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Advances in the Mechanics of Plates and Shells

The Avinoam Libai Anniversary Volume

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ON LATERAL BUCKLING OF END-LOADED CANTILEVERS WITH TRANSVERSE SHEAR DEFORMATIONS

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1. Introduction

The main features of the problem of lateral buckling of transversely loaded beams were successfully treated by A.G. M. Michell [1] and L. Prandtl [2] using appropriate ad hoc considerations. Subsequently, H. Reissner [3] derived the equations governing this problem by appropriate specialization of Kirchhoff's general theory of space-curved beams. In separate ways, Reissner [3] and his assistant M.K. Grober [4] reduced the lateral buckling problem to the solution of a boundary value problem for a third order linear differential equation. Specific application of this theory to the case of a narrow rectangular cross-section beam was carried out by K. Federhofer [5]. More recent development of the lateral buckling problem of beams can be found in [6-9] and references therein. In [9], one-dimensional theories of beams were derived from a three-dimensional theory of elasticity by way of the principle of minimum potential energy. The various cases analyzed in that publication to study the effect of finite deformation with or without warping stiffness all assumed the cantilever is rigid with respect to transverse shear deformation. The present paper complements [9] by studying the impact of non-vanishing transverse shear strains (including nonlinear terms) on the critical load for lateral buckling of cantilevers. For the special case of the vanishing warping deformation, it is found that the nonlinear terms in transverse shear strains and pre-buckling deformation effects should be retained as long as transverse shear deformation is important.

2. Variationally Derived Equations for Cantilevers

Similar to the approach of [9], a set of one-dimensional differential equations governing finite deformations of prismatic elastic bodies is derived from three-dimensional theory of elasticity through the use of the variational equation

$$\delta\left\{\frac{1}{2}\iiint(E\epsilon_z^2+G\gamma_x^2+G\gamma_y^2)dxdydz\right\}=0 \tag{2.1}$$

where the usual Young's modulus E and shear modulus G are known functions of the cross-sectional coordinates x and y. In adopting the particular variational functional (2.1) as the appropriate strain energy of the prismatic body, it is tacitly assumed that the other three strain components of the elastic body are either negligibly small or identically zero for the class of problems of interest.

D. Durban et al. (eds.), Advances in the Mechanics of Plates and Shells, 343–356.

The three non-vanishing strain components ϵ_z, γ_x, and γ_y are defined in terms of the axial displacement component $\tilde{w}$ in the direction of the axial coordinate z along the length of the prismatic body and the two cross-sectional displacement components $\tilde{u}$ and $\tilde{v}$ in the direction of x and y, respectively. For lateral deformation and buckling of cantilevers, we take the following to be the approximate strain-displacement relations

$$\epsilon_z = \tilde{w}_{,z} + \frac{1}{2}(\tilde{u}^2_{,z} + \tilde{v}^2_{,z}) \tag{2.2}$$

$$\gamma_x = \tilde{u}_{,z} + \tilde{w}_{,x} + \eta\tilde{v}_{,x}\tilde{v}_{,z}, \qquad \gamma_y = \tilde{v}_{,z} + \tilde{w}_{,y} + \eta\tilde{u}_{,y}\tilde{u}_{,z} \tag{2.3}$$

where the displacement components are approximated by

$$\tilde{w} = w(z) + \alpha(z)x + \beta(z)y + \lambda(z)\psi(x,y) \tag{2.4}$$

$$\tilde{u} = u(z) - \theta(z)y - \frac{1}{2}\theta^2(z)x, \qquad \tilde{v} = v(z) + \theta(z)x - \frac{1}{2}\theta^2(z)y \tag{2.5}$$

with w, u, v, α, β, θ and λ being functions of z only and the Saint-Venant warping function ψ being a function of x and y only. Analogous to [9], we use the constant parameter η to allow for the inclusion or exclusion of the relevant nonlinear terms in the approximate displacement expressions. In [9], it was assumed that the prismatic body under consideration is rigid with respect to transverse shear with the conditions $\gamma_x = \gamma_y = 0$ incorporated in the variational equations as constraints through the Lagrange multipliers Q_x and Q_y (with the quantity $[G(\gamma_x)^2 + G(\gamma_y)^2]$ replaced by $[Q_x(\tilde{u}_{,z} + \tilde{w}_{,x} + \eta\tilde{v}_{,x}\tilde{v}_{,z}) + Q_y(\tilde{v}_{,z} + \tilde{w}_{,y} + \eta\tilde{u}_{,y}\tilde{u}_{,z})]$). The multipliers Q_x and Q_y have the interpretation of transverse force over the cross section of the prismatic body.

In this paper, we allow for transverse shear deformations so that there are no constraints in the extremization of the energy functional. With the effect of warping already studied extensively in [9], we will be concerned here mainly with a study of the effect of transverse shear deformability, with the nonlinear terms in γ_x and γ_y and with the pre-buckling deformations since their effects on the buckling load of prismatic bodies were not previously treated anywhere.

With the only non-vanishing stress-strain relations

$$\sigma_z = E\epsilon_z, \qquad \tau_x = G\gamma_x, \qquad \tau_y = G\gamma_y \tag{2.6}$$

giving the three non-vanishing stress components, the Euler differential equations of (2.1) are the following seven one-dimensional differential equations of equilibrium:

$$F' = 0, \quad M_x' - Q_x = 0, \quad M_y' - Q_y = 0, \quad R' - S = 0. \tag{2.7}$$

$$(Q_x - \eta\theta Q_y + u'F - \theta' M_y - \theta\theta' M_x)' = 0, \tag{2.8}$$

$$(Q_y + \eta\theta Q_x + v'F + \theta' M_x - \theta\theta' M_y)' = 0, \tag{2.9}$$

$$\begin{aligned}(1+\eta\theta^2)T' \quad &+ \quad (\theta' N - u' M_y + v' M_x)' + \eta(u' Q_y - v' Q_x) \\ &- \quad \theta[v' M_y + u' M_x - \theta\theta' N + (1-\eta)Z]' = 0\end{aligned} \tag{2.10}$$

where $(\ )' = d(\ )/dz$ and where the resultant force and moment quantities in these equations are defined by the relations

$$(F, M_x, M_y, R, N) = \iint (1, x, y, \psi, x^2+y^2)\sigma_z dx\, dy \tag{2.11}$$

$$(Q_x, Q_y, T, S, Z) = \iint (\tau_x, \tau_y, x\tau_y - y\tau_x, \psi_{,x}\tau_x + \psi_{,y}\tau_y, x\tau_x + y\tau_y) dx\, dy \tag{2.12}$$

When the $x-$axis and $y-$axis are axes of geometrical an material symmetry of the cross section (so that $E(-x,y)\psi(-x,y) = -E(x,y)\psi(x,y)$, etc.), we have the following one-dimensional constitutive relations for these resultant quantities (see [6]):

$$F = A_E\epsilon_F + I_{pE}\epsilon_N, \quad N = I_{pE}\epsilon_F + I_{pp}\epsilon_N \tag{2.13}$$

$$M_x = I_x\kappa_x, \quad M_y = I_y\kappa_y, \quad R = A_\psi\kappa_R \tag{2.14}$$

$$Q_x = A_G\Gamma_x, \quad Q_y = A_G\Gamma_y \tag{2.15}$$

$$T = I_{pG}\kappa_T - J\kappa_S, \quad S = J(-\kappa_T + \kappa_S), \quad Z = I_{pG}\kappa_Z \tag{2.16}$$

where

$$\epsilon_F = w' + \frac{1}{2}[(u')^2 + (v')^2], \quad \epsilon_N = \frac{1}{2}(\theta')^2(1+\theta^2) \tag{2.17}$$

$$\kappa_x = \alpha' + (v'\theta' - u'\theta'\theta), \quad \kappa_y = \beta' - (u'\theta' + v'\theta'\theta) \tag{2.18}$$

$$\Gamma_x = u' + \alpha + \eta\theta v', \quad \Gamma_y = v' + \beta - \eta\theta u', \quad \kappa_R = \lambda' \tag{2.19}$$

$$\kappa_T = \theta'(1+\eta\theta^2), \quad \kappa_S = \lambda, \quad \kappa_Z = -(1-\eta)\theta\theta' \tag{2.20}$$

and

$$\{A_E, I_x, I_y, I_{pE}, I_{pp}, A_\psi\} = \iint \{1, x^2, y^2, x^2+y^2, (x^2+y^2)^2, \psi^2\} E dx\, dy \tag{2.21a}$$

$$\{A_G, I_{pG}, J\} = \iint \{1, x^2+y^2, \psi^2_{,x} + \psi^2_{,y}\} G dx\, dy \tag{2.21b}$$

Note that the relations (2.7)-(2.10) and (2.13)-(2.20) are exact consequences of (2.1) - (2.5) while some higher order nonlinear terms in the displacement variables have been neglected in the corresponding relations in [9]. As to be seen later in section 6, at least one term in (2.10), $(-\theta v' M_y)$, which appears to be small of higher order, will actually contribute significantly to the buckling load not anticipated in [7,9].

At the clamped end of the cantilever, $z = L$, we have the no displacement conditions of

$$w(L) = \alpha(L) = \beta(L) = \theta(L) = \lambda(L) = u(L) = v(L) = 0 \tag{2.22}$$

At the loaded end, $z = 0$, the Euler boundary conditions require

$$F(0) = M_x(0) = M_y(0) = R(0) = 0$$

$$[Q_x - \eta\theta Q_y]_{z=0} = F_x, \quad [Q_y + \eta\theta Q_x]_{z=0} = F_y \tag{2.23}$$

$$[(1+\eta\theta^2)T + (1+\theta^2)\theta' N + (\eta - 1)\theta Z]_{z=0} = 0$$

3. Equations for Buckling

For buckling of the prismatic body due to a lateral end force in the y direction ($F_z = F_x = 0, F_y = P$), the non-vanishing stress and displacement measures of the corresponding pre-buckled state are given by

$$Q_y = P, \qquad M_y = Pz \tag{3.1}$$

and

$$Q_y = A_G(v_p' + \beta_p), \qquad M_y = I_y \beta_p' \tag{3.2}$$

From (3.2) and the end conditions $v_p(L) = \beta_p(L) = 0$, we obtain

$$\beta_p = \frac{P}{2I_y}(z^2 - L^2), \quad v_p = \frac{P}{6I_y}\left\{(L^3 - z^3) - 3L^2(L-z) - \frac{6I_y}{A_G}(L-z)\right\} \tag{3.3}$$

The corresponding (linearized) buckling equations are

$$[Q_x - P(\eta\theta + z\theta')])' = 0, \quad M_x' - Q_x = 0, \quad R' - S = 0.$$

$$[T + P\{\eta u - (zu')\} + v_p' M_x]' - (v_p' Pz)'\theta - \eta v_p' Q_x = 0 \tag{3.4}$$

where

$$Q_x = A_G(u' + \alpha + \eta v_p'\theta), \quad M_x = I_x(\alpha' + v_p'\theta')$$

$$T = I_{pG}\theta' - J\lambda, \quad S = J(\lambda - \theta'), \quad R = A_\psi \lambda'. \tag{3.5}$$

Note that terms involving pre-buckling deformations are retained in (3.4) while they were neglected in [9]. Though the influence of pre-buckling displacement was analyzed in [7], it was for the case of no transverse shear deformation and the simple linear relations $u' = -\alpha$ and $v' = -\beta$. The first equation of (3.4) may be integrated once immediately to obtain

$$Q_x = P(\eta\theta + z\theta') \tag{3.6}$$

where the relevant boundary condition in (2.33) (with $F_x = 0$) has been used to determine the constant of integration. With (3.6), the second equation of (3.4) may be written as

$$M_x' - P(\eta\theta + z\theta') = 0 \tag{3.7}$$

Upon substituting (3.5) into (3.4), we obtain an eighth order system of differential equations for the four displacement measures u, θ, α, and λ :

$$\{A_G(u' + \alpha + \eta v_p'\theta) - P(\eta\theta + z\theta')\}' = 0, \qquad \{I_x(\alpha' + v_p'\theta')\}' - P(\eta\theta + z\theta')\} = 0$$

$$\{I_{pG}\theta' - J\lambda + P(\eta u - zu') + v_p' I_x(\alpha' + v_p'\theta')\} - (Pzv_p')'\theta - \eta v_p' P(\eta\theta + z\theta') = 0 \tag{3.8}$$

$$\{A_\psi \lambda'\}' + J(\theta' - \lambda) = 0$$

where we have used (3.7) instead of the second equation of (3.4). The eighth order system (3.8) is supplemented by the eight buckling boundary conditions:

$$u(L) = \theta(L) = \alpha(L) = \lambda(L) = 0 \tag{3.9a}$$

$$[Q_x - \eta P\theta]_{z=0} = M_x(0) = T(0) = R(0) = 0. \tag{3.9b}$$

The homogeneous boundary value problem defined by (3.8) and (3.9) constitutes an eigenvalue problem with the buckling (end force) load P as the eigenvalue parameter.

In addition to omitting terms involving pre-buckling deformations, the analyses in [9] were limited to the special case $\Gamma_x = \Gamma_y = 0$, i.e., the cross section of the prismatic body is rigid and does not allow transverse shear deformations. In that case, we may eliminate u from the first three equations of (3.5) to obtain a sixth order system for θ, α and λ. In this paper, we complement the study in [9] by allowing for transverse shear deformations but not cross-sectional warping so that $\psi = 0$. The latter restriction implies $A_\psi = J = 0$ and the eighth order system (3.4) reduces to a sixth order system:

$$\{A_G(u' + \alpha + \eta v_p'\theta) - P(\eta\theta + z\theta')\}' = 0$$

$$\{I_x(\alpha' + v_p'\theta')\}' - P(\eta\theta + z\theta') = 0 \tag{3.10}$$

$$\{I_{pG}\theta' + P(\eta u - zu')\}' - P(zv_p')'\theta + (1-\eta)v_p' P(\eta\theta + z\theta') + v_p'' I_x(\alpha' + v_p'\theta) = 0$$

while the last differential equation in (3.8) is trivially satisfied. Note that terms involving the pre-buckling displacement v_p in (3.8) were omitted in the buckling analysis of [9]. We keep them in (3.8) to allow for an analysis of the effect of these terms later.

4. Linear Transverse Shear Deformability

We wish to investigate first the effect of transverse shear deformability on the buckling of cantilevers. For this purpose, we will omit terms involving the pre-buckling displacement v_p. In that case, the system (3.10) further reduces to

$$A_G(u' + \alpha) - P(\eta\theta + z\theta') = 0 \tag{4.1a}$$

$$\{I_x\alpha'\}' - P(\eta\theta + z\theta') = 0, \qquad \{I_{pG}\theta' + P(\eta u - zu')\}' = 0 \tag{4.1b, c}$$

The fifth order system (4.1) is supplemented by the boundary conditions

$$u(L) = \theta(L) = \alpha(L) = M_x(0) = T(0) = 0 \tag{4.2}$$

For $\eta = 0$ so that Γ_x and Γ_y are linearly related to the displacement measures, (4.1) may be written as

$$A_G(u' + \alpha) - Pz\theta' = 0, \quad I_{pG}\theta' - Pzu' = 0, \quad (I_x\alpha')' - Pz\theta' = 0. \tag{4.3}$$

where $T(0) = 0$ has been used to determine the unknown constant in the second equation in (4.3) after integration. The first equation is then used to eliminate u' from the second equation to get

$$\left(I_{pG} - \frac{P^2z^2}{A_G}\right)\theta' + Pz\alpha = 0. \tag{4.4}$$

The coupled system of (4.4) and the third equation in (4.3) may be further reduced to a single equation for

$$\left(I_{pG} - \frac{P^2z^2}{A_G}\right)(I_x\alpha')' + P^2z^2\alpha = 0. \tag{4.5}$$

To cast (4.5) in dimensionless form, we set

$$\zeta = \frac{z}{L}, \quad \sigma = \frac{PL^2}{\sqrt{I_xI_{pG}}}, \quad \epsilon_G = \frac{I_x}{A_GL^2}. \tag{4.6}$$

In terms of these dimensionless quantities, (4.5) becomes

$$(1 - \epsilon_G\sigma^2\zeta^2)\alpha^{\circ\circ} + \sigma^2\zeta^2\alpha = 0 \tag{4.7}$$

with $(\ )^\circ = d(\ )/d\zeta$. The second order differential equation (4.7) is supplemented by the two boundary conditions

$$\alpha^\circ(0) = \alpha(1) = 0 \tag{4.8}$$

as previously noted in (4.2) with $\alpha^\circ(0) = 0$ corresponding to $M_x(0) = 0$ (see (3.5)). The parameter ϵ_G is of the order of Ea^2/GL^2, where a is a typical lineal dimension of the cross-section of the cantilever. For a long prismatic body, we have typically $a^2/L^2 << 1$ and, with E/G not large compared to unity, we have $I_x/A_GL^2 << 1$. A perturbation solution of the eigenvalue problem (4.5) and (4.6) is therefore appropriate.

Let

$$\{\alpha(\zeta;\epsilon_G), \sigma(\epsilon_G)\} = \sum_{n=0}^{\infty}\{\alpha_n(\zeta), \sigma_n\}\epsilon_G^n. \tag{4.9}$$

We have from (4.7) and (4.8)

$$[\alpha_0^{\circ\circ} + \sigma_0^2\zeta^2\alpha_0] + \epsilon_G[\alpha_1^{\circ\circ} + \sigma_0^2\zeta^2\alpha_1 + 2\sigma_0\sigma_1\zeta^2\alpha_0 - \sigma_0^2\zeta^2\alpha_0^{\circ\circ}] + O(\epsilon_G^2) = 0 \tag{4.10}$$

and

$$[\alpha_0^\circ(0) + \epsilon_G\alpha_1^\circ(0) + \cdots] = [\alpha_0(1) + \cdots] = 0. \tag{4.11}$$

Since (4.10) and (4.11) must be satisfied for all $\epsilon_G(<< 1)$, we must have

$$\alpha_0^{\circ\circ} + \sigma_0^2\zeta^2\alpha_0 = 0 \tag{4.12}$$

$$\alpha_0^\circ(0) = \alpha_0(1) = 0 \tag{4.13}$$

and

$$\alpha_1^{\circ\circ} + \sigma_0^2\zeta^2\alpha_1 + [2\sigma_0\sigma_1\zeta^2 + \sigma_0^4\zeta^4]\alpha_0 = 0 \tag{4.14}$$

$$\alpha_1^\circ(0) = \alpha_1(1) = 0 \tag{4.15}$$

and so on.

The solution to the $O(1)$ problem (4.12)-(4.13) is known to be

$$\alpha_0 = \alpha_0^{(n)} \equiv \sqrt{\zeta}\, J_{-1/4}(\sigma_0^{(n)}\zeta^2/2) \qquad (n = 1, 2, 3, \cdots) \tag{4.16}$$

where $\sigma_o^{(n)}/2$ is the nth zero of the relevant Bessel function:

$$J_{-1/4}(t) = 0 \tag{4.17}$$

with $\sigma_o^{(1)} \cong 0.4013$. Since, for any particular eigenvalue $\sigma_0^{(n)}$, the solution of the $O(1)$ problem given by (4.16) is also a solution of the homogeneous differential equation corresponding to (4.14), we must satisfy the solvability condition

$$\int_0^1 \{2\sigma_0^{(n)}\sigma_1\zeta^2 + [\sigma_0^{(n)}]^4\zeta^4\}\{\alpha_0^{(n)}\}^2 d\zeta = 0 \tag{4.18}$$

in order for the $O(\epsilon_G)$ problem (4.14)-(4.15) to have a solution. The condition (4.17) determines σ_1 to be

$$\frac{\sigma_1}{\sigma_0^{(n)}} = \frac{\sigma_1^{(n)}}{\sigma_0^{(n)}} \equiv -\frac{1}{2}\left[\sigma_0^{(n)}\right]^2 \frac{\int_0^1[\alpha_0^{(n)}]^2\zeta^4 d\zeta}{\int_0^1[\alpha_0^{(n)}]^2\zeta^2 d\zeta} \tag{4.19}$$

From the results for the weighted integrals of $I_{-1/4}(t)$ (with weight t^n) obtained in [7], we have

$$\frac{\sigma_1^{(1)}}{\sigma_0^{(1)}} = -\sigma_0^{(1)}\frac{\int_0^{\sigma_0^{(1)}/2}[J_{-1/4}(t)]^2t^2dt}{\int_0^{\sigma_0^{(1)}/2}[J_{-1/4}(t)]^2dt} \cong -\sigma_0^{(1)}\frac{0.483}{0.650} \cong -0.743\sigma_0^{(1)} \cong -2.98 \tag{4.20}$$

Thus, the buckling load (corresponding to the lowest eigenvalue $\sigma_0^{(1)}$) of the linear transverse shear strain model is given by

$$\sigma = \sigma^{(1)} \cong \sigma_0^{(1)}[1 - 2.98\epsilon_G + O(\epsilon_G^2)]. \tag{4.21}$$

The $O(\epsilon_G)$ correction term is not insignificant for a cantilever aspect ratio of 1/10.

The results of this section are identical to those obtained in [7], as they should, since the latter study worked with the same set of govering differential equations as (4.3). We reproduce the analysis and results here for subsequent comparisons with those for $\eta = 1$ and those for a model which includes the influence of pre-buckling deformations. The analysis of this section also allows us to omit the details of subsequent calculations for the new studies with $\eta = 1$.

5. Non-Linear Transverse Shear Deformability

For $\eta = 1$ so that Γ_x and Γ_y are nonlinear in the displacement measures, (4.1) may be written as

$$A_G(u' + \alpha) - P(z\theta)' = 0, \quad I_x\alpha' - Pz\theta = 0, \quad (I_{pG}\theta')' - Pzu'' = 0 \tag{5.1}$$

where $M_x(0) = 0$ has been used to determine the unknown constant in the second equation in (5.1) after integration. The first equation is then used to eliminate u from the third equation to get

$$(I_{pG}\theta')' + Pz\left[\alpha - \frac{P}{A_G}(z\theta)'\right]' = 0. \tag{5.2}$$

The coupled system of (5.2) and the second equation in (5.1) may be further reduced to a single equation for

$$\left[\left(I_{pG} - \frac{P^2}{A_G}z^2\right)\theta'\right]' + \frac{P^2}{I_x}z^2\theta = 0. \tag{5.3}$$

In terms of the dimensionless quantities in (4.6), equation (5.3) becomes

$$[(1 - \epsilon_G\sigma^2\zeta^2)\theta^\circ]^\circ + \sigma^2\zeta^2\theta = 0 \tag{5.4}$$

with $(\ )^\circ = d(\ )/d\zeta$. The second order differential equation (5.4) is supplemented by the two boundary conditions

$$\theta^\circ(0) = \theta(1) = 0 \tag{5.6}$$

as previously noted in (4.2) with $\theta^\circ(0) = 0$ corresponding to $T(0) = 0$. Whenever E/G is not large compared to unity, we have for a long prismatic body $\epsilon_G = I_x/A_GL^2 << 1$. A perturbation solution of the eigenvalue problem (5.4) and (5.5), denoted by $\{\bar{\theta}(\zeta, \epsilon_G), \bar{\sigma}(\epsilon_G)\}$, is again appropriate.

With the help of the parametric expansions

$$\{\bar{\theta}(\zeta; \epsilon_G), \bar{\sigma}(\epsilon_G)\} = \sum_{n=0}^{\infty}\{\theta_n(\zeta), \bar{\sigma}_n\}\epsilon_G^n \tag{5.7}$$

we have the following sequence of simpler eigenvalue problems for $\{\theta_n(\zeta), \sigma_n\}$:

The $O(1)$ Problem:

$$\theta_0^{\circ\circ} + \bar{\sigma}_0^2\zeta^2\theta_0 = 0 \tag{5.8}$$
$$\theta_0^\circ(0) = \theta_0(1) = 0$$

The $O(\epsilon_G)$ Problem:

$$\theta_1^{\circ\circ} + \bar{\sigma}_0^2\zeta^2\theta_1 + \{2\bar{\sigma}_1\bar{\sigma}_0\zeta^2\theta_0 - \bar{\sigma}_0^2(\zeta^2\theta_0^\circ)^\circ\} = 0 \tag{5.9}$$
$$\theta_1^\circ(0) = \theta_1(1) = 0$$

and so on.

The solution of the $O(1)$ problem is again

$$\theta_0(\zeta) = \theta_0^{(n)}(\zeta) \equiv \sqrt{\zeta}\ J_{-1/4}(\bar{\sigma}_0^{(n)}\zeta^2/2) \qquad (n = 1, 2, \cdots) \tag{5.10}$$

where $\sigma_0^{(n)}/2$ is the n^{th} zero of $J_{-1/4}(t) = 0$. For buckling, we are interested in the lowest eigenvalue so that $\bar{\sigma}_0 = \sigma_0^{(1)} \approx 4.013$. Similar to the linear transverse shear strain model, $\theta_0^{(1)}$ is also the solution for the homogeneous ODE corresponding to (5.9) for $\bar{\sigma}_0 = \sigma_0^{(1)}$. For the inhomogeneous ODE (5.9) to have a solution, $\bar{\sigma}_1$ must be chosen to satisfy the conventional solvability condition:

$$\int_0^1 \{2\bar{\sigma}_1\sigma_0^{(1)}\zeta^2[\theta_0^{(1)}]^2 - (\sigma_0^{(1)})^2\theta_0^{(1)}[\zeta^2(\theta_0^{(1)})^\circ]^\circ\}d\zeta = 0 \tag{5.11}$$

or

$$\frac{\bar{\sigma}_1}{\sigma_0^{(1)}} = \frac{1}{2}\frac{\int_0^1 \theta_0^{(1)}[\zeta^2(\theta_0^{(1)})^\circ]^\circ d\zeta}{\int_0^1[\theta_0^{(1)}]^2\zeta^2 d\zeta} \tag{5.12}$$

With

$$\theta_0^{(1)}[\zeta^2(\theta_0^{(1)})^\circ]^\circ = [\zeta(\theta_0^{(1)})^2]^\circ - \{1 + [\sigma_0^{(1)}]^2\zeta^4\}[\theta_0^{(1)}]^2 \tag{5.13}$$

we may re-write (5.12) as

$$\frac{\bar{\sigma}_1}{\sigma_0^{(1)}} = -\frac{1}{2}\frac{\int_0^1 \zeta[1 + (\sigma_0^{(1)})^2\zeta^4]\left[J_{-1/4}\left(\frac{1}{2}\sigma_0^{(1)}\zeta^2\right)\right]^2 d\zeta}{\int_0^1 \zeta^3\left[J_{-1/4}\left(\frac{1}{2}\sigma_0^{(1)}\zeta^2\right)\right]^2 d\zeta} \tag{5.14}$$

$$= -\frac{\sigma_0^{(1)}}{4}\frac{\int_0^{\sigma_0^{(1)}/2}\{1 + 4t^2\}[J_{-1/4}(t)]^2 dt}{\int_0^{\sigma_0^{(1)}/2}[J_{-1/4}(t)]^2 t dt}$$

It follows from the results of [7] that

$$\bar{\sigma}_1 \simeq -\frac{1}{4}[\sigma_0^{(1)}]^2\left\{\frac{1.806 + 4(0.483)}{0.650}\right\} \simeq -1.438[\sigma_0^{(1)}]^2 \tag{5.15}$$

We have then the following perturbation solution for the buckling load of the nonlinear transverse shear strain model:

$$\bar{\sigma} \simeq \sigma_0^{(1)}\{1 - 1.438\sigma_0^{(1)}\epsilon_G + O(\epsilon_G^2)\} = \sigma_0^{(1)}\{1 - 5.771\epsilon_G + O(\epsilon_G^2)\}. \qquad (5.16)$$

Thus, when nonlinear terms are included in the strain-displacement relations for the transverse shearing strains, the $O(\epsilon_G)$ correction to the Michell-Prandtl solution for the buckling load is twice the value when the nonlinear terms are not included. For relatively long, homogeneous, isotropic, elastic prismatic bodies with $a/L = 1/10$, the correction term is now about 15% of the Michell-Prandtl solution.

6. The Effect of Pre-Buckling Deformations

In this section, we analyze the effect of pre-buckling deformations in the nonlinear transverse shear strain model for prismatic bodies without warping. For this case, the system (3.10) with $\eta = 1$ applies. Upon integrating the first two equations of (3.10) and oberving the relevant boundary conditions in (3.9), we obtain

$$u' = -(\alpha + v_p'\theta) + \frac{P}{A_G}(z\theta)', \qquad \alpha' + v_p'\theta' = \frac{Pz}{I_x}\theta \qquad (6.1a, b)$$

$$(I_{pG}\theta' - Pzu')' - P(zv_p')\theta + v_p''Pz\theta + Pu' = 0. \qquad (6.1c)$$

We now use the first and second equation of (6.1) to eliminate u' and α' from the third leaving us with a single second order differential equation for θ:

$$\left(I_{pG}\theta' - \frac{P^2z^2}{A_G}\theta'\right)' + \frac{P^2z^2}{I_x}\theta - P\theta(v_p' - zv_p'') = 0 \qquad (6.2)$$

where v_p' is as given in (3.3). This second order differential equation is supplemented by the boundary conditions $\theta'(0) = \theta(L) = 0$. Equation (6.2) can be written in dimensionless form with the help of the dimensionless quantities in (4.6) and

$$\epsilon_v = I_x/I_y. \qquad (6.3)$$

The primary unknown θ is therefore determined by the dimensionless differential equation

$$[(1 - \epsilon_G\sigma^2\zeta^2)\theta^\circ]^\circ + \sigma^2\left[\left(1 - \frac{1}{2}\epsilon_v\right)\zeta^2 - \left(\frac{1}{2}\epsilon_v + \epsilon_G\right)\right]\theta = 0 \qquad (6.4)$$

subject to $\theta^\circ(0) = \theta(1) = 0$.

By setting $\epsilon_G = 0$, we recover the governing differential equation for cantilever buckling obtained in [7] when the prismatic bodies are known to be not transverse shear deformable. By allowing for transverse shear deformability while concurrently retaining terms involving pre-buckling displacement field v_p and its derivatives, we are able to make the following observations for the first time:

1. While the effect of terms involving pre-buckling displacement v_p terms may be $O(1)$ or small of higher order depending on the aspect ratio of the cantilevers cross section, ϵ_v, (whether it is $O(1)$ or small by an order of magnitude), the effect of the transverse shear defomrability is always small of higher order for prismatic bodies as long as E and G are of the same order of magnitude (including homogeneous and isotropic elastic cantilevers) since we always have $a/L << 1$ for prismatic bodies.

2. When the effect of transverse shear deformability is significant and should be included in the model, then terms involving pre-buckling deformation must also be retained for consistency since these contribute a term in the final governing differential equation involving the effect of transverse shear strains, namely, the last term in the coefficient of θ proportional to ϵ_G.

3. When we consider the effect of transverse shear deformability alone, there is a significant difference between the correction to the Michell-Prandtl buckling load given by the linear expression for the transverse shear strains and that by the nonlinear relations. However, the additional contribution from transverse shear deformability indirectly through terms involving pre-buckling displacements has the effect of offsetting the contribution of the nonlinear terms in the expressions for the transverse shear strains.

The validity of the first observation can be seen from the governing differential equation (6.4) for the problem. More specifically, we have from (6.3) the order of magnitude relation $\epsilon_v = I_x/I_y = O(a^2/b^2)$ for homogeneous, isotropic, elastic prismatic bodies where a and b are the lineal dimension in the x-direction and y-direction, respectively. The contribution of terms involving the pre-buckling displacement v_p to (6.4) and the buckling load is small of higher order only if the cantilever cross section has a small aspect ration so that $a^2/b^2 << 1$.

To demonstrate the necessity to retain terms involving pre-buckling deformation when the effect of transverse shear deformability is considered significant, it suffices to limit our analysis to the case $\epsilon_v << 1$. The leading term solution for a perturbation solution in ϵ_v , denoted by $\{\tilde{\theta}, \tilde{\sigma}\}$, is determined by the eigenvalue problem

$$[(1 - \epsilon_G\tilde{\sigma}^2\zeta^2)\tilde{\theta}^\circ]^\circ + \tilde{\sigma}^2(\zeta^2 - \epsilon_G)\tilde{\theta} = 0 \qquad (6.5)$$

$$\tilde{\theta}^\circ(0) = \tilde{\theta}(1) = 0 \qquad (6.6)$$

Note that this problem differs from the one when terms involving the pre-buckling displacement v_p are neglected.

We now seek a regular perturbation solution of this problem in the second small parameter ϵ_G. The leading term solution is again determined by (5.8) and given by (5.10). Instead of (5.9), the $O(\epsilon_G)$ correction term is now determined by

$$\tilde{\theta}_1^{\circ\circ} + (\tilde{\sigma}_0^{(1)})^2\zeta^2\tilde{\theta}_1 + [2\sigma_0^{(1)}\tilde{\sigma}_1\zeta^2 - (\sigma_0^{(1)})^2]\theta_0^{(1)} - (\sigma_0^{(1)})^2[\zeta^2(\theta_0^{(1)})^\circ]^\circ = 0 \quad (6.7a)$$

$$\tilde{\theta}_0^\circ(0) = \tilde{\theta}_1(1) = 0. \quad (6.7b)$$

The differential equation in (6.7) differs from that of (5.9) by a term associated with the pre-buckling displacement v_p even if the aspect ratio a/b is small so that $\epsilon_v = I_x/I_y$ is negligible. This confirms the second observation above indicating the necessity of retaining terms involving the pre-buckling displacement v_p whenever the effect of transverse shear strains on the buckling load is significant.

Regarding the third observation above, we note that the solvability condition for (6.7) now leads to the following expression for $\tilde{\sigma}_1$:

$$\frac{\tilde{\sigma}_1}{\sigma_0^{(1)}} = \frac{\int_0^1 [\theta_0^{(1)}]^2 d\zeta + \int_0^1 [\zeta^2(\theta_0^{(1)})^\circ]^\circ \theta_0^{(1)} d\zeta}{2\int_0^1 \zeta^2[\theta_0^{(1)}]^2 d\zeta}$$

$$= -\sigma_0^{(1)} \frac{\int_0^{\sigma_0^{(1)}/2} [J_{-1/4}(t)]^2 t^2 dt}{\int_0^{\sigma_0^{(1)}/2} [J_{-1/4}(t)]^2 t dt} \cong -0.743\sigma_0^{(1)} \cong -2.98. \quad (6.8)$$

The condition (6.8) is identical to the corresponding expression for the $O(\epsilon_G)$ correction term for σ_1 in (4.19) with $n = 1$ and in (4.20). Hence, for the first two terms of a perturbation solution in ϵ_G without pre-buckling deformations, the linear transverse shear strain model gives a accurate approximation for the correction of the Michell-Prandtl buckling load than the nonlinear model, at least in the case where the aspect ratio a/b is small so that $\epsilon_v = I_x/I_y$ is negligible.

7. On the Moment Equilibrium Equation (2.10)

We noted in section 2 that the seven equilibrium equations (2.7)-(2.10) are the exact consequence of the assumed strain-displacement relations (2.2)-(2.5) and the variational equation (2.1). In previous treatments of this lateral buckling problem for cantilevers, third and higher order nonlinear terms in these equilibrium equations are neglected. In particular, the terms $-\theta[v'M_y + u'M_x - \theta\theta'N]'$ are not included in the analysis of [7], [8] and [9]. While these terms appear to be of higher order in the unknowns, $-\theta[v'M_y]'$, for example, leads to a term $-\theta[v_p'Pz]'$ in the last of the equilibrium equation for the buckled state in (3.4). We see from (6.4) that this term contributes in a qualitatively significant way to the coefficient of θ of the governing differential equation for the determination of the buckling load. In fact, in the absence of the $-\theta[v_p'Pz]'$term, we would have

$$[(1 - \epsilon_G\tilde{\sigma}^2\zeta^2)\tilde{\theta}^\circ]^\circ + \tilde{\sigma}^2[\zeta^2(1 - 2\epsilon_v)]\tilde{\theta} = 0 \quad (7.1)$$

instead of (6.4). Thus, that portion of the effect of transverse shear deformation on the buckling load through pre-buckling deformation would be lost. We saw in section 6 that this effect is comparable to that induced by transverse shear deformability on the buckled state directly. For cases where $\epsilon_v = I_x/I_y = O(a^2/b^2)$ is not small by an order of magnitude, omitting the term $-\theta[v_p'Pz]'$ would change the buckling load significantly, not just a small perturbation since the difference between the differential equation in (6.4) and (7.1) is $-\sigma^2\theta[\epsilon_G - \epsilon_v(3\zeta^2 - 1)/2]$, which is $O(\sigma^2\theta)$ whenever ϵ_v is $O(1)$.

When the term $-\theta[v_p'Pz]'$ is included, the method of reduction of the buckling equations (3.10) and the relevant boundary conditions (3.9) for the linear transverse shear strain model (corresponding to $\eta = 0$) to an eigenvalue problem for a second order differential equation used effectively in [7] no longer applies. It appears that we would have to work with an eigenvalue problem for a fourth order system of two differential equations to determine the buckling load. In contrast, the reduction to a second order differential equation is still possible as shown in section 6 in the nonlinear transverse shear strain model corresponding to the case $\eta = 1$. Since both the nonlinear transverse shear strain terms and pre-buckling deformations contribute significantly to the buckling load when transverse shear strain effects are important, there is no reason to pursue a linear transverse shear strain model that includes pre-buckling deformations.

8. Concluding Remarks

The present study was intended to analyze the relevance or significance of (i) the nonlinear terms in the strain-displacement relations for the transverse shear strains, and (ii) the prebuckling deformation, on the buckling load of cantilevers subject to a transverse end force when the effect of the transverse shear deformations is sufficiently significant to necessitate a correction of the Michell-Prandtl solution. It was found (in sections 4 and 5) that the nonlinear terms in γ_x and γ_y contribute significantly to the buckling load in that they effectively double the magnitude of the correction to the Michell-Prandtl solution whenever warping and pre-buckling deformation are neglected. However, it was shown in section 6 that the the inclusion of terms involving the pre-buckling displacement v_p in the buckling analysis further modifies this correction term, even if the aspect ratio of the cross section of the prismatic body is small so that terms involving $\epsilon_v = I_x/I_y$ may be neglected in the governing differential equation for the buckling problem. To order ϵ_G, the additional modification resulting from the retention of v_p terms is sufficiently substantial that it effectively offsets the effects (on the buckling load) of the nonlinear terms in the expressions for the transverse shear strains, at least for cases when the terms involving $\epsilon_v = I_x/I_y$ are negligible.

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