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LOCATIONAL EQUILIBRIUM MODELS FOR PUBLICLY OWNED RESIDENTIAL LAND

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ABSTRACT. Perturbation and matched asymptotic expansion methods are shown to be useful to the study of land economics. In the context of residential land management theory, the present study complements a previous analysis of locational equilibrium economic models for residential land in Wan [1977] and extends it beyond absentee ownership to include public ownership and mixed ownership. The dependence of the optimal uniform household utility on the (radial) distribution of land fraction for housing space is delineated.

1. Introduction. From demography to economics, the social sciences have benefited from the use of mathematics. With Pascal's notion of "expected value" as a basis for wager in 1654, the use of mathematical methods in the social sciences has gone back several centuries. But, unlike statistics, qualitative theory of differential equations or operations research techniques, the more modern developments in asymptotic methods found their way into the social sciences only recently. In this paper, perturbation and matched asymptotic expansion methods will be shown to be useful to the study of land economics. In the context of residential land management theory, the present study complements a previous work on locational equilibrium problems of residential land use in Wan [1977] and extends it beyond absentee land ownership to include public and mixed ownership. The dependence of optimal household utility on the radial distribution of housing fraction is delineated.

2. Consumer's optimum and locational equilibrium. In the simplest meaningful model of residential land use theory, land values are determined primarily by the amount of land available for housing and differential accessibility of a particular location to the Central Business District (CBD). Essential aspects of the economics of residential land can be understood by working with the standard model of a single abstract featureless circular city with a circular CBD of radius R_c surrounded by an annular region of residential land extending to a distance R_e ($> R_c$) from the center of the circular city. The abstract city is inhabited by N_o identical households all having the same consumer's utility function and each sending a commuter to work in the CBD and earning the same annual wage Y. (See Kanemoto [1980], Wan [1989] and references therein for analyses of models that allow for nonuniform household utilities and incomes). The bulk of travel costs arises from traveling to and from the edge of the CBD along symmetrically distributed radial streets (with polar symmetry giving us a spatially one-dimensional problem). The standard model makes the simplifying assumption that nonradial travel and traveling within the CBD are free. Such an assumption reflects the availability of free (or negligibly small cost of) public transportation in the downtown area of cities such as Seattle and negligible nonradial travel costs of the residential area relative to the commuting cost. As such, travel costs are incurred in this model city only through commuting to and from the edge of the CBD.

Each household in the model city uses its annual income y ($\geq Y$), for housing, for a single composite consumption good and for the commuting cost to work in the CBD. When saving is not an option, a household at location X has as its budget constraint

$$rs + pc + t = y \tag{2.1}$$

where c and s are the amount of consumption good and residential land for that household per annum, respectively, and where r is the annual rent per unit area of land at location X, t is the annual travel cost for a household at the same location, and p is the constant price of a unit of the consumption good. Each household chooses a location (characterized by the *dimensionless* radial distance $x \equiv X/R_c$ from the city center with $x \leq R \equiv R_e/R_c$), an amount of land for housing s per annum and an amount of consumption good c per annum to maximize the household utility function U subject to the household budget constraint (2.1). With no loss of generality, we take $p = 1$ and write the budget equation (2.1) as

$$rs + c = y\left(1 - \frac{t}{y}\right) \equiv yw(x). \tag{2.2}$$

Household satisfaction clearly depends on the amounts of space s and consumption good c available to the household. Hence, the household utility is expected to be a function of c and s. The quantities c and s are expected to vary with the location of the household since the net family income after travel cost, yw, available for housing and consumption good will vary with X.

For a featureless city, it would be difficult to say whether a household would have an explicit preference for, or aversion to locational proximity to the CBD, other than the implicit dependence on location through the travel cost (via c and s). Given this ambiguity, we take the utility function to depend on c and s only so that we have $U(c, s)$. In fact, to obtain explicit results, many previous studies assume a Cobb-Douglas type of utility function with U depending on c and s through $\xi \equiv s^{\sigma} c^{(1-\sigma)}$ where σ is a prescribed constant with $0 < \sigma < 1$. Typically, $U(\xi)$ is assumed to be monotone increasing and strictly concave with respect to its argument ξ.

It is straightforward to obtain the first order necessary conditions for a maximum U subject to the budget constraint (2.2). In locational equilibrium where all households achieve identical utility independent of their location (for otherwise some would relocate to achieve the higher utility of another household), these first order conditions give three relations for the four quantities c, s, r and t (or w). In terms of the dimensionless after-travel cost income function w, we have for the maximum (equilibrium) value of the utility function $U(\xi)$ of the Cobb-Douglas type,

$$\begin{aligned} t &= y(1-w), \qquad r = r_c w^{\alpha+1} \\ s &= \frac{y}{r}\sigma w = \frac{y}{r_c}\sigma w^{-\alpha}, \qquad c = \alpha r s = (1-\sigma) y w \end{aligned} \tag{2.3}$$

where $\alpha + 1 = 1/\sigma$ and where $X = R_c$ is the unknown unit ground rent at the edge of the CBD (see Wan [1989, 1993], Solow [1972] for a derivation of (2.3)).

The classical theory of residential land use takes the travel cost $t(X)$ to depend only on the location of the household away from the edge of the CBD through the linear relation

$$t = t_o + \tau_o(X - R_c) \equiv t_d \tag{2.4}$$

where t_o is the per annum travel cost within the CBD and τ_o is the fixed annual cost for moving one commuter one unit distance within the residential area. In the conventional model, t_o is taken to be zero as previously explained (with transportation being free within the CBD). With t given as an explicit function of location, the expressions in (2.3) give r, s and c up to the unknown constant r_c, which is then determined by the fraction of land in the residential area allocated for housing (with the remaining land area for streets) as explained in the next section.

3. Congestion cost of transportation. In Solow [1972, 1973], Solow added a new feature to extend the above classical economic model of residential land use making it more realistic. The new element was to allow the commuting cost of a household at location X to be the sum of the distance cost and a congestion cost. The latter is an increasing function of the number of households, $N(x)$, living outside the circle of radius X and the fraction $b(x)$ of land allocated for housing in the ring between X and $X + dX$ (with the remaining land area, $2\pi[1-b]XdX$, for roads). With travel being free within the CBD so that $t = 0$ at $X = R_c$, we take

$$\begin{aligned} t &= t_d + t_c \\ &= \int_{R_c}^{X} \left\{ \tau + \frac{aN}{2\pi X(1-b)} \right\} dX \end{aligned} \tag{3.1}$$

where the prescribed constant a is the annual household travel cost per unit radial distance induced by the traffic density at X, and where $\tau = dt_d/dX$ is the known fixed annual household travel cost per unit radial distance within the residential area. In other words, t_d is the annual cost for a household at location X to travel the distance $(X - R_c)$ to work in the CBD absent of any traffic. Similar to previous investigations, we will work with the following differential form of (3.1)

$$\frac{dt}{dX} = \tau + \frac{aN}{2\pi X[1-b]} = -\,y\frac{dw}{dX} \tag{3.2}$$

throughout this paper after using (2.2) to express t in terms of w.

Since $N(x)$ is also an unknown, we need another condition to determine t (or w) and N. This second condition comes from a conservation law for space: *The amount of space occupied by the households living in an annular region of the residential district must equal the total amount of space in the same region allocated for housing.* For the annular region extending radially from X to $X + dX$ (of area $2\pi X dX$), this requirement implies $-s dN = 2\pi b X\, dX$. We may also write this relation in the form of a differential equation:

$$\frac{dN}{dX} = -\frac{2\pi X b}{s} = -\frac{2\pi r_c}{y}(\alpha + 1) X b w^{\alpha} \tag{3.3}$$

keeping in mind $(\alpha + 1) = 1/\sigma$. For the two first order differential equations (3.2) and (3.3), we have the initial conditions $t = 0$ and $N = N_o$ at $X = R_c$ since households living at the edge of the CBD pay no travel cost (as travel is free within the CBD) and all households must reside outside the CBD. Given a distribution b of the fraction of land for housing at different locations, the initial value problem defined by the two differential equations (3.2), (3.3) and the two initial conditions for t and N determine t (or w) and N as functions of the radial distance X from the city center up to the unknown parameter r_c. Finally, the condition that there is no household residing outside the city limit, $N(1; r_c) = 0$, determines r_c. The common maximum household utility $\overline{U}$ is then given by

$$\overline{U} = U(\sigma^{\sigma}(1 - \sigma)^{1-\sigma} y r_c^{-\sigma}) \tag{3.4}$$

With (3.3) and $N(1; r_c) = N_o$ written as

$$N(x; r_c) = N_o - \frac{2\pi r_c}{y\sigma} \int_{R_c}^{X} X b w^{\alpha}\, dX, \tag{3.5a}$$

the condition $N(R; r_c) = 0$ requires

$$N(R; r_c) = N_o - \frac{2\pi r_c}{y\sigma} \int_{R_c}^{R_e} X b w^{\alpha}\, dX = 0. \tag{3.5b}$$

Since w may also depend on r_c, equation (3.5b) generally provides a nonlinear equation for r_c to be solved by some iterative method.

It should be noted that, without the congestion component of travel cost, i.e., when $a = 0$, we have from (2.4) (with $t_o = 0$ and a constant unit distance cost, $\tau = \tau_o$)

$$w = 1 - \frac{t}{y} = 1 - \frac{\tau_o}{y}(X - R_c) \tag{3.6}$$

which does not depend on r_c. The condition $N(R; r_c) = 0$ then determines r_c to be

$$\frac{1}{r_c} = \frac{2\pi}{N_o y \sigma} \int_{R_c}^{R_e} Xb\left[1 - \frac{\tau_o}{y}(X - R_c)\right]^{\alpha} dX \tag{3.7}$$

Note that, unlike (3.5b), the righthand side does not depend on the unknown r_c. For the case of worker's wage constituting the entire household income (so that y is just the known annual wage Y), r_c (and hence the entire solution of the problem) is completely determined once $b(x)$ is prescribed. It follows from (3.4) and (3.7) that $\overline{U}$ is maximized by taking $b(x) = 1$. This is clearly unrealistic since traveling to the CBD would be impossible for any household living away from the edge of the CBD, $X = R_c$ (as there is no road to get there). As such, the introduction of a congestion component of commuting cost constitutes a fundamental contribution to the theory of residential land economics.

4. Public ownership. Previous applications of perturbation and asymptotic methods to residential land use problems were restricted to the case where all the available residential land is owned by one or more *absentee landlords* who collect rent and spend it elsewhere. An example of a single absentee landlord is The Irvine Company which effectively owns all the land in the city of Irvine. In that case, the annual household income is simply the wage earned so that $y = Y$. In this paper, we consider also the other extreme case of *public ownership* where the residential land in the city is owned by the city which collects rent and distributes the proceeds equally among the N_o families as a lump-sum dividend. In reality, the situation is of course much more complex. Some households own land and collect and spend rent; others pay rent, and still others are owner-occupiers. Only the two extreme cases were considered in previous investigations (with asymptotic methods applied only to the case of absentee ownership). In this paper, we allow for a simple form of mixed ownership by returning a fraction c_D of the total rent collected from all households residing in the city as equal shares of dividend to these households. Note that $c_D = 1$ corresponds to total public ownership and $c_D = 0$ to the case of absentee ownership.

To allow for the public (and mixed) ownership case, we let the lump-sum dividend for each household be D_e with

$$D_e = \frac{c_D}{N_o} \int_{R_c}^{R_e} 2\pi r b X \, dX \quad 0 \leq c_D \leq 1. \tag{4.1}$$

As indicated above, we allow for $0 < c_D < 1$ for mixed ownership. The annual income of each household is now the sum of the annual wage Y and the (yet unknown) dividend received from rent redistribution:

$$y = Y + D_e. \tag{4.2}$$

To avoid working with a mixture of differential and integral relations, we introduce the dividend function $D(X)$ defined by

$$D(X) = \frac{c_D}{N_o} \int_{R_c}^{X} 2\pi r b Z \, dZ \tag{4.3}$$

so that

$$\frac{dD}{dX} = \frac{2\pi c_D r_c}{N_0} X b w^{\alpha+1}, \tag{4.4}$$

$$D(R_c) = 0 \tag{4.5}$$

with the condition (4.1) taking the form

$$D(R_e) = D_e. \tag{4.6}$$

We see from (4.1) that the amount of dividend for each household depends on the total rent collected (which in turn depends on the unknown r_c) and is therefore an unknown quantity.

It is convenient to think of the present model of the theory of residential land to consist of the three first order differential equations (3.2), (3.3) and (4.4), and the three initial conditions

$$t(1; r_c, D_e) = 0, \qquad N(1; r_c, D_e) = N_o, \qquad D(1; r_c, D_e) = 0 \tag{4.7}$$

for the three unknown functions t, or w, N and D. In terms of w, the first condition in (4.7) corresponds to $w = 1$ at $x = 1$. The two terminal conditions (see (3.5b) and (4.6)),

$$N(R; r_c, D_e) = 0 \quad \text{and} \quad D(R; r_c, D_e) = D_e \tag{4.8}$$

provide two nonlinear equations for the determination of the two remaining unknown parameters r_c and D_e. The solution of this boundary value problem allows us to assess how the choice of b affects the optimal household utility $\bar{U}$ (see (3.4)).

5. Dimensionless form and perturbation solution. For the solution of the boundary value problem stated in Section 4, we introduce, in addition to

$$x = X/R_c, \qquad R = R_e/R_c, \qquad w(x) = 1 - \frac{t}{y} \tag{5.1 a,b,c}$$

with $(\ )' = d(\ )/dx$ as previously given, the following dimensionless quantities:

$$b_c = b(x = 1), \qquad b_e = b(x = R) \tag{5.1 d,e}$$

$$p(x) = \frac{x(1-b)}{1-b_c}, \qquad q(x) = \frac{xb}{b_c}, \qquad T(x) = \frac{\tau}{\tau_o} \tag{5.2}$$

$$\begin{aligned} u(x;\nu,\Delta;\varepsilon) &= \frac{1}{N_o} N(x; r_c, D_e), \\ \delta(x;\nu,\Delta;\varepsilon) &= \frac{1}{Y} D(x; r_e, D_e) \end{aligned} \tag{5.3}$$

$$\varepsilon\eta = \frac{R_c\tau_o}{Y}, \qquad \varepsilon(1-\eta) = \frac{aN_o}{2\pi Y(1-b_c)} \tag{5.4}$$

$$\nu = \frac{2\pi b_c r_c R_c^2}{\sigma Y N_o}, \qquad \Delta = \frac{D_e}{Y} = \frac{y - Y}{Y} \tag{5.5}$$

with

$$\Delta = \frac{D_e}{Y} = c_D \sigma\nu \int_1^R w^{\alpha+1} q(x)\, dx. \tag{5.6}$$

We then rewrite the differential equations (3.2), (3.3) and (4.4) and the auxiliary conditions (4.7) and (4.8) in terms of these dimensionless quantities to get the dimensionless differential equations

$$w' = -\frac{\varepsilon}{1+\Delta}\left\{\eta T(x) + (1-\eta)\frac{u}{p}\right\} \tag{5.7}$$

$$u' = -\ \frac{\nu q(x) w^{\alpha}}{1+\Delta} \tag{5.8}$$

$$\delta' = c_D \sigma \nu q(x) w^{\alpha+1} \tag{5.9}$$

and the dimensionless auxiliary conditions

$$w(1;\nu,\Delta;\varepsilon) = 1, \qquad u(1;\nu,\Delta;\varepsilon) = 1, \qquad \delta(1;\nu,\Delta;\varepsilon) = 0 \tag{5.10}$$

$$u(R;\nu,\Delta;\varepsilon) = 0, \qquad \delta(R;\nu,\Delta;\varepsilon) = \Delta. \tag{5.11}$$

where we have noted the dependence of w, u and δ on ν (the dimensionless r_c) and Δ (the dimensionless D_e) as well as on the (small) travel cost parameter ε. The quantity $\varepsilon\eta$ is of the order of magnitude of the fraction of the household annual income for the distance component of the household commuting cost (without traffic). The quantity $\varepsilon(1-\eta)$ is the corresponding fraction for the congestion component. Hence, $\varepsilon = \varepsilon\eta + \varepsilon(1-\eta)$ is a measure of the fraction of the household annual income spent on commuting to work which is normally small compared to unity so that $0 < \varepsilon \ll 1$. This observation suggests that we seek a perturbation solution for the governing differential equations in powers of the parameter ε :

$$\begin{aligned} \{w,u,\delta\} &= \sum_{n=0}^{\infty}\{w_n(x), u_n(x), \delta_n(x)\}\varepsilon^n, \\ \{\nu,\Delta\} &= \sum_{n=0}^{\infty}\{\nu_n, \Delta_n\}\varepsilon^n. \end{aligned} \tag{5.12}$$

Since the differential equations and auxiliary conditions must be satisfied identically in ε, the parametric series (5.12) give rise to a sequence of boundary value problems (BVP) for the coefficients of the various expansions. The new BVP are significantly simpler in our case as they are all linear problems!

The $O(1)$ problem. The leading terms of the series in (5.12) are determined by the differential equations

$$w_o' = 0, \qquad u_o' = -\frac{\nu_o q(x) w_o^\alpha}{1+\Delta_o}, \qquad \delta_o' = \sigma c_D \nu_o q(x) w_o^{\alpha+1} \tag{5.13}$$

and the auxiliary conditions

$$w_o(1) = 1, \qquad u_o(1) = 1, \qquad \delta_o(1) = 0 \tag{5.14}$$

$$u_o(R) = 0, \qquad \delta_o(R) = \Delta_o. \tag{5.15}$$

They correspond to the case $\varepsilon = 0$. The solution to this linear boundary value problem is

$$\begin{gathered} w_o(x) = 1, \qquad u_o(x) = 1 - \frac{Q(x)}{Q(R)}, \qquad \delta_o(x) = \sigma c_D \nu_o Q(x) \\ \Delta_o = \frac{c_D \sigma}{1 - c_D \sigma}, \qquad \nu_o = \frac{1}{(1 - c_D\sigma) Q(R)} \end{gathered} \tag{5.16}$$

with

$$Q(x) = \int_1^x q(\zeta)\, d\zeta \tag{5.17}$$

where $q(x) = xb/b_c$ is assumed to be independent of ε for simplicity. For the case of a fixed fraction function with $b(x) = b_c$, we have $q(x) = x$ so that the leading term of the normalized unit rent at the edge of the CBD, ν_o, is independent of b_c. For other housing fraction functions treated in previous investigations, the integral of $q(x)$ can also be evaluated explicitly.

Evidently, when annual commuting cost is only a small fraction of the annual income, the locational equilibrium configuration is approximated to leading order by ignoring all travel cost (both the congestion cost and the distance cost) components. In particular, we have

$$r_c \sim \frac{\sigma N_o y}{A} = \frac{\sigma N_o}{A}(Y + D_e) \tag{5.18}$$

where we have from (5.16)

$$\frac{D_e}{Y} \sim \Delta_o = \frac{c_D\sigma}{1 - c_D\sigma}, \tag{5.19a}$$

with

$$A = 2\pi \int_{R_c}^{R_e} bX\, dX \tag{5.19b}$$

being the residential area available for housing space in the annular region, $R_c \leq X \leq R_e$. Here Δ_o is independent of ε; it follows that the unit land rent at the edge of the CBD decreases with increasing area of housing space A and hence with increasing b. Now, the optimal utility $\overline{U}$ is a (monotone increasing and strictly concave) function of $r_c^{-\sigma}$. It follows from (5.18)

$$\overline{U} \sim U([(1-\sigma)(1+\Delta_o)Y]^{1-\sigma} A^\sigma / N_o^\sigma). \tag{5.20}$$

Thus, to a leading term approximation for small commuting cost, the optimal equilibrium household utility increases with available housing space and therefore with increasing $b(x)$.

The leading term approximation cannot remain adequate as $b(x)$ increases. Otherwise we could increase household utility by taking larger and larger $b(x)$; in fact, we would attain maximum household utility by taking $b(x) \equiv 1$. But when more land is allocated to housing, the available roads to carry the commuting traffic will become congested and the commuting cost for households living away from the edge of the CBD will increase due to the higher congestion cost. With

$$\varepsilon = \varepsilon\eta + \varepsilon(1-\eta) = \frac{R_c\tau_o}{Y} + \frac{aN_o}{2\pi(1-b_c)} \tag{5.21}$$

the effect of the parameter ε must be included to account more accurately for the true impact of $b(x)$. The trade-off between a higher congestion cost and more housing suggests an optimal $b(x) < 1$ (see (6.4)). For the perturbation analysis through Section 8, we assume that we have $\varepsilon < 1$ for the range of $b(x)$ of interest.

The $O(\varepsilon)$ problem. The coefficients of the ε terms in the various expansions of (5.12) are determined by the differential equations

$$w_1' = -\frac{1}{1+\Delta_o}\left\{\eta T(x) + (1-\eta)\frac{u_o}{p(x)}\right\} \tag{5.22a}$$

$$u_1' = -q(x)w_o^{\alpha}\left\{\nu_1 + \nu_o\alpha\frac{w_1}{w_o} - \frac{\nu_o\Delta_1}{1+\Delta_o}\right\} \tag{5.22b}$$

$$\delta_1' = c_D q(x)[\sigma\nu_1 w_o^{\alpha+1} + \nu_o w_o^{\alpha} w_1] \tag{5.22c}$$

and the auxiliary conditions

$$\begin{aligned} &w_1(1) = 0, \qquad u_1(1) = 0, \qquad \delta_1(1) = 0,\\ &u_1(R) = 0, \qquad \delta_1(R) = \Delta_1. \end{aligned} \tag{5.23}$$

The righthand side of the equation (5.22a) for w_1' is known from the leading term solution; together with the initial condition $w_1(1) = 0$, it determines $w_1(x)$ completely. With this result, $u_1(x)$ and $\delta_1(x)$ are also determined by the differential equations (5.22b) and (5.22c) along with the initial conditions $u_1(1) = 0$ and $\delta_1(1) = 0$ up to the two unknown parameters ν_1 and Δ_1. The expressions for $w_1(x)$, $u_1(x)$ and $\delta_1(x)$ are in the form of quadratures that depend on $p(x)$ and $q(x)$. They will not be listed here. With these expressions, the remaining two auxiliary conditions at $x = R$ determine the two remaining unknown parameters to be:

$$\frac{\Delta_1}{\nu_o} = \frac{c_D\sigma Q_w(R)}{1 - c_D\sigma}, \qquad \frac{\nu_1}{\nu_o} = \nu_o(c_D - \alpha)Q_w(R) \tag{5.24}$$

where

$$Q_w(x) = \int_1^x w_1(x)q(x)\,dx. \tag{5.25}$$

By setting $c_D = 0$, we recover the results for these two parameters in the absentee ownership case obtained in Wan [1977].

Again, for housing fraction functions treated in previous investigations, the various integrals can be evaluated explicitly in terms of elementary functions. We can continue the solution process to calculate higher order correction terms such as $\{w_2(x), u_2(x), \ldots, \Delta_2\}$, $\{w_3(x),$

$u_3(x), \ldots, \Delta_3\}$, etc. For a sufficiently small ε, the present two-term perturbation solution is generally an adequate approximation of the exact solution. For $\varepsilon < 1$ but not small compared to unity, the same two-term perturbation solution provides a useful initial estimate for the exact solution in an iterative numerical solution scheme (Ascher et al. [1979]).

6. The optimal equilibrium household utility. The expression for the optimal equilibrium household utility (3.4) can be written as

$$\overline{U} = U([Y(1-\sigma)]^{(1-\sigma)}[2\pi R_c^2/N_o]^\sigma [b_c/\nu]^\sigma [1+\Delta]). \tag{6.1}$$

Among the components of the argument of the monotone increasing utility function $U(\cdot)$ in (6.1), we see from the $O(\varepsilon)$ correction terms in (5.24) that only ν and Δ depend on the normalized housing fraction $b(x)/b_c$. For ε small compared to unity, we use the two-term perturbation solutions for ν and Δ to write

$$\begin{aligned}\left(\frac{b_c}{\nu}\right)^\sigma (1+\Delta) &\sim \left(\frac{b_c}{\nu_o}\right)^\sigma (1+\Delta_o)\left\{\frac{1+\varepsilon c_D \sigma Q_r(R)}{1+\varepsilon\sigma(c_D-\alpha)Q_r(R)} + O(\varepsilon^2)\right\} \\ &\sim \left(\frac{b_c}{\nu_o}\right)^\sigma (1+\Delta_o)[1+\varepsilon\sigma\alpha Q_r(R) + O(\varepsilon^2)] \end{aligned} \tag{6.2}$$

where ν_o and Δ_o are as given in (5.16) and

$$Q_r(R) = \frac{Q_w(R)}{(1-c_D\sigma)Q(R)} \tag{6.3}$$

with the functions Q and Q_w as given previously by (5.17) and (5.25). Correspondingly, (6.1) becomes

$$\overline{U} \sim U\left(\left[\frac{Y(1-\sigma)}{1-c_D\sigma}\right]^{1-\sigma}\left[\frac{2AQ(R)}{N_o(R^2-1)}\right]^\sigma [1+\varepsilon\sigma\alpha Q_r(R)]\right) \tag{6.4}$$

where A is given by (5.19b). For a constant "housing fraction" function so that $b(x) = b_c$ does not vary with x, we have $q(x) = x$ and $Q(R) = (R^2-1)$. Moreover, the quantities $Q_w(R)$ and $Q(R)$ are independent of b_c (see (5.17) and (5.25)). For other housing fraction functions previously investigated in Wan [1977] and Solow [1972, 1973], numerical solutions obtained for these studies (see Acknowledgment) indicate that these two quantities do not vary significantly with $b(x)$. Thus, the dependence of the optimal equilibrium household utility on $b(x)$ is mainly through b_c^σ (since A is proportional to b_c as shown in (5.19)) and ε as shown in (6.4). Clearly $\overline{U}$ increases with b_c^σ as we already have learned from the leading term solution. The expression (5.21) for the dimensionless transportation cost parameter ε shows that this parameter also increases with b_c. However the quantity Q_r is negative since $w_1(x)$ is negative (according to the differential equation (5.22a) and the corresponding initial condition in (5.23)). With σ less than unity (as we typically spend no more than a quarter of our income on housing in North America), the effect of ε becomes more dominant with increasing b_c and eventually causes $\overline{U}$ to reach a maximum value; further increases in b_c would lower $\overline{U}$. When the two-term perturbation solution is applicable, the determination of the value of b_c that maximizes $\overline{U}$ is a simple calculus problem.

It should be noted that the general results for the public (or mixed) ownership case does not differ qualitatively from those for the absentee ownership case investigated previously with the commuting cost parameter ε increases with b_c for both cases. In theory, the numerical value of ε that maximizes $\overline{U}$ as given by (6.4) may not be small compared to unity and the two-term perturbation solution may not be adequate (or applicable). For cases where perturbation solutions in ε are inappropriate, accurate numerical solutions for the relevant boundary value problem (BVP) is possible by existing BVP solvers such as COLSYS or its newer edition COLNEW (Ascher et al. [1979]). Some such results for absentee ownership cases have been reported in Wan [1979, 1993]. In reality, however, the household commuting cost is expected to remain a relatively small fraction of the household income (and this is certainly the case for all sample cities discussed in Solow [1973]). Such a restriction imposes a limitation on the magnitude of b_c (to considerably less than unity) and allows the profile of the housing fraction function $b(x)$ to have a significant role in the maximum value $\overline{U}$.

A more general and systematic analysis of optimizing the household utility $U(c, s)$ through an appropriate choice of $b(x)$ (and not just b_c) is possible with or without locational household equilibrium (see Kanemoto [1980], Wan [1993]). While there is an exact optimal solution for Cobb-Douglas type utility functions, true optimality that involves differential household utility is of little interest to a society where people are free to relocate. More relevant is the *second best* allocation problem which chooses a housing fraction function $b(x)$ to optimize U under the constraint of a uniform equilibrium utility for all households. This more difficult problem has already been studied by perturbation and numerical methods in Wan [1993] for the absentee ownership case and will be investigated for public ownership and mixed ownership cases in a future study. The corresponding optimal household utility is "second best" compared to optimality without the constraint of a uniform utility for all households.

7. A free boundary problem. Some cities do not have a physical or political outer boundary but are free to expand. The land beyond the residential area, $X > R_e$, is typically used for farming. For such a city, the residential area will continue to expand as long as the unit land rent at the outer edge of the annular residential region is higher than the income from the same unit land area from farming. Locational equilibrium is attained when

$$r(X = R_e) = r_A \quad \text{or} \quad \nu w^{\alpha+1}(R) = \nu_A \tag{7.1}$$

where r_A is the *agricultural rent* for a unit of farm land and ν_A is the corresponding unit area land rent normalized by the same factor as the one for the residential unit rent.

With the dimensionless outer edge R of the residential area being also an unknown in a city with no political or physical outer boundary, we should consistently seek R as a parametric series expansion in powers of ε in our perturbation solution scheme as well since the value of R also depends on the parameter ε. Previous treatments of this problem expanded each composite function of R as a single parametric series in ε. For example, with

$$R(\varepsilon) = R_o + R_1\varepsilon + R_2\varepsilon^2 + \cdots \tag{7.2}$$

we have

$$\begin{aligned} w(R(\varepsilon);\varepsilon) &= w_o(R_o + R_1\varepsilon + \cdots) + w_1(R_o + R_1\varepsilon + \cdots)\varepsilon + \cdots \\ &= w_o(R_o) + [w_1(R_o) + w_o'(R_o)R_1]\varepsilon + O(\varepsilon^2). \end{aligned} \tag{7.3}$$

Here, we will work with a different approach that is also suitable for a numerical solution of the free boundary problem by BVP solvers. The new approach is to reformulate the BVP as one for the unit interval by re-scaling the independent variable x. In particular, we let

$$z = \frac{x-1}{R-1}, \quad \text{or} \quad x = 1 + (R-1)z. \tag{7.4}$$

Evidently, the new independent variable ranges over $[0, 1]$ as x ranges over $[1, R]$. We now consider w, u and δ as functions of z (without giving them new names) and, with $(\ )^{\bullet} \equiv d(\ )/dz$, write the three differential equations (5.7)–(5.9) as

$$w^{\bullet} = -\,\varepsilon \frac{R-1}{1+\Delta}\left\{\eta T(x) + (1-\nu)\,\frac{u}{p}\right\} \tag{7.5}$$

$$u^{\bullet} = -\,\frac{R-1}{1+\Delta}\,\nu q w^{\alpha} \tag{7.6}$$

$$\delta^{\bullet} = c_D \sigma \nu (R-1) q w^{\alpha+1} \tag{7.7}$$

where p and q as defined in (5.2) are now expressed in terms of z with the help of (7.4). For simplicity, we limit further discussion to the conventional case where $T(x) \equiv 1$ so that distance cost rate τ does not vary with location; extending the analysis to the case where τ varies with X is straightforward. We will continue to assume that the functions $q(x)$ and $p(x)$ do not depend on ε. When written in terms of z however, they now depend on ε through $R(\varepsilon)$, e.g., $q(x) = q(1 + [R(\varepsilon) - 1]z)$. We must therefore expand p and q in parametric series in ε when we seek a perturbation solution using z as the independent variable:

$$\{p(x), q(x)\} = \sum_{n=0}^{\infty} \{p_n(z), q_n(z)\}\varepsilon^n. \tag{7.8}$$

With (7.4), the auxiliary conditions (5.10)–(5.11) and (7.1) now take the form

$$\begin{array}{lll} w(z=0) = 1, & u(z=0) = 1, & \delta(z=0) = 0 \\ u(z=1) = 0, & \delta(z=1) = \Delta, & \nu w^{\alpha+1}(z=1) = \nu_A \end{array} \tag{7.9}$$

We may now apply the perturbation method to the new BVP on $[0, 1]$ with the help of the parametric series (7.2) and (7.8) and the corresponding series for $w(z;\varepsilon)$, $u(z;\varepsilon)$ and $\delta(z;\varepsilon)$ where we have deliberately suppressed the dependence of these three quantities on Δ and ν for brevity.

The $O(1)$ problem. The leading term solution of the problem defined by (7.5)–(7.7) and (7.9) is determined by the differential equations

$$w_o^\bullet = 0, \qquad u_o^\bullet = -\frac{R_o - 1}{1+\Delta_o}\nu_o q_o w_o^\alpha, \\ u_o^\bullet = \sigma c_D \nu_o (R_o - 1) q_o w^{\alpha+1} \tag{7.10}$$

and the auxiliary conditions

$$w_o(0) = 1, \qquad u_o(0) = 1, \qquad \delta_o(0) = 0 \tag{7.11}$$

$$u_o(1) = 0, \qquad \delta_o(1) = \Delta_o, \qquad \nu_o w_o^{\alpha+1}(1) = \nu_A \tag{7.12}$$

The solution to this boundary value problem is

$$w_o(z) = 1, \qquad \delta_o(z) = \sigma c_D \nu_o (R_o - 1) Q_o(z; R_o), \\ u_o(z) = 1 - \frac{R_o - 1}{1+\Delta_o}\nu_o Q_o(z; R_o) \tag{7.13}$$

where

$$Q_o(z; R_o) = \int_0^z q_o(\zeta; R_o)\, d\zeta \tag{7.14}$$

with

$$\nu_o = \nu_A \tag{7.15}$$

The solution (7.15) for ν_o follows from the last condition in (7.12). At first glance, it appears to be radically different from the corresponding expression for the fixed boundary case where ν_o is determined by the total residential land available for housing. However, the total land available for housing in the free boundary case is determined by the agricultural rent; hence, it is consistent with the fixed boundary case for ν_o to be determined by ν_A.

The other two conditions in (7.12) give

$$\begin{aligned} \Delta_o &= c_D\sigma\nu_o(R_o-1)Q_o(1;R_o), \\ 1+\Delta_o &= \nu_o(R_o-1)Q_o(1;R_o) \end{aligned} \tag{7.16}$$

for R_o and Δ_o. Since ν_o is already known to be ν_A, we solve these last two relations for Δ_o to obtain:

$$\Delta_o = \frac{c_D\sigma}{1-c_D\sigma}. \tag{7.17}$$

This expression is identical to the corresponding expression for the fixed outer boundary case given in (5.19). The same two relations in (7.16) also give a condition for determining R_o:

$$\nu_A(R_o-1)Q_o(1;R_o)[1-c_D\sigma] = 1 \tag{7.18}$$

which is generally nonlinear in R_o. Its solution may be obtained by an iterative numerical method.

For housing fraction functions treated in previous investigations, the integral $Q_o(z;R_o)$ as given by (7.14) can be evaluated explicitly. To illustrate how this quantity may depend on R_o, consider the *linear* housing fraction function studied by Solow [1972]:

$$b = b_c + (b_e-b_c)\,\frac{X-R_c}{R_e-R_c} = b_c + (b_e-b_c)z \tag{7.19}$$

with $b=b_c$ and $b(R)=b_e$. Note that a linear $b(x)$ expressed in terms of z does not depend on $R(\varepsilon)$ explicitly. However, when expressed in terms of z, the corresponding function $q(x)=xb$ becomes

$$\begin{aligned} q(x) &= [1+(R-1)z]\left[1-\left(1-\frac{b_e}{b_c}\right)z\right] \\ &= \left[1-\left(1-\frac{b_e}{b_c}\right)z\right]\{[1+(R_o-1)z]+\varepsilon R_1 z + O(\varepsilon^2)\} \\ &= q_o(z;R_o)+\varepsilon q_1(z;R_o,R_1)+O(\varepsilon^2) \end{aligned} \tag{7.20}$$

The corresponding quantity $Q_o(1;R_o)$ as defined by (7.14) becomes

$$Q_o(1;R_o) = \left[1-\frac{1}{2}\left(1-\frac{b_e}{b_c}\right)\right] + \frac{1}{6}\left[3-2\left(1-\frac{b_e}{b_c}\right)\right](R_o-1). \tag{7.21}$$

Thus, for the case of a linear housing fraction distribution, the condition (7.18) is a quadratic equation for $(R_o - 1)$ which can be solved for its only positive root (since we must have $R_o > 1$).

We may also obtain results for a constant housing fraction function by setting $b_e = b_c$. For this case, the expression for $q(x)$ reduces to

$$q(x) = [1 + (R-1)z] \tag{7.22}$$

as it should. It follows that $Q_o(1; R_o) = 1 + (R_o - 1)/2$ and, from (7.18),

$$R_o = \sqrt{1 + \frac{2}{\nu_A(1 - c_D\sigma)}} \tag{7.23}$$

The result in (7.23) is realistic as it predicts a residential area decreasing with increasing agricultural land rent and with increasing absentee ownership.

As in the fixed boundary case, the effect of congestion cost of transportation mitigates against the excessive allocation of residential area land for housing. With the help of (7.15) and (7.17), the expression (3.4) (or (6.1)) for $\overline{U}$ becomes

$$\begin{aligned} \overline{U} &= U([Y(1-\sigma)]^{1-\sigma}(1+\Delta)[2\pi R_c^2/N_o]^{\sigma}[b_c/\nu]^{\sigma}) \\ &\sim U([Y(1-\sigma)]^{1-\sigma}(1+\Delta_o)[2\pi R_c^2/N_o]^{\sigma}[b_c/\nu_o]^{\sigma}) \\ &= U([Y(1-\sigma)]^{1-\sigma}[2\pi R_c^2/N_o]^{\sigma}[b_c/\nu_A]^{\sigma} c_D\sigma/[1-c_D\sigma]) \end{aligned} \tag{7.24}$$

We see from (7.24) that the leading term approximation for $\overline{U}$ increases with b_c, reaching its maximum at the unrealistic value of $b_c = 1$. We must again consider the effect of $O(\varepsilon)$ terms in the perturbation solution for a more realistic solution of the problem, however small the effect of congestion cost of transportation may be.

The $O(\varepsilon)$ problem. Routine calculations show that the $O(\varepsilon)$ terms of the parametric expansions are determined by the three differential equations

$$w_1^{\bullet} = -\frac{R_o - 1}{1 + \Delta_o}\left[\eta + (1-\eta)\frac{u_o}{p_o}\right], \tag{7.25}$$

$$u_1^\bullet = \frac{1}{1+\Delta_o}[\nu_o q_1(R_o-1)+\nu_o q_o R_1+\nu_1 q_o(R_o-1)$$

(7.26)

$$+ \alpha\nu_o q_o(R_o-1)w_1 - \frac{\Delta_1}{1+\Delta_o}\nu_o q_o(R_o-1)]$$

$$\delta_1^\bullet = c_D\sigma[\nu_o q_1(R_o-1)+\nu_o q_o R_1$$

(7.27)

$$+ \nu_1 q_o(R_o-1)+(\alpha+1)\nu_o q_o(R_o-1)w_1]$$

and the auxiliary conditions

(7.28) $$w_1(0)=0, \qquad u_1(0)=0, \qquad \delta_1(0)=0,$$

(7.29) $$u_1(1)=0, \qquad \delta_1(1)=\Delta_1, \qquad \nu_1+(\alpha+1)\nu_o w_1(1)=0$$

The righthand side of (7.25) is known from the leading term solution; the function $w_1(z)$ can therefore be found by simple integration with the help of the first initial condition in (7.28). For housing fraction functions treated previously, this integral can be evaluated in terms of elementary functions. With this result, it follows immediately from the last condition in (7.29)

(7.30) $$\nu_1 = -\frac{\nu_A}{\sigma}w_1\Big|_{z=1}$$

It is evident from (7.25) that $w_1(1)$ is negative. Hence, the dimensionless unit land area rent at the edge of the CBD is increased by the inclusion of a small congestion cost.

With $w_1(z)$ determined up to the parameters Δ_1 and R_1, the solution of the remaining two differential equations can also be found up to the same two parameters by simple integration with the help of the two remaining initial conditions of (7.28). We write the solutions as

$$u_1(z) = \frac{1}{1+\Delta_o}\Big\{\nu_o(R_o-1)Q_1(z;R_o,R_1)+\alpha\nu_o(R_o-1)Q_w(z;R_o)$$

(7.31)

$$+\left[\nu_o R_1+\left(\nu_1-\frac{\nu_o\Delta_1}{1+\Delta_o}\right)(R_o-1)\right]Q_o(z;R_o)\Big\}$$

$$\begin{aligned}\delta_1(z) = {} & c_D\sigma\nu_o(R_o-1)Q_1(z;R_o,R_1)\\ & + [\nu_o R_1 + \nu_1(R_o-1)]Q_o(z;R_o)\\ & + (\alpha+1)\nu_o(R_o-1)Q_w(z;R_o)\end{aligned} \tag{7.32}$$

where

$$\begin{aligned}\{Q_o(z;R_o),& Q_1(z;R_o,R_1),Q_w(z;R_o)\}\\ & = \int_0^z \{q_o(z;R_o),q_1(z;R_o,R_1),q_o(z;R_o)w_1(z;R_o)\}\,dz.\end{aligned} \tag{7.33}$$

The first two auxiliary conditions in (7.29) now give two conditions for the determination of R_1 and D_1 :

$$\begin{aligned}(R_o-1)Q_1(1;R_o,R_1) & + \left[R_1 - \frac{\Delta_1}{1+\Delta_0}(R_o-1)\right]Q_o(1;R_o)\\ & = -\,\alpha(R_o-1)Q_w(1) - \frac{\nu_1}{\nu_A}(R_o-1)Q_o(1;R_o),\end{aligned} \tag{7.34}$$

$$\begin{aligned}& c_\Delta\sigma\nu_A\{(R_o-1)Q_1(1;R_o,R_1)+R_1Q_o(1;R_o)\} - \Delta_1\\ & = -c_D\sigma\{\nu(R_o-1)Q_o(1;R_o)+(\alpha+1)\nu_A(R_o-1)Q_w(1;R_o)\}\end{aligned} \tag{7.35}$$

where we have made use of (7.15) for ν_o. It is rather remarkable that one of these relations can be used to eliminate R_1 from the other to give the following simple result for Δ_1:

$$\Delta_1 = \frac{c_D\sigma}{(1-c_D\sigma)^2} \tag{7.36}$$

The corresponding elimination of Δ_1 gives a nonlinear relations for R_1:

$$\begin{aligned}(R_o-1)& Q_1(1;R_o,R_1)+R_1Q_o(1;R_o)\\ & = (R_o-1)\left\{\left[\frac{\Delta_1}{1+\Delta_o}+\frac{\nu_1}{\nu_A}\right]Q_o(1;R_o)+\alpha Q_w(1)\right\}\end{aligned} \tag{7.37}$$

For the special case of a *constant* housing fraction function so that $b(x)\equiv b_c$, this relation is linear and has the unique exact solution

$$\begin{aligned}R_1 = {} & -\frac{1}{2R_o}\{2\alpha(R_o-1)Q_w(1;R_o)\}\\ & -(R_o^2-1)\left[(\alpha+1)w_1(1;R_o)-\frac{c_D\sigma}{1-c_D\sigma}\right]\end{aligned} \tag{7.38}$$

Similar to R_o, the quantity R_1 does not depend on b_c for the constant housing fraction distribution.

8. Optimal equilibrium household utility for the free boundary problem. Upon substituting the results for the two-term perturbation solution for the various quantities into the expression for the monotone increasing utility function $\overline{U}$ given by (3.4) (or (6.1)), we have

$$\begin{aligned}\overline{U} &= U(([Y(1-\sigma)]^{1-\sigma}[2\pi R_c^2/N_o]^{\sigma}[b_c/\nu]^{\sigma}[1+\Delta]) \\ &\sim U([Y(1-\sigma)]^{1-\sigma}[2\pi R_c^2/N_o]^{\sigma}(1+\Delta_o)[b_c/\nu_A]^{\sigma} \\ &\qquad [1+\varepsilon\Delta_1/(1+\Delta_o)]/[1+\varepsilon\nu_1/\nu_A]^{\sigma})\end{aligned} \tag{8.1}$$

In view of (7.30) and (7.36), the last expression may be rewritten as

$$\overline{U} \sim U([Y(1-\sigma)]^{1-\sigma}[2\pi R_c^2/N_o]^{\sigma}[b_c/\nu_A]^{\sigma}[1+\varepsilon Q_\rho(R_o)]/[1-c_D\sigma]) \tag{8.2}$$

where

$$Q_\rho(R_o) = w_1(1;R_o) + \frac{c_D\sigma}{1-c_D\sigma} \tag{8.3}$$

with an error of the order of ε^2. Generally, the dependence of $\overline{U}$ on b_c through $w_1(1;R_o)$ will vary with the particular housing fraction function. For the case of a constant housing fraction function $b(x) = b_c$, the quantity $w_1(1;R_o)$ does not vary with b_c so that the dependence of the optimal equilibrium utility $\overline{U}$ on $b(x)$ is similar to that of fixed boundary problems as described in Sections 5 and 6 except for a different weight factor for the congestion cost parameter ε.

9. Matched asymptotic expansion solution for small σ. In most investigations, the parameter σ is taken to be 1/4, which is roughly the fraction of household income spent on housing in the United States and Canada. If σ is interpreted strictly as the fraction of *net* income (after travel cost) to be spent on ground rent, then σ is typically smaller than 1/20 or α is typically larger than 10. For $\alpha \gg 1$, the quantity ε/σ, and therewith $\varepsilon\nu$ (see (9,3)) may not be small even if ε is small, and perturbation solution of the type (5.1) may no longer be appropriate. To see this, we use (5.7) to express $u(x)$ in terms of $w'(x)$

$$u = -\frac{p(x)}{\varepsilon(1-\eta)}[(1+\Delta)w' + \varepsilon\eta T(x)] \tag{9.1}$$

and then use the result to eliminate $u(x)$ from (5.8) to obtain a second order nonlinear ODE for w with ν and Δ as two unknown parameters:

$$[p(x)w']' - \frac{\varepsilon\nu(1-\eta)}{(1+\Delta)^2} q(x)w^\alpha = -\frac{\varepsilon\eta}{1+\Delta}[p(x)T(x)]' \tag{9.2}$$

where

$$\varepsilon\nu = \frac{2\pi R_c^2 b_c r_c}{N_o Y}\left(\frac{\varepsilon}{\sigma}\right) \tag{9.3}$$

Evidently, σ and ε may be of comparable magnitude so that ε/σ may not be small compared to unity. Hence, perturbation methods for small ε may not be appropriate for $\sigma \ll 1$ as (5.24) shows that ν_1/ν_o increases with α and thereby has the effect of "disordering" the terms in the perturbation series for $\nu(\varepsilon)$. On the other hand, a *matched asymptotic expansions* solution for the relevant boundary value problem with $\sigma = (\alpha+1)^{-1}$ as the small parameter is possible for $\sigma \ll 1$, independent of the magnitude of ε.

To apply the method of matched asymptotic expansions, we introduce a new dimensionless unit land rent parameter

$$\lambda = 2\pi(1-\eta)(1+\alpha^{-1})\frac{R_c^2 b_c r_c}{N_o Y} \tag{9.4}$$

and write the differential equation (9.2) as

$$[p(x)w']' - \frac{\alpha\lambda\varepsilon}{(1+\Delta)^2} q(x)w^\alpha = -\frac{\varepsilon\eta}{1+\Delta} p'(x). \tag{9.5}$$

where we have again limited our analysis to the case $T(x) \equiv 1$ for simplicity.

The outer solution. A useful observation which leads to an appropriate outer solution is the fact that we have $w(x) < 1$ for $x > 1$. For $\alpha \gg 1$, $w^\alpha(x)$ is exponentially small for $x > 1$ with $\alpha w^\alpha(x)$ tending to zero as α tends to ∞. Thus, the outer solution for the differential equation (9.5) is given by $w_o(x)$ which satisfies the differential equation

$$[p(x)w_o'(x)]' = -\frac{\varepsilon\eta}{1+\Delta_o} p'(x) \tag{9.6}$$

except for exponentially small terms. The differential equation (9.6) can be integrated once to give $p(x)w_o'(x) = -\varepsilon\eta p(x)/(1+\Delta_o) + C_1$. The constant of integration C_1 is determined to be zero by the condition $u(R) = 0$, or $w_o'(R) = -\varepsilon\eta/(1+\Delta_o)$; this leaves us with

$$w_o'(x) = -\frac{\varepsilon\eta}{1+\Delta_o} \tag{9.7}$$

and therewith

$$w_o(x) = -\frac{\varepsilon\eta}{1+\Delta_o}(x-1) + C_2 \tag{9.8}$$

where C_2 is another constant of integration. It is not possible to choose this constant to satisfy the two remaining auxiliary conditions on $w(1)$ and $w'(1)$, i.e., the first two conditions of (5.10), an *inner* (asymptotic expansion) solution is therefore needed near the edge of the CBD.

The inner solution. For the layer solution near $x = 1$, we introduce the stretched variable

$$\zeta = \alpha(x-1) \quad \text{or} \quad x = 1 + \frac{\zeta}{\alpha}. \tag{9.9}$$

and consider the unknowns as functions of ζ. The structure of the relevant inner solution must be such that the contribution of the w^α term to the differential equation (9.5) is significant but not dominant. This requires $w(x)$ not to differ much from unity in the layer and the gradient of the decrease be sufficiently sharp for $w'(x)$ to be $O(1)$ within the layer adjacent to the CBD. This requirement is met by

$$w(x) = 1 + \frac{W(\zeta)}{\alpha} \tag{9.10}$$

With $(\ )' = \alpha(\ )^\circ$, where $(\ )^\circ$ indicates differentiation with respect to ζ, the differential equation (9.5) becomes

$$\begin{aligned} &p(1+\zeta/\alpha)W^{\circ\circ} + \left[\frac{p'(1+\zeta/\alpha)}{\alpha}\right]W^\circ \\ &\quad - \frac{\varepsilon\lambda}{(1+\Delta)^2}q\left(1+\frac{\zeta}{\alpha}\right)\left(1+\frac{W}{\alpha}\right)^\alpha = -\frac{\varepsilon\eta p'(1+\zeta/\alpha)}{\alpha(1+\Delta)} \end{aligned} \tag{9.11}$$

Since $p'(x)$ is $O(1)$ and $x = 1 + \zeta/\alpha$, the differential equation for the leading term inner solution is obtained by letting α tend to ∞ in (9.11) while ζ remains bounded to obtain

$$W_o^{\circ\circ} - \frac{\varepsilon\lambda_o}{(1+\Delta_o)^2} e^{W_o} = 0 \tag{9.12}$$

where we have made use of the fact that $p(x) \sim p(1) = 1$ as α tends to ∞ and where λ_o is the leading term approximation for λ for large α.

The second order differential equation (9.12) is autonomous and admits an exact solution. Upon multiplying both sides of (9.12) by W^o and integrating once with respect to ζ, we obtain

$$(W_o^\circ)^2 - \frac{2\varepsilon\lambda_o}{(1+\Delta_o)^2} e^{W_o} = \mu^2, \tag{9.13}$$

where we have made use of the auxiliary condition $u(1) = 1$ (or $W^\circ(0) = -\varepsilon/(1+\Delta)$) to obtain

$$\mu^2 = \frac{\varepsilon^2 - 2\varepsilon\lambda_o}{(1+\Delta_o)^2}. \tag{9.14}$$

The first order differential equation (9.13) is separable and may be integrated to give

$$W_o(\zeta) = -2\ln(B_o \sin h[\mu(\zeta+\zeta_o)/2]) \tag{9.15}$$

with $\mu B_o = \sqrt{2\varepsilon\lambda_o}/(1+\Delta_o)$ (keeping in mind that we must have $[W_o]^\circ < 0$ near the edge of the CBD). The constant of integration ζ_o is determined by the initial condition $W_o(\zeta = 0) = 0$ to be

$$\mu\zeta_o = 2\ln\left(\left[1 + \sqrt{1+B_o^2}\right]/B_o\right) \tag{9.16}$$

The remaining constant C_2 (in the outer solution) is to be determined by matching the leading term inner solution (9.15) with the outer solution (9.8).

Matching. The matching of the inner and outer solution is accomplished by way of an *intermediate variable*

$$\theta = \sqrt{\alpha}(x-1). \tag{9.17}$$

With

$$1+\frac{1}{\alpha}W_o(\zeta) = 1-(2/\alpha)\ln(B_o\sinh[\mu(\zeta+\zeta_o)/2])$$
$$\sim 1-(2/\alpha)\ln[B_o e^{\mu(\zeta+\zeta_o)/2}]$$
$$= 1-\frac{1}{\alpha}\left\{\mu\theta\sqrt{\alpha}+2\ln\left[1+\sqrt{1+B_o}\right]\right\} \tag{9.18}$$
$$\sim 1-\frac{\mu\theta}{\sqrt{\alpha}}.$$

for the leading term inner solution, and with the outer solution of (9.8) given in terms of θ by

$$w_o(x) = C_2 - \frac{\varepsilon\eta}{1+\Delta_o}\frac{\theta}{\sqrt{\alpha}}, \tag{9.19}$$

matching of the leading term inner and outer solutions through the intermediate variable θ gives

$$C_2 = 1, \qquad \mu^2 = \frac{\varepsilon^2-2\varepsilon\lambda_o}{(1+\Delta_o)^2} = \left[\frac{\varepsilon\eta}{1+\Delta_o}\right]^2. \tag{9.20}$$

The second relation appears to involve two unknown parameters λ_o and Δ_o but actually determines λ_o explicitly: $\lambda_o = \varepsilon(1-\eta^2)/2$.

The optimal equilibrium household utility. It follows from (9.4) and $\lambda_o = \varepsilon(1-\eta^2)/2$ that the leading term solution for r_c is given by

$$4\pi(1-\eta)\left(1+\frac{1}{\alpha}\right)\frac{R_c^2 b_c r_c}{N_o Y} \sim 2\lambda_o = \varepsilon(1-\eta^2) \tag{9.21}$$

or

$$r_c \sim \frac{N_o Y}{4\pi R_c^2 b_c}\varepsilon(1+\eta) = \frac{aN_o^2(1+\eta)}{8\pi^2 R_c^2(1-\eta)b_c(1-b_c)}. \tag{9.22}$$

We can also calculate the leading term approximation of the dimensionless dividend parameter Δ from the expression

$$\Delta = \frac{D_e}{Y} = c_D\sigma\nu\int_1^R [w(x)]^{\alpha+1}q(x)dx \equiv c_D\sigma\nu Q_\alpha(R) \tag{9.23}$$

where

$$(9.24)\qquad \begin{aligned} \sigma\nu &= \frac{2\pi b_c R_c^2 r_c}{Y N_0} \sim \frac{\varepsilon}{2}(1+\eta) \\ &= \frac{a N_o (1+\eta)}{4\pi Y(1-\eta)(1-b_c)} \equiv \frac{\gamma_o}{1-b_c}. \end{aligned}$$

It follows from (3.4) that

$$(9.25)\qquad \begin{aligned} \overline{U} &= U(\sigma^\sigma (1-\sigma)^{1-\sigma} Y(1+\Delta) r_c^{-\sigma}) \\ &\sim U(\Lambda[b_c(1-b_c)/\gamma_o]^\sigma [1 + c_D \gamma_o Q_\alpha(R)/(1-b_c)]) \end{aligned}$$

where

$$(9.26)\qquad \Lambda = [(1-\sigma)Y]^{1-\sigma}[2\pi\sigma R_c^2/N_o]^\sigma.$$

Note that the results obtained above reduce to those obtained in Wan [1977] for the absentee ownership case (with $c_D = 0$). To a leading term approximation for that case, the maximum optimal equilibrium household utility is attained at $b_c = (1/2)$ if $\tau_o = 0$ (so that $\eta = 0$). For the case of public ownership, ($c_D = 1$) and $\eta = 0$, this maximand b_c is slightly modified by a terms proportional to γ_o which is small compared to unity. The modification will become significant only if $(1 - b_c)$ is $O(\gamma_o)$.

10. Concluding remarks. In the preceding sections we demonstrated how perturbation and asymptotic methods can be useful in analyzing residential land economic models. In the process, we obtained results for models not previously studied by these methods. The perturbation and asymptotic results of this paper have enabled us to see the differences between the two extreme cases of absentee ownership and public ownership. A first attempt to study a mixed ownership model is accomplished by returning only a fraction $c_D (< 1)$ of the total rent collected to the households as dividends. In all cases, an effort was made to analyze how the leading term optimal equilibrium household utility depends on the housing fraction distribution $b(x)$, and how it may be maximized by varying the housing fraction at the edge of the CBD. A more systematic search for the choice of $b(x)$ which maximizes $\bar{U}$ for a second best residential land allocation will be the subject of a future study.

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