Z. angew. Math. Phys. 51 (2000) 509–529
0044-2275/00/040509-21 $ 1.50+0.20/0

Zeitschrift für angewandte Mathematik und Physik ZAMP

Outer solution for elastic torsion by the method of boundary layer residual states

Frederic Y. Wan

Dedicated to Professor Eric Reissner for his guidance and support as the author's teacher and colleague.

Abstract. By the method of boundary layer residual state (BLRS), it is possible to specify the unknown parameters in the general form of the outer asymptotic solution of the governing differential equations for linear boundary value problems (BVP) without any reference to the inner asymptotic solutions of the same problem and the matching procedure. The method accomplishes this task by rationally assigning a portion of the prescribed boundary data to the outer solution. Specifically, the method requires certain weighted averages of the outer solution to be equal to the same averages of the data over the (localized) boundary where the data is prescribed. These weighted averages are consequences of a reciprocity relation inherent in the BVP and the stipulation that the difference between the outer solution and the exact solution (called the *residual solution*) of the BVP be a boundary layer phenomenon.

The weighted average requirements are only necessary conditions for the residual state to be a boundary layer. Unfortunately, there are generally countably infinite number of (2) states, many more than the available degrees of freedom in the outer solution to satisfy them. We must show that there is no over-determination or non-uniqueness of the outer asymptotic solution, the abundance of necessary conditions notwithstanding. The present note describes an approach to assuring a well-specified outer solution (up to the expected accuracy) by way of the problem of Saint-Venant torsion. The same approach also also applies to other linear BVP, deducing the appropriate outer solution whenever the determination of the relevant inner solutions is not practical.

Keywords. Saint Venant's principle, match asymptotic expansions, elastic beam torsion, boundary layer residual solution.

1. Introduction

In the context of the method of matched asymptotic expansions, theories of beams, plates and shells are known to be the truncated *outer* (asymptotic expansions of the exact) solutions of the relevant boundary value problems (BVP) in three-dimensional elasticity theory [1,2,10,11,14-17,22]. It is rarely possible for an outer solution to fit the prescribed boundary data of the BVP exactly. Appropriate *inner* (asymptotic expansion) solutions are needed to satisfy the relevant boundary conditions. Such inner solutions are to be matched to the outer solution of the problem via an intermediate variable to provide a composite asymptotic solution for the problem. Because of the complexity of the three-dimensional problems, it is often not feasible to obtain the relevant inner solutions for a particular problem and to execute the matching procedure for the correct asymptotic solution as required by this approach. With the help of the reciprocal theorem in elasticity theory, a different approach has been developed in [4-7] to assign an appropriate portion of the prescribed boundary data to the outer solution for *flat plate* problems without any reference to the inner solutions and matching. This new *method of boundary layer residual states* (BLRS) is based on the requirement that the *residual data* of the problem (defined to be the difference between the actual prescribed data and the portion of that data assigned to the (entire) outer solution induces only a boundary layer *residual solution*, i.e., an exponentially decaying (residual) elastostatic state of the plate. The method has since been extended to shell problems in [3,8] and also applied to beam problems in [20-22].

The BLSR method for three dimensional problems of elastostatics generates a group of necessary conditions for the residual state to be a boundary layer, each specifying a surface integral relation between the prescribed data and the outer solution for each localized area of surface load of the elastic body. More specifically, each of the necessary conditions requires a certain weighted average of the outer solution over the (localized) loaded area to be the same as the corresponding weighted average of the prescribed boundary data. The admissible set of weight functions involved is determined by the reciprocal theorem to be a solution of a prescribed auxiliary BVP of elastostatics for the same elastic body (designated as a (2)-state of the original problem of interest). The solution of this auxiliary BVP is known to be non-unique (so that there are many such (2)-states) and it is possible in principle that the totality of these weighted average conditions may overspecify the outer solution. Hence, the uniqueness property of the asymptotic expansion for a given function (see p.17 of [13]) cannot be invoked to validate our result as the correct outer asymptotic solution of the problem.

A new approach to validating the correctness of the outer solution by the BLRS method and the nature of its key results will be described and illustrated in this paper by way of the problem of axial torsion of prismatic bodies. The result will be applied to provide (under certain assumptions on the mathematical structure of the relevant BVP) an alternative to the minimum energy criterion for the validation of Saint-Venant's principle and ultimately the Saint-Venant torsion solution (see [9] and references therein) as the correct outer asymptotic solution of the actual problem of elastic torsion in the three-dimensional linear elasticity theory.

2. Torsion of prismatic bodies

The deformation of a slender cylindrical body in torsion is expected to consist of an interior component which is significant throughout the entire slender body and boundary layer components which decay rapidly (exponentially) away from the ends (see [12,18,21] and references therein). In the context of the method of matched asymptotic expansions, the interior solution component corresponds to the outer asymptotic solution. It is the part of the actual solution of primary interest in many applications. When stress data are prescribed at both ends of the prismatic body, the Saint-Venant torsion solution (see [9,12,18,21]) is generally taken to be a good approximate interior solution for the actual solution to the torsion problem. The method of the Saint-Venant solution in that case effectively assigned a portion of the stress data to an exact (particular) solution of the equations of elasticity by an unproved conjecture of Saint-Venant known today as Saint-Venant's principle. This method of assignment as is now known to be identical to the assignment by the BLRS method using the simplest admissible (2)-state for the problem [21]. As noted in [21], Saint-Venant's principle is not applicable to other types of boundary data while the method of BLRS continues to be successful.

There are many other admissible (2)-states for the torsion problem. In order to validate the Saint-Venant torsion solution we must show that a different set of (2)-states does not lead to a different outer solution for the problem. The possible complications of over specification or non-uniqueness will be addressed in this paper. We begin by stating the BVP for the torsion of prismatic, linearly elastic bodies.

We will work with the Cartesian coordinates (x_1, x_2, x_3) with x_3 in the direction along the length (and through the centroid of the uniform cross section) of the cylindrical body. In terms of the displacement components $\{u_j\}$, the strain components $\{e_{ij}\}$ and stress components $\{\sigma_{ij}\}$, the relevant equations of linear elasticity theory for isotropic bodies are:

(i) the strain-displacement relations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad (i, j = 1, 2, 3), \tag{2.1}$$

(ii) the stress-strain relations

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e \delta_{ij} \qquad (i, j = 1, 2, 3) \tag{2.2}$$

where μ and λ are the Lamé parameters (which are constants for the homogeneous bodies considered herein), δ_{ij} is the Kronecker delta and $e = e_{11} + e_{22} + e_{33}$, and

(iii) the equilibrium equations

$$\sigma_{1k,1} + \sigma_{2k,2} + \sigma_{3k,3} = -\,p_k \qquad (k = 1, 2, 3). \tag{2.3}$$

where $\{p_k(x_1, x_2, x_3)\}$ are known interior load components which are absent for the Saint-Venant torsion problem.

The governing differential equations are supplemented by appropriate boundary conditions on the boundary surface of the body. For simplicity, we limit further consideration to general cylinders with a uniform solid (simply-connected) cross section A bounded by the curve C. We characterize this edge curve of the solid cross section by $f(x_1, x_2) = 0$. The cylindrical surface $S = \{f(x_1, x_2) = 0,\ 0 < x_3 < L\}$ with a (positive outward) unit normal vector ν is stress free so that

$$\nu_1\sigma_{1k} + \nu_2\sigma_{2k} = 0 \qquad \text{on } S \tag{2.4}$$

for $k = 1, 2, 3$ with $\nu_k = \nu \cdot \mathbf{i}_k$. At a constant x_3 end, say ℓ ($= 0$ or L), we first consider the classical case of prescribed stress data:

$$\sigma_{3k}(x_1, x_2, \ell) = \tau_k^\ell(x_1, x_2) \qquad (k = 1, 2, 3;\ \ell = 0, L). \tag{2.5}$$

Other types of end data for which Saint-Vernant's principle is not useful will be discussed later in sections 7 and 8 (see also [21]).

For pure torsion, we require the prescribed stress data to have no resultant forces or moments other than an axial torque so that

$$\iint_A \tau_k^\ell dx_1 dx_2 = \iint_A (x_3\tau_j^\ell - x_j\tau_3^\ell) dx_1 dx_2 = 0 \tag{2.6}$$

with $k = 1, 2, 3,\ j = 1, 2$, and $\ell = 0, L$. For global equilibrium, we should have the same resultant torque M_t at both ends of the body:

$$\iint_A (x_1\tau_2^0 - x_2\tau_1^0) dx_1 dx_2 = \iint_A (x_1\tau_2^L - x_2\tau_1^L) dx_1 dx_2 = M_t\ . \tag{2.7}$$

A classical (exact) solution of the governing equations and the traction free conditions on the cylindrical surface S is given by

$$u_1 = -\ \theta_0 x_2 x_3, \qquad u_2 = \theta_0 x_1 x_3, \qquad u_3 = \theta_0\psi(x_1, x_2) \tag{2.8}$$

where θ_0 is an unknown constant (*angle of twist*) and ψ is the unknown *warping function* still to be specified. The only nonvanishing stress components corresponding to (2.8) are

$$\begin{aligned} \sigma_{31} = \sigma_{13} &= \mu\theta_0(\psi_{,1} - x_2) = \theta_0\Phi_{,2} \\ \sigma_{32} = \sigma_{32} &= \mu\theta_0(\psi_{,2} + x_1) = -\ \theta_0\Phi_{,1} \end{aligned} \tag{2.9}$$

with the only nontrivial equilibrium equation, $\sigma_{13,1} + \sigma_{23,2} = 0$, satisfied by way of the *stress function* $\Phi(x_1, x_2)$. Compatibility among the strain components (through $\psi_{,12} = \psi_{,21}$) requires that Φ be the solution

$$\left(\frac{\Phi_{,1}}{\mu}\right)_{,1} + \left(\frac{\Phi_{,2}}{\mu}\right)_{,2} = -\,2 \tag{2.10}$$

For a simply-connected cross section, the stress free condition on the cylindrical surface can be shown to require Φ to satisfy the Dirichlet condition on C:

$$\Phi = 0 \qquad \text{on } C. \tag{2.11}$$

Except for a rigid body rotation, the solution given by (2.8)-(2.11) is now known also to be the outer solution of Navier's equations of linear elasticity theory for the problem of elastic torsion (see Appendix).

The boundary value problem defined by (2.10) and (2.11) determines the stress function Φ. It remains to choose the unknown parameter θ_0 so that the end stress boundary conditions (2.5) is satisfied. It is not difficult to show by a parametric series in powers of $\epsilon = b/L$ (where b is the maximum span of the cross section of the slender body) that for any constant θ_0, equations (2.8)-(2.11) constitute a formal outer asymptotic expansion of the exact solution appropriate for the torsion problem.

Evidently, the stress components σ_{31}, σ_{32} and $\sigma_{33} = 0$ cannot be made to fit an arbitrary set of admissible stress data at each of the two ends of the prismatic body (even those with $\tau_3^0 = \tau_3^L = 0$). For slender cylindrical bodies, Saint-Venant's principle requires only that the resultant axial torque of the internal stress components be the same as that of the prescribed end stresses:

$$\iint_A (x_1\sigma_{32} - x_2\sigma_{31})dx_1dx_2 = M_t. \tag{2.12}$$

This resultant torque condition can be transformed into the following integral relation for Φ:

$$2\theta_0 \iint_A \Phi(x_1, x_2)dx_1dx_2 = M_t. \tag{2.13}$$

It determines θ_0 and therewith the Saint-Venant torsion solution completely. Note that the solution (2.8) and (2.9), with θ_0 given by (2.13) and with Φ determined by (2.10) and (2.11), does not satisfy the end conditions at $x_3 = 0$ and $x_3 = L$ in detail, only having the same weighted average as (2.7) over the end sections as the prescribed end data. Saint Venant's principle asserts the residual solution to be significant only in an $O(b)$ boundary layer adjacent to the two ends of the slender body

3. The method of boundary layer residual states for torsion

As indicated earlier, Saint-Venant's principle had been an unproved conjecture but useful over the years. Its application to the torsion problem was shown by Sternberg and Knowles to result in a minimum strain energy solution for the relevant stress BVP of elastostatics [19]. However, the work of [19] did not deal with the boundary layer nature of the residual solution. More recently, it was shown that Saint-Venant's principle is a set of necessary conditions for the residual solution to be a boundary layer [21]. This was established by the BLRS method using the simplest (2)-state for the weighing functions in the following form of the reciprocal theorem of elasticity:

$$\iint_A \left[\sigma_{31}^R u_1^{(2)} + \sigma_{32}^R u_2^{(2)} + \sigma_{33}^R u_3^{(2)}\right]_{x_3=\ell} dx_1 dx_2 = 0 \tag{3.1}$$

for $\ell = 0$ and $\ell = L$, valid up to *exponentially small terms* (E.S.T.).

Briefly, we sketch the derivation of (3.1) for the end $\ell = 0$. For this end of a prismatic body without interior loading and no surface traction on S, we take the reciprocal theorem in the following form

$$\iint_A \left[\sum_{k=1}^{3} \sigma_{3k}^{(1)} u_k^{(2)}\right]_{x_3=0}^{z} dx_1 dx_2 = \iint_A \left[\sum_{k=1}^{3} \sigma_{3k}^{(2)} u_k^{(1)}\right]_{x_3=0}^{z} dx_1 dx_2 \tag{3.2}$$

for any two elastostatic states of the body and $0 < z < L$. We take the (1)-state in (3.2) to be the residual state, i.e., the difference between the outer solution and the exact solution of the torsion problem. An *admissible* (2)-state for the $x_3 = 0$ end in this case are taken to

(i) satisfy the homogeneous equations of elasticity,
(ii) be traction free on S,
(iii) have at worst an algebraic growth as $L \to \infty$ (for a fixed x_3/L), and
(iv) be traction free at the end $x_3 = 0$.

For such a (2)-state and a location z sufficiently far away from both ends of the prismatic body, (3.2) requires (3.1) as a *necessary condition for the residual state to be exponentially small away from the ends.*

A judicious application of (3.1) (for $\ell = 0$) allows us to determine the outer asymptotic solution for our stress boundary value problem without any reference to the inner solution or matching. An elementary (2)-state needed for (3.1) readily suggests itself. The rigid body rotation

$$u_1^{(2)} = -\, x_2, \qquad u_2^{(2)} = x_1, \qquad u_3^{(2)} = 0 \tag{3.3}$$

satisfies all four requirements for an admissible (2)-state. It reduces (3.1) for $\ell = 0$ to the same requirement (2.12) or (2.13) as Saint-Venant's principle. Note that the same (2)-state (3.3) may be used at $x_3 = L$ but the resulting necessary condition does not give a new requirement in views of (2.7).

A necessary condition similar to (3.1) can also be established for other types of end data at an end $x_3 = \ell$ (see [21] and sections 7 and 8 herein). The main purpose of the present discussion, however, is to address the possibility of distinctly different solutions of the form (2.8) for (2.5) resulting from other admissible (2)-states.

4. The correct outer solution for Saint-Venant torsion

We will work with a vector form of the necessary condition (3.1) and the reciprocal relations (3.2) by setting

$$\mathbf{u} = u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3, \qquad \mathbf{T} = \sigma_{31}\mathbf{i}_1 + \sigma_{32}\mathbf{i}_2 + \sigma_{33}\mathbf{i}_3. \tag{4.1}$$

Let $\mathbf{T}^R$ and $\mathbf{T}^R + \Delta\mathbf{T}^R$ be the decaying residual states induced by the application of (3.1) with the (2)-states $\mathbf{u}^{(2)}$ and $\mathbf{u}^{(2)} + \Delta\mathbf{u}^{(2)}$, respectively, for the stress data at $x_3 = 0$ (and an analogous choice of (2)-state at $x_3 = L$) with (3.1) itself taking the form

$$\iint_A \left[\mathbf{T}^R \cdot \mathbf{u}^{(2)}\right]_{x_3=0} dx_1 dx_2 = 0 \tag{4.2}$$

$$\iint_A \left[(\mathbf{T}^R + \Delta\mathbf{T}^R) \cdot (\mathbf{u}^{(2)} + \Delta\mathbf{u}^{(2)})\right]_{x_3=0} dx_1 dx_2 = 0, \tag{4.3}$$

except for E.S.T. The reciprocal relation (3.2) in the form

$$\iint_A \left[\mathbf{T}^{(1)} \cdot \mathbf{u}^{(2)}\right]_{x_3=0}^{z} dx_1 dx_2 = \iint_A \left[\mathbf{T}^{(2)} \cdot \mathbf{u}^{(1)}\right]_{x_3=0}^{z} dx_1 dx_2 \tag{4.4}$$

can be used to establish the following lemma:

[IV.1] *If the difference between the (corresponding components of the) two (2)-states for the $x_3 = 0$ end is exponentially small at large distances away from the ends, then the difference in the induced residual state $\Delta\mathbf{T}^R$ satisfies the integral condition*

$$\iint_A \left[\Delta\mathbf{T}^R \cdot \mathbf{u}^{(2)}\right]_{x_3=0} dx_1 dx_2 = 0 \tag{4.5}$$

except for E.S.T. Moreover, the difference $\Delta\mathbf{T}^0$ between the corresponding two outer solutions satisfies the same integral condition so that

$$\iint_A \left[\Delta\mathbf{T}^0 \cdot \mathbf{u}^{(2)}\right]_{x_3=0} dx_1 dx_2 = 0 \tag{4.6}$$

again except for E.S.T.

Note that the two (2)-states may be very different adjacent to the end sections; they are nearly the same only at a distance far away from the loaded ends involved. It is rather remarkable that the "near identity" of the (2)-states away from the ends should be sufficient to guarantee the "near identity" of the *weighted averages* of the two induced outer solutions over a loaded end.

A proof of [IV.1] will be sketched in the next section. Here we apply [IV.1] to the Saint-Venant torsion solution. With

$$\{\mathbf{T}^0, \mathbf{T}^0 + \Delta\mathbf{T}^0\} = \{\theta_0, \theta_0 + \Delta\theta_0\}[\Phi_{,2}\mathbf{i}_1 - \Phi_{,1}\mathbf{i}_2], \tag{4.7}$$

the condition (4.6) becomes

$$\Delta\theta_0 \iint_A [\Phi_{,2}\mathbf{i}_1 - \Phi_{,1}\mathbf{i}_2] \cdot [-x_1\mathbf{i}_1 + x_2\mathbf{i}_2] dx_1 dx_2 = 0 \tag{4.8}$$

(up to E.S.T.) which can be transformed into

$$2\ \Delta\theta_0 \iint_A \Phi dx_1 dx_2 = 0 \tag{4.9}$$

or $\Delta\theta_0 = 0$, except for E.S.T. It follows that the two outer solutions differ only by exponentially small terms.

Note that the [IV.1] does not imply the identity of all outer solutions determined by the BLRS method except for E.S.T. For the slender cylindrical body problem, there are admissible (2) states which differ from the rigid body rotation (3.3) not by an exponentially small term away from the ends. For examples, there are five other rigid body displacements and rotations other than (3.3) that qualify as (2)-states for prescribed stress end conditions. However, they assign only end data to outer solutions associated with axial extension, pure bending and flexure, but not torsion. This observation (together with the requirement on the prescribed stress end data to have no resultant forces and moments other than the axial torque (see (2.6)) eliminates any impact these few other non-layer (2)-states may have on the form of the outer asymptotic solution of the torsion problem.

5. Proof of [IV.1]

With the two outer solutions $\{\mathbf{u}^0, \mathbf{T}^0\}$ and $\{\mathbf{u}^0 + \Delta\mathbf{u}^0, \mathbf{T}^0 + \Delta\mathbf{T}^0\}$ determined by the method of BLSR (as in Section (3)) using $\mathbf{u}^{(2)}$ and $\mathbf{u}^{(2)} + \Delta\mathbf{u}^{(2)}$, respectively, the corresponding pairs $\{\mathbf{u}^{(2)}, \mathbf{u}^R\}$ and $\{\mathbf{u}^{(2)} + \Delta\mathbf{u}^{(2)}, \mathbf{u}^R + \Delta\mathbf{u}^R\}$ satisfy (4.2) and (4.3) respectively. Applications of the reciprocal relation (3.2) for a subdomain of the prismatic body, say $0 \leq x_3 \leq z < L$, give

$$\begin{aligned} 0 &= \iint_A \Big[\mathbf{T}^R \cdot \mathbf{u}^{(2)}\Big]_{x_3=0} dx_1 dx_2 = \iint_A \Big[\mathbf{T}^R \cdot \mathbf{u}^{(2)} - \mathbf{T}^{(2)} \cdot \mathbf{u}^R\Big]_{x_3=0} dx_d x_2 \\ &= \iint_A \Big[\mathbf{T}^R \cdot \mathbf{u}^{(2)} - \mathbf{T}^{(2)} \cdot \mathbf{u}^R\Big]_{x_3=z} dx_1 dx_2 \end{aligned} \tag{5.1}$$

(up to E.S.T.) since $\mathbf{T}^{(2)} = 0$ at $x_3 = 0$. Similarly, we have

$$\begin{aligned} 0 &= \iint_A \Big[(\mathbf{T}^R + \Delta\mathbf{T}^R)\cdot(\mathbf{u}^{(2)} + \Delta\mathbf{u}^{(2)})\Big]_{x_3=0} dx_1 dx_2 = \;\cdots\cdots \\ &= \iint_A \Big[(\mathbf{T}^R + \Delta\mathbf{T}^R)\cdot(\mathbf{u}^{(2)} + \Delta\mathbf{u}^{(2)}) \\ &\qquad - (\mathbf{u}^R + \Delta\mathbf{u}^R)\cdot(\mathbf{T}^{(2)} + \Delta\mathbf{T}^{(2)})\Big]_{x_3=z} dx_1 dx_2 \end{aligned} \tag{5.2}$$

(up to E.S.T.). Observing (5.1) simplifies (5.2) to

$$\begin{aligned} 0 = \iint_A \Big[&\Delta\mathbf{T}^R\cdot\mathbf{u}^{(2)} + (\mathbf{T}^R + \Delta\mathbf{T}^R)\cdot\Delta\mathbf{u}^{(2)} - \Delta\mathbf{u}^R\cdot\mathbf{T}^{(2)} \\ &- (\mathbf{u}^R + \Delta\mathbf{u}^R)\cdot\Delta\mathbf{T}^{(2)}\Big]_{x_3=z} dx_1 dx_2 \end{aligned} \tag{5.3}$$

For a cross section location z $(0 < z < L)$ sufficiently far away from the two ends, $\Delta\mathbf{T}^{(2)}$ and $\Delta\mathbf{u}^{(2)}$ are exponentially small (by hypothesis); hence, (5.3) reduces to

$$0 = \iint_A \Big[\Delta\mathbf{T}^R\cdot\mathbf{u}^{(2)} - \mathbf{T}^{(2)}\cdot\Delta\mathbf{u}^R\Big]_{x_3=z} dx_1 dx_2 \tag{5.4}$$

except for E.S.T. We now apply the reciprocal relation (3.2) again to transform (5.4) back to an integral over the cross section at $x_3 = 0$:

$$\begin{aligned} 0 &= \iint_A \Big[\Delta\mathbf{T}^R\cdot\mathbf{u}^{(2)} - \mathbf{T}^{(2)}\cdot\Delta\mathbf{u}^R\Big]_{x_3=0} dx_1 dx_2 \\ &= \iint_A \Big[\Delta\mathbf{T}^R\cdot\mathbf{u}^{(2)}\Big]_{x_3=0} dx_1 dx_2 \end{aligned} \tag{5.5}$$

except for E.S.T., keeping in mind $\mathbf{T}^{(2)} = 0$ at $x_3 = 0$. Equation (4.5) is therefore proved.

With

$$\Delta\mathbf{T}^R = \Big[\mathbf{T} - (\mathbf{T}^0 + \Delta\mathbf{T}^0)\Big] - \Big[\mathbf{T} - \mathbf{T}^0\Big] = -\,\Delta\mathbf{T}^0, \tag{5.6}$$

it follows from (5.5) that

$$\iint_A \Big[\Delta\mathbf{T}^0\cdot\mathbf{u}^{(2)}\Big]_{x_3=0} dx_1 dx_2 = 0 \tag{5.7}$$

(except for E.S.T.) which is effectively the condition (4.6) in [IV.1].

It should be evident from the proof of [IV.1] that the condition (5.7) (or (4.6)) holds also if $\mathbf{u}^{(2)}$ is replaced by $\mathbf{u}^{(2)} + \Delta\mathbf{u}^{(2)}$.

6. Saint-Venant torsion for a circular shaft by different (2)-states

Up to a rigid body rotation of the cross section, the outer (Saint-Venant) solution for the torsion of cylinders with a circular cross section is given by the circumferential displacement

$$v^0(r, x_3) = \theta_0 r x_3 \tag{6.1}$$

which depends only on the radial and axial coordinates r and x_3 (and not on the circumferential coordinate). The only nontrivial stress component associated with (6.1) is

$$\sigma_{3\theta} = \sigma_{3\theta} = G v^0_{,3} = G\theta_0 r. \tag{6.2}$$

The method of BLSR determines θ_0 by way of an appropriate (2)-state $v^{(2)} = r$ (which is a rigid body rotation and therefore induces no stress whatsoever). It gives an expression for θ_0 in terms of the applied loading (see (2.12)):

$$\theta_0 \int_0^{2\pi}\int_0^b [Gr][r] r dr d\theta = \int_0^{2\pi}\int_0^b \tau_\theta^0 r^2 dr d\theta \tag{6.3a}$$

or

$$\theta_0 = \frac{4}{Gb^4}\int_0^b \tau_\theta^0 r^2 dr \tag{6.3b}$$

where τ_θ^0 is the prescribed stress distribution on the end section $x_3 = 0$. (Again, a similar necessary condition at $x_3 = L$ for a boundary layer residual state imposes no new requirement on θ_0 given (2.7).)

Suppose now another (2)-state, denoted by $v^{(2)} + \Delta v^{(2)}$ is used to determine the angle of twist to be $\theta_0 + \Delta\theta_0$ so that we have the following new outer solution:

$$v^0 + \Delta v^0 = (\theta_0 + \Delta\theta_0) r x_3, \quad \sigma^0_{3\theta} + \Delta\sigma^0_{3\theta} = (\theta_0 + \Delta\theta_0) G r. \tag{6.4}$$

If $\Delta v^{(2)}$ is a boundary layer so that it is exponentially small away from the two ends (and with $\Delta v^0 = \Delta\theta_0 r x_3$), the condition (4.6) requires

$$\Delta\theta_0 \int_0^{2\pi}\int_0^b \Big[(Gr)(r)\Big]_{x_3=0} r dr d\theta = 0 \tag{6.5}$$

(except for E.S.T.) where b is the radius of the solid circular cross section. It follows that we have $\Delta\theta_0 = 0$. This result is consistent with (4.9) of which (6.5) is a special case.

For a circular cross section, we have the additional luxury of examining the actual contribution of $\Delta v^{(2)}$ to the new value of the angle of twist. With other non-layer solutions for the elastic circular cylinder all "orthogonal" to the Saint-Venant torsion solution (for any value of θ_0), any modification $\Delta v^{(2)}$ of $v^{(2)} = r$

that affects the vallue of θ_0 must be a boundary layer phenomenon (see [21]). Adjacent to the end $x_3 = 0$, $\Delta v^{(2)}$ is necessarily of the form

$$\Delta v_n^{(2)} = \frac{\cosh(\lambda_n x_3)}{\cosh(\lambda_n L)} J_1(\lambda_n r) \tag{6.6}$$

where $\lambda_n b \equiv t_n$ is the nth nonvanishing root of $J_2(t) = 0$ to ensure that circular shaft is stress free on its cylindrical surface $r = b$ [21]. The factor $1/\cosh(\lambda_n L)$ is introduced to ensure that $\Delta v_n^{(2)}$ is exponentially small at any cross section $x_3 = z$ away from both end sections $x_3 = 0$ and $x_3 = L$.

With $v^{(2)} + \Delta v_n^{(2)}$ and $\ell = 0$, the necessary condition (3.1) for a decaying residual state may be written as

$$\int_0^{2\pi}\!\!\int_0^b \left[\sigma_{3\theta}^R(v^{(2)} + \Delta v_n^{(2)})\right]_{x_3=0} r dr d\theta = 0 \tag{6.7a}$$

or

$$\begin{aligned}(\theta_0 + \Delta\theta_0) \int_0^{2\pi}\!\!\int_0^b [Gr]\left[r + \frac{J_1(\lambda_n r)}{\cosh(\lambda_n L)}\right] r dr d\theta \\ = \int_0^{2\pi}\!\!\int_0^b \tau_\theta^0 \left[r + \frac{J_1(\lambda_n r)}{\cosh(\lambda_n L)}\right] r dr d\theta.\end{aligned} \tag{6.7b}$$

Given (6.3a), we may re-write (6.7b) as

$$\begin{aligned}\Delta\theta_0 \int_0^b G\left[r + \frac{J_1(\lambda_n r)}{\cosh(\lambda_n L)}\right] r^2 dr \\ = \frac{1}{\cosh(\lambda_n L)} \int_0^b [\tau_\theta^0 - \theta_0 G r] J_1(\lambda_n r) r dr.\end{aligned} \tag{6.8}$$

It follows that we have

$$\Delta\theta_0 = O\left(\frac{1}{\cosh(\lambda_n L)}\right) \tag{6.9}$$

so that $\Delta\theta_0$ is exponentially small as predicted by [IV.1].

For the present case of a circular shaft, exact explicit layer type $\Delta v_n^{(2)}$ makes it possible to exhibit the exponentially small nature of the correction to θ_0. For other types of cross section geometries, it is not possible to determine $\Delta\theta_0$ explicitly. The more general result in [IV.1] allows us to validate the Saint-Vanent solution (2.8) - (2.12) for the torsion problem of prismatic elastic bodies for all types of cross sections. In sections 7 and 8, we see that this general result can be easily extended to torsion problems where the end data are not all prescribed in terms of stresses. Analogous results can also be obtained for the application of the method of BLSR to elastostatic BVP for other beam, plate and shell structures.

7. Elastic torsion with end data involving displacement components

Torsion of prismatic bodies can also be induced by prescribed end displacements (instead of end stresses) or an admissible combination of end stress and displacement components. For example, we may have at an end $x_3 = \ell$ ($\ell = 0$ or L),

$$\sigma_{33} = \tau_3^\ell(x_1, x_2), \qquad u_j = v_j^\ell(x_1, x_2) \qquad (j = 1, 2) \tag{7.1}$$

or

$$u_3 = v_3^\ell(x_1, x_2), \qquad \sigma_{3j} = \tau_j^\ell(x_1, x_2) \qquad (j = 1, 2) \tag{7.2}$$

Of course, we may also have cases with all three displacement components prescribed at the same end. At least for the end data (7.1) and prescribed data involving only prescribed displacements at the same end, Saint-Venant's principle is not useful for obtaining an outer asymptotic solution for the BVP in three-dimensional elasticity theory, since we do not know the relevant end stress distributions and hence not the resulting twisting moment M_t . For problems involving prescribed end displacements, the outer solution should be taken in the form:

$$u_1^0 = -x_2(\theta_0 x_3 + \alpha_0), \qquad u_2^0 = x_1(\theta_0 x_3 + \alpha_0), \qquad u_3^0 = \theta_0 \psi(x_1, x_2) \tag{7.3}$$

(instead of (2,8)) where the additional terms involving α_0 is merely a rigid body rotation of all (uniform) cross sections. When only stress data are prescribed at both ends of the prismatic body, the application of Saint-Venant principle (to give the usual Saint-Venant torsion solution) does not specify this rigid rotation. As shown in [21], this rigid rotation would be specified in cases where end displacements are involved. In particular, we have from [21] the following two new results for the determination of the outer solution (7.3) for the torsion problem when the prescribed set of end data is of the form (7.1) and (7.2):

[VII.1] *For the residual mixed end data u_1^R, u_2^R and σ_{33}^R at $x_3 = \ell$ to induce only a boundary layer (residual) state, they must satisfy the integral condition*

$$\iint_A \left[\sigma_{31}^{(2)} u_1^R + \sigma_{32}^{(2)} u_2^R - u_3^{(2)} \sigma_{33}^R\right]_{x_3=\ell} dx_1 dx_2 = 0 \tag{7.4}$$

(except for E.S.T.) for any regular elastostatic (2)-state of the prismatic body that is stress free on the cylindrical surface S and satisfies the homogeneous end conditions

$$x_3 = \ell: \quad u_1^{(2)} = u_2^{(2)} = \sigma_{33}^{(2)} = 0. \tag{7.5}$$

[VII.2] *For the residual mixed end data σ_{31}^R, σ_{32}^R and u_3^R at $x_3 = \ell$ to induce only a boundary layer (residual) state, it is necessary that they satisfy the condition*

$$\iint_A \left[u_1^{(2)} \sigma_{31}^R + u^{(2)} \sigma_{32}^R - \sigma_{33}^{(2)} u_3^R\right]_{x_3=\ell} dx_1 dx_2 = 0 \tag{7.6}$$

(except for E.S.T.) for any regular elastostatic (2)-state of the prismatic body that is stress free on S and satisfies the homogeneous end conditions

$$x_3 = \ell: \quad \sigma_{31}^{(2)} = \sigma_{32}^{(2)} = u_3^{(2)} = 0. \tag{7.7}$$

There is also a corresponding result for the case where all data at $x_3 = \ell$ are prescribed in terms of the displacement components (see [21]), but we will not be concerned with that case in this paper.

Sketches of the proof of these results were given in [21]. Though not specifically mentioned there, the existence of stress singularities at the rims of the end sections of the prismatic body where there is an abrupt change of boundary conditions does not play a role in the final results. However, such singularities do have to be taken into account and addressed technically in the formal proofs of these results as was done in [4] for the corresponding plane strain problems.

Suppose (7.1) is prescribed at $x_3 = 0$. Then the elastostatic state

$$u_1^{(2)} = -\, x_3 x_2, \qquad u_2^{(2)} = x_3 x_1, \qquad u_3^{(2)} = \psi(x_1, x_2) \tag{7.8}$$

where ψ is the warping function of Saint-Venant torsion for the same prismatic body, constitutes a required (2)-state for [VII.1]. It follows that (7.4) gives the following algebraic condition for α_0:

$$\alpha_0 = \frac{1}{M_t^{(2)}} \iint_A \left[v_1^0 \Phi_{,2} - v_2^0 \Phi_{,1} - \tau_3^0 \psi \right] dx_1 dx_2 \tag{7.9}$$

where Φ is the stress function for the Saint-Venant torsion solution of the prismatic body and

$$M_t^{(2)} = 2 \iint_A \Phi \; dx_1 dx_2. \tag{7.10}$$

If end data of the type (7.1) is also prescribed at the end $x_3 = L$, then a similar algebraic condition would also be available for the two constants α_0 and θ_0, now using

$$u_1^{(2)} = -(x_3 - L) x_2, \quad u_2^{(2)} = (x_3 - L) x_1, \quad u_3^{(2)} = \psi(x_1, x_2). \tag{7.11}$$

With (7.9), this second condition determines θ_0 for a complete specification of the outer solution of the torsion problem (7.3) for the given mixed end conditions (7.1):

$$\theta_0 = \frac{1}{M_t^{(2)} L} \iint_A \Big[(v_1^L - v_1^0) \Phi_{,2} - (v_2^L - v_1^0) \Phi_{,1} - (\tau_3^L - \tau_3^0) \psi \Big]_{x_3 = L} dx_1 dx_2 \tag{7.12}$$

Suppose we have (7.2) prescribed at the other end $x_3 = L$ instead. Then the elastostatic state

$$u_1^{(2)} = -x_2, \quad u_2^{(2)} = x_1, \quad u_3^{(2)} = 0 \tag{7.13}$$

constitutes an acceptable (2)-state for the necessary condition (7.6) of [VII.2] at that end. It follows from [VII.2] that we must have

$$\iint_A \left[-x_2\sigma_{31}^R + x_1\sigma_{32}^R\right]_{x_3=L} dx_1 dx_2 = 0 \tag{7.14}$$

(except for E.S.T.) which is just Saint-Venant's principle. [As observed in in [21], this means the constraint of no end warping at $x_3 = L$ has no effect on the outer (torsion) solution as long as shear stresses are prescribed at the same end.] The condition (7.14) implies

$$\theta_0 = \frac{1}{M_t^{(2)}} \iint_A \left[-x_2\tau_1^L + x_1\tau_2^L\right] dx_1 dx_2 = \frac{M_t}{M_t^{(2)}}. \tag{7.15}$$

The results by the method of BLRS above show for the first time that Saint-Venant's principle is a necessary condition for the residual solution to be a boundary layer whenever both transverse shear stress components are prescribed at an end section (even if the axial displacement should be prescribed as the third condition). In addition, when one or more of the non-axial displacement components are prescribed at an end section (including pure displacement end data), we have now a corresponding set of necessary conditions for the residual solution to be a boundary layer phenomenon. Similar to the case of the classical Saint-Venant torsion solution, there is again an abundance of admissible (2)-states for these new necessary conditions. As in the classical case, we must show that they do not overspecify the corresponding outer torsion solution.

8. The correct outer torsion solutions

Given the development in section 4, it is not surprising that there should be analogues to [IV.1] which allow us to conclude no overspecification of the outer solution by taking advantage of the boundary layer nature of the (2)-state differentials. For the mixed end conditions (7.1) for example, we have the following analogue of [IV.1]:

[VIII.1] *If the difference between (corresponding components of the) two (2)-states for problems with prescribed data of the form (7.1) at the $x_3 = 0$ end is exponentially small away from the ends, then the difference between the induced residual states satisfies*

$$\iint_A \left[\sigma_{31}^{(2)}\Delta u_1^R + \sigma_{32}^{(2)}\Delta u_2^R - u_3^{(2)}\Delta\sigma_{33}^R\right]_{x_3=0} dx_1 dx_2 = 0 \tag{8.1}$$

(except for E.S.T.), and the difference between the induced outer solutions satisfies

$$\iint_A \left[\sigma_{31}^{(2)}\Delta u_1^0 + \sigma_{32}^{(2)}\Delta u_2^0 - u_3^{(2)}\Delta\sigma_{33}^0\right]_{x_3=0} dx_1 dx_2 = 0. \tag{8.2}$$

The proof of this result is similar to that for [IV.1]. Starting with the necessary condition (7.4) for the end data (7.1) for the $x_3 = 0$ end, we apply the reciprocal theorem of elasticity to transform (7.4) into

$$\begin{aligned} 0 &= \iint_A \left[\sigma_{31}^{(2)} u_1^R + \sigma_{32}^{(2)} u_2^R - u_3^{(2)} \sigma_{33}^R\right]_{x_3=0} dx_1 dx_2 \\ &= \iint_A \left[\mathbf{T}^{(2)} \cdot \mathbf{u}^R - \mathbf{u}^{(2)} \cdot \mathbf{T}^R\right]_{x_3=0} dx_1 dx_2 \\ &= \iint_A \left[\mathbf{T}^{(2)} \cdot \mathbf{u}^R - \mathbf{u}^{(2)} \cdot \mathbf{T}^R\right]_{x_3=z} dx_1 dx_2 \end{aligned} \tag{8.3}$$

keeping in mind that the conditions in (7.5) hold at $x_3 = 0$. (As in [21], we have omitted the technical details on the treatment of the stress singularities along the rims of the end sections of the prismatic body.) Observing the condition (8.3) in a corresponding condition with a different (2)-state $\{\mathbf{u}^{(2)} + \Delta\mathbf{u}^{(2)}, \mathbf{T}^{(2)} + \Delta\mathbf{T}^{(2)}\}$, we obtain (as in section 5)

$$0 = \iint_A \left[\mathbf{T}^{(2)} \cdot \Delta\mathbf{u}^R - \mathbf{u}^{(2)} \cdot \Delta\mathbf{T}^R\right]_{x_3=z} dx_1 dx_2 \tag{8.4}$$

except for E.S.T., where we have made use of the fact that, for a cross section location z $(0 < z < L)$ away from the two ends, $\Delta\mathbf{T}^{(2)}$ and $\Delta\mathbf{u}^{(2)}$ are exponentially small (by the hypothesis). We now apply the reciprocal relation (3.2) again to transform (8.4) back to an integral over the cross section at $x_3 = 0$ so that we have

$$\begin{aligned} 0 &= \iint_A \left[\mathbf{T}^{(2)} \cdot \Delta\mathbf{u}^R - \mathbf{u}^{(2)} \cdot \Delta\mathbf{T}^R\right]_{x_3=0} dx_1 dx_2 \\ &= \iint_A \left[\sigma_{31}^{(2)} \Delta u_1^R + \sigma_{32}^{(2)} \Delta u_2^R - u_3^{(2)} \Delta\sigma_{33}^R\right]_{x_3=0} dx_1 dx_2 \end{aligned} \tag{8.5}$$

except for E.S.T. keeping in mind that the (2)-state satisfies the homogeneous end conditions (7.5) for the present problem. We have then proved (8.1) except for some technical details. The condition (8.2) follows from (8.1) by an argument similar to the corresponding development in section 4.

Analogous results can also be obtained for problems involving end data of the form (7.2) and for pure displacement end data. We summarize them in the following theorems:

[VIII.2] *If the difference between (corresponding components of the) two (2)-states for problems with prescribed data of the form (7.2) at the $x_3 = 0$ is exponentially small away from the ends, then the difference between the induced residual states satisfies the integral condition*

$$\iint_A \left[u_1^{(2)} \Delta\sigma_{31}^R + u_2^{(2)} \Delta_{32}^R - \sigma_{33}^{(2)} \Delta u_3^R\right]_{x_3=0} dx_1 dx_2 = 0 \tag{8.6}$$

except for E.S.T., and the difference between the induced outer solutions satisfies

$$\iint_A \left[u_1^{(2)} \Delta\sigma_{31}^0 + u_2^{(2)} \Delta\sigma_{32}^0 - \sigma_{33}^{(2)} \Delta u_3^0 \right]_{x_3=0} dx_1 dx_2 = 0 \tag{8.7}$$

except for E.S.T.

[VIII.3] *Under the hypotheses of [VIII.2], the difference between the induced residual states for the case of pure displacement data at the end $x_3 = 0$ satisfies the integral condition*

$$\iint_A \left[\mathbf{T}^{(2)} \cdot \Delta\mathbf{u}^R \right]_{x_3=0} dx_1 dx_2 = 0 \tag{8.8}$$

except for E.S.T., and the difference between the induced outer solutions satisfies

$$\iint_A \left[\mathbf{T}^{(2)} \cdot \Delta\mathbf{u}^0 \right]_{x_3=0} dx_1 dx_2 = 0 \tag{8.9}$$

except for E.S.T.

The proofs of these results are similar to that for [IV.1] and [VIII.1] and will not be given here. Still other combinations of end conditions are possible; they can be handled analogously.

Fortunately, most differences between admissible (2)-states are exponentially small away from the two end sections. There are of course the five rigid body displacements and rotations (other than to (7.13)) as well as the five Saint-Venant solutions for axial extension, pure bending and flexure. Even if these non-layer solutions may be pieced together for a different (2)-state for end data that induce torsion, such a (2)-state would not give rise to a new condition on θ_0 and α_0 because the corresponding necessary condition ((7.4), (7.6) (8.8) with $\Delta\mathbf{u}^R$ replaced by $\mathbf{u}^R$) would be trivially satisfied. In this sense, the six Saint-Venant solutions are mutually "orthogonal" and can be de-coupled from each other. For this reason, results of the type [IV.1], [VIII.1] - [VIII.3] are just what is needed to settle the question of overspecification.

9. Concluding remarks

In this paper, the method of boundary layer residual state (BLRS) developed in [3-8,20,21] for problems in linear elastrostatics involving thin or slender elastic bodies is reviewed briefly by way of the problem of Saint-Venant torsion of prismatic bodies. Given the multiplicity of admissible (2)-states which can be used in the necessary conditions for the residual state to be a boundary layer, we need to be sure that the correct outer solution is in fact obtained by the BLRS method using an adequate number of the admissible (2)-states.

To address the question of the correct outer solution, the difference between two outer solutions of the torsion problem obtained by using any two (2)-states, which are identical away from the ends of the prismatic body except for exponentially small terms, is shown to satisfy the same vanishing weighted average condition as the residual solution of the problem. From this follows the identity of these two outer solutions except for exponentially small terms; hence the same Saint-Venant torsion solution is obtained within the expected accuracy inherent in the BLRS method. This result is then extended to cover torsion induced by other combinations of prescribed end data.

With suitable modification, the same approach is expected to apply to the outer solution of more general linear boundary value problems which exhibits boundary layer phenomena but whose inner solution components are not tractable. The approach reported in this paper would be useful for these BVP even when the outer solution can only be obtained numerically.

Note that if only a finite number of terms in the outer asymptotic expansion is available (or can be used), the necessary conditions of the BLRS method (such as (3.1)) are still applicable but now the accuracy of the solution obtained would be determined by the terms neglected in the outer solution. Finally, if the needed (2)-states are not available in terms of elementary or special functions, they can be generated numerically, only once for all future applications (with different prescribed boundary data distributions). If the terms of the outer solution are to be determined by a sequence of (simpler) BVP, the same necessary conditions would provide the appropriate boundary conditions for these BVP (which may have to be solved numerically).

References

[1] K.O. Friedrichs and R.F. Dressler, A boundary layer theory for elastic plates, *Comm. Pure and Appl. Math.* **14** (1961), 1-33.

[2] R.D. Gregory, The general form of the three-dimensional elastic field inside an isotropic plate with free faces, *J. Elasticity* **28** (1992), 1-28.

[3] R.D. Gregory, T.I. Milac and F.Y.M. Wan, The axisymmetric deformation of a thin, or moderately thick, elastic spherical cap, Studies in *Appl. Math.* **100** (1998), 67-94.

[4] R.D. Gregory and F.Y.M. Wan, Decaying states of plane strain in a semi-infinite strip and boundary conditions for plate theory, *J. Elasticity* **14** (1984), 27-64.

[5] R.D. Gregory and F.Y.M. Wan, On plate theories and Saint-Venant's principle, *Int'l. J. Solids Structures* **21** (1985), 1005-1024.

[6] R.D. Gregory and F.Y.M. Wan, Edge effects in the stretching of plates, *Local Effects in the Analysis of Structures*, P. Ladevèze, Ed., Elsevier, Amsterdam, 1985, 35-54.

[7] R.D. Gregory and F.Y.M. Wan, The interior solution for linear problems of elastic plates, *J. Appl. Mech.* **55** (1988), 551-559.

[8] R.D. Gregory and F.Y.M. Wan, Correct asymptotic theories for axisymmetric deformations of thin and moderately thick cylindrical shells, *Int'l. J. Solids Structures* **30** (1993), 1957-1981.

[9] C.O. Horgan and J.K. Knowles, Recent development concerning Saint-Venant's principle, *Adv. in Appl. Mechanics* **23** (1983), Academic Press, 179-269.

[10] M.W. Johnson, A boundary layer theory for unsymmetric deformations of circular cylindrical elastic shells, *J. Math. Phys.* **42** (1963), 167-188.
[11] M.W. Johnson and E. Reissner, On the foundations of the theory of thin elastic shells, *J. Math. Phys.* **27** (1958), 371-392.
[12] A.E.H. Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th Ed., Dover Publications, 1944.
[13] F.W.J. OIver, *Introduction to Asymptotic and Special Functions*, Academic Press, 1974.
[14] E.L. Reiss, A theory for the small rotationally symmetric deformations for cylindrical shells, *Comm. Pure. Appl. Math.* **13** (1960), 531-550.
[15] E.L. Reiss, On the theory of cylindrical shells, *Quart. J. Mech. and Appl. Math.* **15** (1962), 325-338.
[16] E.L. Reiss, Symmetric bending of thick circular plates, *J. Soc. Indust. Appl. Math.* **10** (1962), 596-609.
[17] E.L. Reiss and S. Locke, On the theory of plane stress, *Quart. Appl. Math.* **19** (1961), 195-203.
[18] I.S. Sokolnikoff, *Mathematical Theory of Elasticity*, McGraw Hill, 1956.
[19] E. Sternberg and J.K. Knowles, Minimum energy characterizations of Saint-Venant's solution to the Relaxed Saint-Venant Problem, *Arch. Rational Mech. Anal.* **21** (1965), 89-107.
[20] F.Y.M. Wan, Interior solution for slender and thin elastic bodies with prescribed displacement data, *Analytical and Computational Models of Shells*, A.K. Noors, T. Belytschko and J.C. Simo, eds., ASME Publications, (1989), 55-70.
[21] F.Y.Wan, On torsion of prismatic bodies, *Studies in Appl. Math.* **89** (1993), 63-94.
[22] G.E.O. Widera, H. Fan and P. Afshari, Applicability of asymptotic beam theories to thin (thick) walled pipes, *Analytical and Computational Models of Shells*, A.K. Noors, T. Belyschko and J.C. Simo, eds., ASME, Publications, (1989), 33-54.

Appendix. Outer solution for elastic torsion

Let

$$x = \frac{x_1}{b}, \quad y = \frac{x_2}{b}, \quad z = \frac{x_3}{L}, \quad \epsilon = \frac{b}{L} \qquad (A.1)$$

where b characterizes the lineal dimension of the uniform cross section of the prismatic body and L is the length of the body. In terms of the dimensionless coordinates, we take

$$\mathbf{u}(x_1, x_2, x_3) = \{U(x, y, z), V(x, y, z), W(x, y, z)\}. \qquad (A.2)$$

Navier's equations of linear elasticity without body loads may then be written as

$$(1 - 2\nu)(U_{,xx} + U_{,yy} + \epsilon^2 U_{,zz}) + (U_{,xx} + V_{,yx} + \epsilon W_{,zx}) = 0 \qquad (A.3)$$

$$(1 - 2\nu)(V_{,xx} + V_{,yy} + \epsilon^2 V_{,zz}) + (U_{,xy} + V_{,yy} + \epsilon W_{,zy}) = 0 \qquad (A.4)$$

$$(1-2\nu)(W_{,xx}+W_{,yy}+\epsilon^2 W_{,zz})+\epsilon(U_{,xz}+V_{,yz}+\epsilon W_{,zz})=0 \qquad (A.5)$$

With

$$\{U,V,W\}=\sum_{n=0}^{\infty}\{U_n,V_n,W_n\}\epsilon^n, \qquad (A.6)$$

the leading term perturbation solutions U_0, V_0 and W_0 are determined by

$$L_1[U_0,V_0]\equiv 2(1-\nu)U_{0,xx}+(1-2\nu)U_{0,yy}+V_{0,xy}=0 \qquad (A.7)$$

$$L_2[U_0,V_0]\equiv 2(1-\nu)V_{0,yy}+(1-2\nu)V_{0,xx}+U_{0,xy}=0 \qquad (A.8)$$

$$W_{0,xx}+W_{0,yy}=0 \qquad (A.9)$$

It is not difficult to verify that (A.7) and (A.8) are effectively the two equilibrium equations for plane strain deformation:

$$\sigma^{(0)}_{1j,x}+\sigma^{(0)}_{2j,y}=0 \qquad (j=1,2) \qquad (A.10)$$

Together with the stress free boundary conditions

$$\nu_1\sigma^{(0)}_{1j}+\nu_2\sigma^{(0)}_{2j}=0 \qquad (j=1,2) \qquad (A.11)$$

on S from (2.4), we have

$$\sigma^{(0)}_{11}=\sigma^{(0)}_{12}=\sigma^{(0)}_{21}=\sigma^{(0)}_{22}=0 \qquad (A.12)$$

and by the plane strain condition also

$$\sigma^{(0)}_{33}=0 \qquad (A.13)$$

for the leading term solution.

The conditions (A.12) and (A.13) require

$$W_0(x,y,z)=W_0(x,y)=\theta_0 w(x_1,x_2) \qquad (A.14)$$

and

$$U_0(x,y,z)=U_0(y,z), \qquad V_0(x,y,z)=V_0(x,z) \qquad (A.15)$$

with

$$V_{0,x} + U_{0,y} = 0 \ . \tag{A.16}$$

The plane strain deformation (A.7) and (A.8) effectively requires the leading term solution for the transverse shear components $\sigma_{3j}^{(0)} = \sigma_{j3}^{(0)}, j = 1, 2$, to be independent of z . The stress strain relations for the transverse shear stresses imply that u_j should be linear in $x_3 = Lz$ since all other terms in these relations are now known to be independent of z:

$$U_0(y,z) = \xi_1(y)z + \xi_0(y), \qquad V_0(x,z) = \eta_1(x)z + \eta_0(x). \tag{A.17}$$

The condition (A.16) then requires

$$\xi_1^{\bullet}(y) + \eta_1'(x) = 0, \qquad \xi_0^{\bullet}(y) + \eta_0'(x) = 0 \tag{A.18}$$

where $(\quad)^{\bullet} = (\quad)_{,y}$ and $(\quad)' = (\quad)_{,x}$ so that

$$U_0 = -(\bar{\theta}_0 z + \bar{\alpha}_0)y, \qquad V_0 = (\bar{\theta}_0 z + \bar{\alpha}_0)x \tag{A.19}$$

In (A.19), we have omitted various rigid body displacements not related to axial torsion.

Altogether, the leading term outer solution consists of the displacement expressions (A.19), and the stress-strain relations

$$\sigma_{31}^{(0)} = G\Big(-\frac{\bar{\theta}_0}{L}y + \theta_0 w_{,1}\Big) = G\theta_0(-x_2 + w_{,1}) \tag{A.20}$$

$$\sigma_{32}^{(0)} = G\Big(\frac{\bar{\theta}_0}{L}x + \theta_0 w_{,2}\Big) = G\theta_0(x_1 + w_{,2}) \tag{A.21}$$

(after setting $\bar{\theta}_0/Lb = \theta_0$) with (A.9) taking the form

$$w_{,11} + w_{,22} = 0 \tag{A.22}$$

Having the leading term outer solution (A.14) and (A.19), we can now move on to obtain the $O(\epsilon)$ terms in the parametric series in (A.6). The governing PDE for these terms are:

$$\begin{gathered} L_1[U_1, V_1] + W_{0,zx} = 0, \qquad L_2[U_1, V_1] + W_{0,zy} = 0 \\ (1-2\nu)(W_{1,xx} + W_{1,yy}) + (U_{0,xz} + V_{0,yz}) = 0 \end{gathered} \tag{A.23}$$

Since W_0 is independent of z, while U_0 and V_0 are independent of x and y, respectively, the governing PDE for U_1, V_1 and W_1 are homogeneous. Together with the homogeneous boundary conditions in (A.11), the $O(\epsilon)$ terms in (A.6) are identical to $O(1)$ terms up to two unknown constants (θ_0 and $\alpha_0 = \bar{\alpha}_0/b$). In fact, by continuing the process, we find that the same conclusion applies to all higher order terms in the perturbation solution (A.6). While there are outer solutions associated with axial extension, bending and flexure (which are known to be "orthogonal" to the outer solution for torsion in the sense described in section 8), the solution given by (A.14) and (A.19) constitutes the exact outer solution for the problem of axial torsion.

Frederic Y. Wan
Department of Mathematics
University of California, Irvine
Irvine, CA 92697
U.S.A.