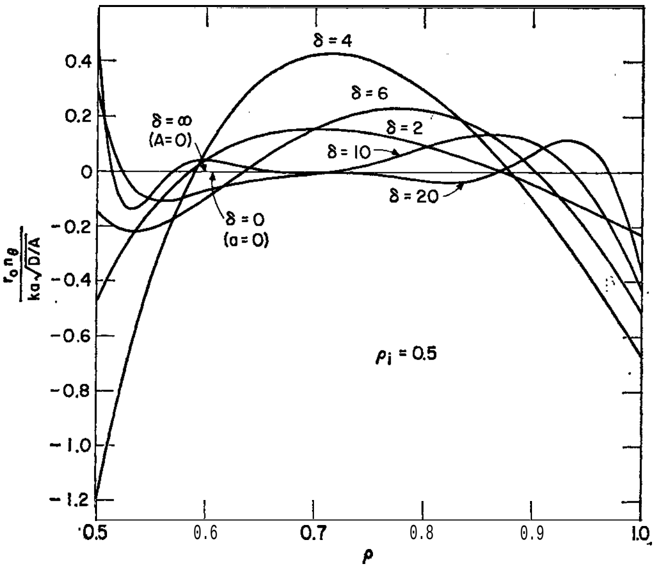

Fig. 3 Variation of stress resultant N_θ across shell width for $\nu_b = 0.3$, $r_i/r_0 = 0.5$, and $8 = \pi/2$

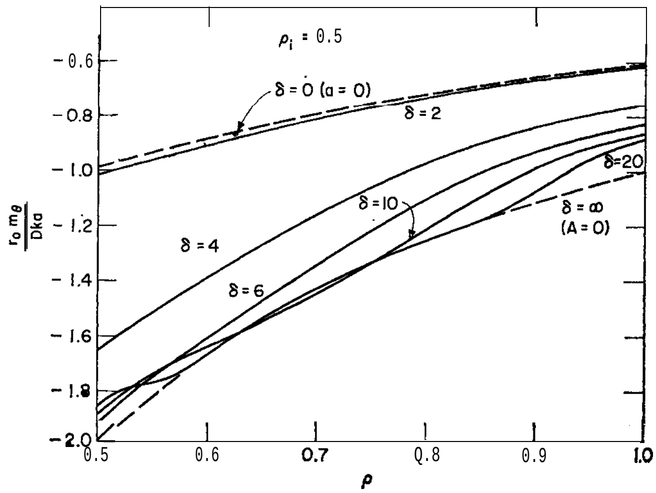

Fig. 4 variation of stress couple M_θ across shell width for $\nu_b = 0.3$, $r_i/r_0 = 0.5$, and $8 = 0$

layer effect and may therefore be considered as an *edge-zone state.* These two distinct stress states can be seen from the graphs for n_θ and m_θ shown in Figs. 3 and 4.

The presence of the edge-zone state can also be seen from the structure of the differential equations (10) which can be written as one equation of the form

$$\frac{\sqrt{DA}}{2a}\Delta\Delta\left(W+\sqrt{\frac{A}{D}}f\right)=\frac{1}{r^2}\left[\frac{1}{r}\left(W+\sqrt{\frac{A}{D}}f\right)\right]' \quad (29)$$

in which the dimensionless quantity $\sqrt{DA}/a$ may be (and in general is for shells) small compared to unity. It follows (as can also be seen from (19)) that the dimensionless characteristic length for the edge-zone solution is of the order $1/\delta = 0(\sqrt[3]{h/a})$ which is the same as that suggested by the differential equation for the edge effect for shells with *negative Gaussian curvature* as given in [1, p. 424].

To see the relative order of the direct and bending stress near the edges of the shell, we consider

$$\sigma_{\theta D} = \left(\frac{N_\theta}{h}\right)_{\substack{\theta=\frac{\pi}{2}\\ \rho=1}} = \left(\frac{n_\theta}{h}\right)_{\rho=1},$$

$$\sigma_{\theta B} = \left(\frac{6M_\theta}{h^2}\right)_{\substack{\theta=0\\ \rho=1}} = \left(\frac{6m_\theta}{h^2}\right)_{\rho=1} \quad (30a)$$

so that

$$\frac{\sigma_{\theta D}}{\sigma_{\theta B}} = \left(\frac{hn_\theta}{6m_\theta}\right)_{\rho=1} \quad (30b)$$

From Figs. 3 and 4, we see that both r_0m_θ/Dka and $r_0n_\theta/ka\sqrt{D/A}$ are 0(1) for $\delta \geq 2$. Therefore, we have

$$\frac{\sigma_{\theta D}}{\sigma_{\theta B}} = \frac{h}{6\sqrt{DA}}\left[\left(\frac{r_0n_\theta}{ka\sqrt{D/A}}\right)\left(\frac{kaD}{r_0m_\theta}\right)\right] = 0\left(\frac{h}{6\sqrt{DA}}\right) \quad (30c)$$

For a homogeneous and isotropic medium, we have $\sqrt{DA} = O(h)$. Therefore, the edge direct and bending stress are of the same order of magnitude. It is not difficult to check that the transverse shearing stresses are at least an order of magnitude smaller.

Inplane Displacement Components

We now use the strain-displacement relations (5) and the stress-strain relations (4) to determine the expression for the displacement components u_r and u_θ. The expression for ϵ_r can be written as

$$u_r' = \epsilon_r = A[(r^{-1}f)' - \nu_s f''] \sin 8$$

or

$$u_r = A(r^{-1}f - \nu_s f')\sin\theta + H^\cdot(\theta) \quad (31)$$

where $H(\theta)$ is an arbitrary function of θ.

The expression for ϵ_θ can be written as

$$\begin{aligned} u_\theta^\cdot &= r\epsilon_\theta - u_r - ar^{-1}w^\cdot \\ &= \{A[rf'' - (1-\nu_s)r^{-1}f] + ar^{-1}W\}\sin 8 \\ &\qquad - H^\cdot(\theta) - \frac{1}{2}ka^2(\theta\ \sin\ \theta)^\cdot \end{aligned}$$

or upon integration

$$\begin{aligned} u_\theta &= -\{A[rf'' - (1-\nu_s)r^{-1}f] + ar^{-1}W\}\cos\theta \\ &\quad - H(\theta) - \frac{1}{2}ka^2\theta\sin 8 + K(r) \end{aligned} \quad (32)$$

where $K(r)$ is an arbitrary function of r.

To determine $H(\theta)$ and $K(r)$, we make use of the last equation of (5) which, upon using (31), (32), (9), and the last equation of (4), becomes

$$\begin{aligned} &\{A[r^2f''' - 3f' + 3r^{-1}f] - 2ar^{-1}W\}\cos\theta \\ &\qquad + r^2(r^{-1}K)' + H^{\cdot\cdot} + H + ka^2\theta\sin\theta = 0 \end{aligned} \quad (33)$$

We now use (19) and (20) to reduce the foregoing equation to

$$(H^{\cdot\cdot} + H + ka^2\theta\sin 8 - 2ac_1\cos\theta) + r^2(r^{-1}K)' = 0 \quad (34)$$

or

$$\begin{aligned} K &= K_0r + K_1 \\ H &= \frac{1}{4}ka^2(\theta^2\cos\theta - \theta\sin\theta) + c_1a\theta\sin\theta + K_1 \end{aligned} \quad (35)$$

With these, the expressions for u_r and u_θ now take the form

$$\begin{aligned} u_r &= A(r^{-1}f - \nu_s f')\sin\theta + c_1a(\sin\theta + \theta\cos\theta) \\ &\quad - \frac{1}{4}ka^2[(1+\theta^2)\sin\theta - \theta\cos\theta] \\ u_\theta &= -\{A[rf'' - (1-\nu_s)r^{-1}f] + ar^{-1}W\}\cos\theta \\ &\quad - \frac{1}{4}ka^2[\theta^2\cos\theta + \theta\sin\theta] + K_0r \end{aligned} \quad (36)$$

Note that K_1 does not appear in the final expressions and can therefore be set equal to zero. Note also that the K_0-term is a rigid-body rotation about the z-axis.

The final expressions for u_r and u_θ along with the expression for w are significant in that they suggest the appropriate form of the displacement field which allows the reduction of the problem of pure bending of nonshallow helicoidal shell to a two-point boundary-value problem in ordinary differential equations [6].

APPENDIX

Pure Bending of Ring Plate Sectors

For the limiting case of a flat plate for which a = 0, the governing equations of the section, "Formulation," uncouple into two groups. The generalized plane-stress equation for F and the homogeneous boundary conditions lead to a trivial solution $F = 0$. For the transverse bending of the plate, the governing partial differential equation is

$$D\nabla^2\nabla^2 w = 0 \tag{37}$$

In terms of w, we have

$$\begin{aligned} M_r &= -D(w'' + \nu_b r^{-1}w' + \nu_b r^{-2}w^{\cdot\cdot}),\\ M_\theta &= -D(\nu_b w'' + r^{-1}w' + r^{-2}w^{\cdot\cdot}),\\ M_{r\theta} &= -D(1-\nu_b)(r^{-1}w^{\cdot})',\\ Q_r &= -D(\nabla^2 w)', \quad Q_\theta = -Dr^{-1}(\nabla^2 w)^{\cdot} \end{aligned} \tag{38}$$

At the radial edges $r = r_i$ and $r = r_0$, the Kirchhoff conditions require that

$$M_r = Q_r + r^{-1}M_{r\theta}^{\cdot} = 0 \tag{39}$$

while the overall equilibrium conditions at $\theta = \pm\theta_0$ become

$$\begin{aligned} &\int_{r_i}^{r_0} (Q_\theta + M_{\theta r}')dr - [2M_{r\theta}]_{r_i}^{r_0} = 0,\\ &-\int_{r_i}^{r_0} [(Q_\theta + M_{\theta r}')r\cos\theta + M_\theta \sin\theta]dr + [2rM_{r\theta}\cos\theta]_{r_i}^{r_0} = 0\\ &\int_{r_i}^{r_0} [(Q_\theta + M_{\theta r}')r\sin\theta - M_\theta\cos\theta]dr - [2rM_{r\theta}\sin\theta]_{r_i}^{r_0} = M \end{aligned} \tag{40}$$

Guided by the result of [2], we consider a solution of (37) in the form

$$w(r,\theta) = k_0\left\{[c_1 r^{-1} + c_2 r^3 + c_3 r \ln r]\cos\theta + \frac{1}{2} r\theta\sin\theta\right\} \tag{41}$$

where $k_0, c_1, c_2,$ and c_3 are four arbitrary constants. Correspondingly, we have

$$\begin{aligned} M_r &= m_r(r)\cos\theta = -Dk_0[2(1-\nu_b)c_1 r^{-3} + 2(3+\nu_b)c_2 r + (1+\nu_b)c_3 r^{-1} + \nu_b r^{-1}]\cos\theta\\ M_\theta &= m_\theta(r)\cos\theta = -Dk_0[-2(1-\nu_b)c_1 r^{-3} + 2(1+3\nu_b)c_2 r + (1+\nu_b)c_3 r^{-1} + r^{-1}]\cos\theta\\ M_{r\theta} &= m_{r\theta}(r)\sin\theta = D(1-\nu_b)k_0[-2c_1 r^{-3} + 2c_2 r + c_3 r^{-1}]\sin\theta\\ Q_r &= q_r(r)\cos\theta = -Dk_0[8c_2 - 2c_3 r^{-2} - r^{-2}]\cos\theta\\ Q_\theta &= q_\theta(r)\sin\theta = -Dk_0[8c_2 + 2c_3 r^{-2} + r^{-2}]\sin\theta \end{aligned} \tag{42}$$

The conditions of no stress at the radial edges are satisfied by taking

$$c_1 = \frac{(1-\nu_b)r_0^2 r_i^2}{8(r_0^2 + r_i^2)}, \qquad c_2 = \frac{(1-\nu_b)^2}{8(3+\nu_b)(r_0^2+r_i^2)},\qquad c_3 = -\frac{1+\nu_b}{4} \tag{43}$$

Through an analysis similar to that of the section, "Reduction," it can be shown that the first two conditions of (41) are satisfied identically by (42) and (43) while the last condition becomes

$$\begin{aligned} M &= -\int_{r_i}^{r_0} m_\theta dr\\ &= Dk_0\{(1-\nu_b)c_1 r^{-2} + (1+3\nu_b)c_2 r^2 + [c_3(1+\nu_b)+1]\ln r\}_{r_i}^{r_0} \end{aligned} \tag{44}$$

which can be put in the form of a linear relation between the constant k_0 and the applied moment M

$$M = \frac{D(1-\nu_b)}{4(3+\nu_b)} f_b k_0 \tag{45}$$

where

$$\begin{aligned} f_b(r_0, r_i, \nu_b) &= (3+\nu_b)^2 \ln\left(\frac{r_0}{r_i}\right) - \frac{(1-\nu_b)^2(r_0^2 - r_i^2)}{r_0^2 + r_i^2}\\ &= -\left[(3+\nu_b)^2 \ln \rho_i + \frac{(1-\nu_b)^2(1-\rho_i^2)}{1+\rho_i^2}\right] \end{aligned} \tag{46}$$

Equations (45) and (46) are exactly the same as those obtained in [2]. The foregoing outline of a derivation of these results in a form different from that in [2] serves the purpose of indicating the appropriate θ-dependence of the solution for the shell problem, as well as supplies the stress distributions in the plate (which were not given in [2]) to be used as a reference state for the shallow helicoidal shells.

References

1 Gol'denveizer, A. L., *Theory of Elastic Thin Shells,* Pergamon Press, New York, 1961.

2 Goodier, J. N., and Wilhoit, J. C., "Elastic Stress Discontinuities in Ring Plates," *Proceedings of the 4th Annual Midwestern Conference on Solid Mechanics,* 1959, pp. 152-170.

3 Maunder, L., and Reissner, E., "Pure Bending of Pretwisted Rectangular Plates," *J. Mech. Phys. Solids,* Vol. 5, 1957, pp. 261-266.

4 Reissner, E., "On Finite Bending and Twisting of Circular Ring Sector Plates and Shallow Helicoidal Shells," *Quarterly of Applied Mathematics,* **Vol.** 11, 1953, pp. 473-483.

5 Timoshenko, S., *Strength of Materials, Part II,* D. Van Nostrand, Princeton, N. J., 1956.

6 Wan, F. Y. M., "A Class of Unsymmetric Stress Distributions in Helicoidal Shells," *Quarterly of Applied Mathematics,* Vol. 24, pp. 374-379.

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