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Static-geometric duality and stress concentration in twisted and sheared shallow spherical shells

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1 Introduction

With reference to our previous work (Reissner and Wan, 1994), the present note has two main objectives. The first of these is a simplification in the use of the static-geometric duality by means of alternate forms of the general solution of the two simultaneous partial differential equations which govern the solution of stress concentration problems for twisted or sheared shallow spherical shells with a stress free hole, a rigid insert or a circular edge with a roller support near the apex of the shells. The second objective is to establish the complete solutions, including numerical data, for a specific stress concentration problem, intermediate to the problem of a shell with a circular hole and one with a rigid circular insert, which gives rise to the simplest set of edge conditions in the class of problems to which these developments apply.

The significant simplification of the solution process is achieved through the reduction of all twelve stress concentration factors of interest to expressions involving only an auxiliary function χ or its Laplacian, where χ embodies the boundary layer portion of the complete solution of the boundary value problem (BVP). Conceptually, this is not unexpected since stress concentration is concerned with edge zone behavior of the shell. The new forms of the general solution for the governing differential equations also enable us to decouple the determination of the function χ from solution of the general BVP. This is accomplished by deriving appropriate boundary conditions for χ without involving the interior portion of the solution encapsulated in the potential functions ϕ and ψ. Together, the two results allow us to calculate the stress concentration factors by determining the boundary layer solution χ alone without any reference to the interior solution components.

The simplified solution process also exploits the static-geometric duality inherent in the group of boundary value problems for the shells of interest. In particular, applications of this elegant duality reduce the number of χ functions required for the six BVP from six to three and the calculation for actual number of stress concentration factors from twelve to six.

In contrast to our previous work which limited consideration to direct asymptotic analysis of boundary value problems, we focused here on exact analyses of these problems with the results applicable to the entire range of the shell parameter $\mu^4 = a^4/DBR^2$ (where D and $1/B$ are the bending and stretching stiffness factors of the shell material, R is the radius of spherical midsurface and a is the radius of the circular opening near the apex of the shell) as defined in our previous work. These include exact numerical values for the stress concentration factors for shells with a roller-supported edge (with three term asymptotic expansions of the exact solutions provided for comparison). Unlike the previous effort, we also consistently keep distinct in our analysis the two effective Poissons ratios, ν_M and ν_N, for our shells in order to allow for effective applications of the static-geometric duality.

2 The differential equations and a modified solution representation

With reference to equations (1) to (11) in (Reissner & Wan, 1994) and using the same notation therein, we now take the solutions of the two governing differential equations in (6) there in the following form:

$$w = \frac{Ta^2}{2D(1-\nu_M)} w^T, \quad F = \frac{Ta^2}{2D(1-\nu_M)} F^T \qquad \text{(1a,b)}$$

$$w^T = \phi + \chi, \quad F^T = \sqrt{\frac{D}{B}}\psi + \frac{RD}{a^2}\nabla^2\chi \qquad \text{(1c,d)}$$

for the problem of transverse twisting and

$$F = -Sa^2F^S, \quad w = -Sa^2w^S \qquad \text{(2a,b)}$$

$$F^S = \phi + \chi, \quad w^S = \sqrt{\frac{B}{D}}\psi - \frac{RB}{a^2}\nabla^2\chi \qquad \text{(2c,d)}$$

for the problems of membrane shearing. For later convenience, we use the symbol F in place of the symbol K used in (Reissner & Wan, 1994).

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Table 1

μ	k_m^{SH}	k_b^{SH}	k_m^{TH}	k_b^{TH}
0	4	0	0	$\frac{4+4\nu}{3+\nu}$
>>1	$\frac{\mu^2}{2}$	$\frac{\mu^2}{2}\sqrt{\frac{3+3\nu}{1-\nu}}$	$\sqrt{\frac{1-\nu^2}{3}}$	$1+\nu$

In the expression (1) and (2), we have

$$\nabla^2(\phi,\psi) = 0, \quad \nabla^4\chi + \mu^4\chi = 0 \tag{3a,b}$$

with the dimensionless parameter $\mu^4 = a^4/R^2DB$ and

$$\nabla^2(\,) = (\,)_{,\rho\rho} + \frac{1}{\rho}(\,)_{,\rho} + \frac{1}{\rho^2}(\,)_{,\theta\theta} \tag{3c}$$

where $\rho = r/a$ is a dimensionless radial coordinate.

In conformity with Table 1 in (Reissner & Wan, 1994), w^T and F^T are the static-geometric duals of F^S and w^S.

3 Boundary conditions

Consistent with the analysis in (Reissner & Wan, 1994), the boundary conditions at the edge of a stress free circular hole of radius a can be written as

$$\rho = 1: \begin{cases} F = 0, \quad F_{,\rho} = 0 & \text{(4a,b)} \\ \nabla^2 w - (1-\nu_M)(\rho^{-1}w_{,\rho} + \rho^{-2}w_{,\theta\theta}) = 0 & \text{(4c)} \\ (\nabla^2 w)_{,\rho} + (1-\nu_M)(\rho^{-2}w_{,\rho} - \rho^{-3}w)_{,\theta\theta} = 0 & \text{(4d)} \end{cases}$$

and at the edge of a rigid circular insert as

$$\rho = 1: \begin{cases} w = 0, \quad w_{,\rho} = 0 & \text{(5a,b)} \\ \nabla^2 F - (1+\nu_N)(\rho^{-1}F_{,\rho} + \rho^{-2}F_{,\theta\theta}) = 0 & \text{(5c)} \\ (\nabla^2 F)_{,\rho} + (1+\nu_N)(\rho^{-2}F_{,\rho} - \rho^{-3}F)_{,\theta\theta} = 0 & \text{(5d)} \end{cases}$$

As these conditions are homogeneous, they imply the same stipulations for w^T and F^T and for w^S and F^S. That the above conditions on F in one case came out as the static-geometric duals of the conditions for w in the other case was one of the significant aspects of the analysis in (Reissner & Wan, 1994).

A third problem, formulated in (Reissner & Wan, 1994) as a particularly simple counterpart of the two problems of the hole and of the rigid insert, involved a shell with the edge of its circular hole transversely fixed and tangentially free. The edge is said to be on a (tangential) roller support. For this case, the boundary conditions are

$$\rho = 1: \quad w = 0, \quad w_{,\rho} = 0, \quad F = 0, \quad F_{,\rho} = 0 \tag{6a–d}$$

Note that the conditions (6) are their own static-geometric duals.

Any of the three sets of conditions at $r = a$, (4), (5) or (6), is complemented by conditions of loading, which may be obtained from the known elementary solutions for a shell without a hole or insert, of the form

$$\rho \to \infty: \quad w^T \to \tfrac{1}{2}\rho^2 \sin 2\theta, \quad F^T \to 0 \tag{7a,b}$$

for the problem of transverse twisting, and

$$\rho \to \infty: \quad w^S \to 0, \quad F^S \to \tfrac{1}{2}\rho^2 \sin 2\theta \tag{8a,b}$$

for the problem of membrane shearing. Both sets of boundary conditions (7) and (8) are satisfied upon taking and ϕ and ψ in the form

$$\phi = \left(\frac{1}{2}\rho^2 + c_1\rho^{-2}\right)\sin 2\theta, \quad \psi = c_2\rho^{-2}\sin 2\theta \tag{9a,b}$$

provided that χ is a boundary layer phenomenon.

The form of (4), (5) or (6) requires that

$$\chi(\rho,\theta) = g(\rho)\sin 2\theta \tag{10}$$

The introduction of (9a,b) and (10) into equations (4) to (6) gives, as a consequence of (1c,d) for the case of a transverse twisting, three different sets of edge conditions for the function χ, corresponding to a free circular hole, a rigid circular insert and a roller supported circular edge, respectively. Repeating this for the three membrane shear problem gives three new sets of edge conditions for χ. Evidently, these three new sets of conditions are the static geometric duals of the three sets already obtained above.

For the transverse twisting problem with a circular hole, the set of four free edge conditions $r = a$ may be taken in the form

$$\rho = 1: \begin{cases} c_2\rho^{-2}s + \mu^{-2}\nabla^2\chi = 0, \\ \quad -2c_2\rho^{-3}s + \mu^{-2}(\nabla^2\chi)_{,\rho} = 0 & \text{(11a,b)} \\ \nabla^2\chi + (1-\nu_M)[4\rho^{-2}\chi - \rho^{-1}\chi_{,\rho} \\ \quad + (1+6c_1\rho^{-4})s] = 0 & \text{(11c)} \\ (\nabla^2\chi)_{,\rho} + (1-\nu_M)[4\rho^{-3}\chi - 4\rho^{-2}\chi_{,\rho} \\ \quad -(2\rho^{-1} - 12c_1\rho^{-5})s] = 0 & \text{(11d)} \end{cases}$$

where we have set $s = \sin 2\theta$. The corresponding conditions for the membrane shearing problem with a rigid insert follow form (11) with no changes other than a replacement of the quantity ν_M in (11c,d) by the quantity $-\nu_N$ and the quantity μ^2 by $-\mu^2$ (since the static geometric dual of $\mu^2 = (a/\sqrt{BR})(a/\sqrt{DR})$ is $(a/\sqrt{-DR})$ $(a/\sqrt{-BR})$ $= -\mu^2$).

Similarly, the four clamped edge conditions for the problem of transverse twisting with a circular insert at $r = a$ may be taken in the form

$$\rho = 1: \begin{cases} (\frac{1}{2}\rho^2 + c_1\rho^{-2})s + \chi = 0, \\ \quad (\rho - 2c_1\rho^{-3})s + \chi_{,\rho} = 0 & \text{(12a,b)} \\ \mu^4\chi + (1+\nu_N)[\rho^{-1}(\nabla^2\chi)_{,\rho} - 4\rho^{-2}\nabla^2\chi] \\ \quad -6\mu^2c_2\rho^{-4}(1+\nu_N)s = 0 & \text{(12c)} \\ \mu^4\chi_{,\rho} + (1+\nu_N)[4\rho^{-2}(\nabla^2\chi)_{,\rho} - 4\rho^{-3}\nabla^2\chi] \\ \quad -12\mu^2c_2\rho^{-5}(1+\nu_N)s = 0 & \text{(12d)} \end{cases}$$

They are the static-geometric duals of the edge conditions for membrane shearing with a hole centered at the apex.

For the transverse twisting problem with a roller-supported circular edge, the four boundary conditions given by (6) now become (in view of (9) and (10))

$$\rho = 1: \begin{cases} (\frac{1}{2}\rho^2 + c_1\rho^{-2})s + \chi = 0, \\ \quad (\rho - 2c_1\rho^{-3})s + \chi_{,\rho} = 0 & \text{(13a,b)} \\ c_2\rho^{-2}s + \mu^{-2}\nabla^2\chi = 0, \\ \quad -2c_2\rho^{-3}s + \mu^{-2}(\nabla^2\chi)_{,\rho} = 0\ . & \text{(13c,d)} \end{cases}$$

Rather remarkably, they are identical to the roller-supported edge conditions for the problem of membrane shearing with μ^2 replaced by $-\mu^2$.

4
The Function χ for the transverse twisting problem

For the transverse twisting problem for shells with a free circular hole, we eliminate c_2 from the edge conditions (11a,b) and c_1 from the edge conditions (11c,d) to obtain

$$\rho = 1: \begin{cases} (\nabla^2\chi)_{,\rho} + 2\nabla^2\chi = 0 & (14a) \\ 2\nabla^2\chi + (1 - \nu_M)[\chi_{,\rho} + 2\chi + 2s] = 0 & (14b) \end{cases}$$

The differential equation (3b), the two (combined) edge conditions (14a,b), and the two limiting conditions

$$\rho \to \infty \ : \quad \chi \to 0, \quad \nabla^2\chi \to 0 \qquad (15a,b)$$

which follow from (7a,b), completely determine χ for this problem as a function of ρ and θ. Note that χ also depends on the two parameters μ^4 and ν_M which appear in (3b) and (14b), respectively. We denote this function by $\chi^{\mathrm{TH}}(\rho, \theta; \nu_M, \mu^4)$ with T and H indicating the Problem of transverse twisting (T) in the presence of a free circular hole (H). The remaining two constants c_2 and c_1 can then be obtained from (11a) and (11c), respectively.

Similarly, for the transverse twisting of shells with a rigid insert, we eliminate c_1 from (12a,b) and c_2 from (12c,d) to obtain

$$\rho = 1: \begin{cases} \chi_{,\rho} + 2\chi + 2s = 0 & (16a) \\ \mu^4[\chi_{,\rho} - 2\chi] + 2(1 + \nu_N)[(\nabla^2\chi)_{,\rho} + 2\nabla^2\chi] = 0 & (16b) \end{cases}$$

The PDE (3b), the two edge conditions (16a,b) and the limiting conditions (15a,b) at infinity, determine χ as a function of ρ and θ with μ^4 and ν_N as two parameters so that $\chi = \chi^{\mathrm{TI}}(\rho, \theta; \nu_N, \mu^4)$.

Finally, upon eliminating c_1 and c_2, we obtain from (13a–d) the following two edge conditions on χ for an edge with a roller support:

$$\rho = 1: \begin{cases} \chi_{,\rho} + 2\chi + 2s = 0, \\ (\nabla^2\chi)_{,\rho} + 2\nabla^2\chi = 0 \ . \end{cases} \qquad (17a,b)$$

The differential equation (3b), the two edge condition (17a,b) and the two limiting conditions (15a,b) do not contain ν_N or ν_M explictly. They determine the function $\chi = \chi^R(\rho, \theta; \mu^4)$ for shells with a roller support loaded by transverse twisting at infinity.

5
The membrane shear problem and static-geometric duality

The relevant sets of boundary conditions for the membrane shear problem follow immediately from static-geometric duality inherent in the structure of the various boundary value problems. In particular, the relevant set of boundary conditions for the membrane shear problem for a shell with a circular hole and one with a circular rigid insert are the static geometric duals of the set for the transverse twisting problem for a shell with a circular rigid insert and one with a circular stress free hole, respectively. The two problems with roller support are identical to each other since ν_N and ν_M do not appear and μ^2 appears only in the form of μ^4. Hence the three sets of boundary conditions for the membrane shear problem will not be listed here.

Instead, we note only that, for the case of a circular rigid insert, the relevant boundary conditions at $\rho = 1$ for the determination of the corresponding χ are

$$\rho = 1: \begin{cases} (\nabla^2\chi)_{,\rho} + 2\nabla^2\chi = 0, \\ 2\nabla^2\chi + (1 + \nu_N)[\chi_{,\rho} + 2\chi + 2s] = 0 \end{cases} \qquad (18a,b)$$

(while the *DE* (3b) and far field conditions (15a,b) remain applicable). The BVP for χ for this case is identical to the one for χ^{TH} except for ν_M in the boundary conditions (14a,b) replaced by $-\nu_N$ in (18a,b). Denoting the present χ function by χ^{SI} (with S and I indicating membrane shear and rigid insert respectively), it follows from the static-geometric duality of the two BVP that we have

$$\chi^{\mathrm{SI}}(\rho, \theta; \nu_N, \mu^4) = \chi^{\mathrm{TH}}(\rho, \theta; -\nu_N, \mu^4) \ . \qquad (18c)$$

Similarly, the relevant boundary conditions for χ for the membrane shear problem at the edge of a circular hole are

$$\rho = 1: \begin{cases} \chi_{,\rho} + 2\chi + 2s = 0 & (19a) \\ \mu^4[\chi_{,\rho} - 2\chi] + 2(1 - \nu_M)[(\nabla^2\chi)_{,\rho} + 2\nabla^2\chi] = 0 & (19b) \end{cases}$$

Comparing (19a,b) with (16a,b) leads to

$$\chi^{\mathrm{SH}}(\rho, \theta; \nu_M, \mu^4) = \chi^{\mathrm{TI}}(\rho, \theta; -\nu_M, \mu^4) \ . \qquad (19c)$$

Regarding the problem of membrane shear with a roller supported edge, the boundary conditions (17a,b) (as well as the far field conditions (15a,b)) remain applicable. Since effective Poissons ratios, ν_M and ν_N, do not appear in this BVP problem (or in the corresponding problem for transverse twisting), the solution for χ in this case is also given by χ^R so that

$$\chi^{\mathrm{SR}}(\rho, \theta; \mu^4) = \chi^{\mathrm{TR}}(\rho, \theta; \mu^4) = \chi^R(\rho, \theta; \mu^4) \ . \qquad (20)$$

6
Stress concentration factors

The transverse twisting and membrane shear problems as formulated in the preceding sections are of interest for the stress concentration around the edge of the hole or insert. For each problem, we examine the relevant stress concentration factor, the ratio of the magnitude of the dominant stress measure at the edge $r = a$ and its far field magnitude. For all three transverse twisting problems, the relevant far field stress magnitude is $M_\infty^T = T/2$ (with $M_{\theta\theta}(\infty, \theta) = -M_{rr}(\infty, \theta) = T \sin 2\theta/2$). For all three membrane shear problems, the relevant far field stress magnitude is $N_\infty^s = S$ (with $N_{\theta\theta}(\infty, \theta) = -N_{rr}(\infty, \theta) = -S \sin 2\theta$). We examine separately the stress concentration of direct and bending stresses for each of the six BVP. The six pairs of stress concentration factors for the six BVP are given by

$$k_b^{\mathrm{TH}} = \frac{M_\theta^{\mathrm{TH}}(1)}{M_\infty^T}, \quad k_m^{\mathrm{TH}} = \frac{h}{6}\frac{N_\theta^{\mathrm{TH}}(1)}{M_\infty^T} \qquad (21a,b)$$

$$k_b^{\mathrm{TI}} = \frac{M_r^{\mathrm{TI}}(1)}{M_\infty^T}, \quad k_m^{\mathrm{TI}} = \frac{h}{6}\frac{N_r^{\mathrm{TI}}(1)}{M_\infty^T} \qquad (21c,d)$$

$$k_b^{\mathrm{SH}} = \frac{6}{h}\frac{M_\theta^{\mathrm{SH}}(1)}{N_\infty^S}, \quad k_m^{\mathrm{SH}} = \frac{N_\theta^{\mathrm{SH}}(1)}{N_\infty^S} \tag{22a,b}$$

$$k_b^{\mathrm{SI}} = \frac{6}{h}\frac{M_r^{\mathrm{SI}}(1)}{N_\infty^S}, \quad k_m^{\mathrm{SI}} = \frac{N_r^{\mathrm{SI}}(1)}{N_\infty^S} \tag{22c,d}$$

$$k_b^{\mathrm{TR}} = \frac{M_r^{\mathrm{TR}}(1)}{M_\infty^T}, \quad k_m^{\mathrm{TR}} = \frac{h}{6}\frac{N_\theta^{\mathrm{TR}}(1)}{M_\infty^T} \tag{23a,b}$$

$$k_b^{\mathrm{SR}} = \frac{6}{h}\frac{M_r^{\mathrm{SR}}(1)}{N_\infty^S}, \quad k_m^{\mathrm{SR}} = \frac{N_\theta^{\mathrm{SR}}(1)}{N_\infty^S} \tag{23c,d}$$

where the quantity $C_z^{xy}(1)$ is $C_{zz}^{xy}(1,\theta)$ for $\theta = \pi/4$ or $-\pi/4$ with the sign chosen so that the stress concentration factors are all positive. (The superscripts TH denote the transverse twisting problem (T) for shells with a free circular hole (H), etc.)

As they are defined, the stress concentration factors involve different combinations of the three functions ϕ, ψ and χ for the different physical problems. Rather remarkably, all these different combinations can be rearranged so that they are proportional to the appropriate χ function or its Laplacian $\nabla^2\chi$! Since the determination of χ in all cases has been shown in the last two sections to be uncoupled from the constants c_1 and c_2, the stress concentration factors can therefore be obtained from the solution of the boundary value problem for χ alone without any reference to that portion of the solution of the BVP involving the potential functions ϕ and ψ ! Furthermore, given the static-geometric dualities among the BVP observed in section (5), only three of the six χ functions need to be found to obtain the stress concentration factors for all six problems.

To express the stress concentration factors in terms of the χ function or its Laplacian, we make use of the relations

$$M_{rr} + M_{\theta\theta} = -\frac{D}{a^2}(1+\nu_M)\nabla^2 w, \qquad N_{rr} + N_{\theta\theta} = \frac{1}{a^2}\nabla^2 F \tag{24a,b}$$

The critical step in the transformation of the expressions for the stress concentration factors is the use of the known boundary conditions at the edge $r = a$. In particular, along a free edge, we have

$$M_{\theta\theta}(1) = M_{\theta\theta}(1,\pm\pi/4) + M_{rr}(1,\pm\pi/4) = -\frac{D}{a^2}(1+\nu_M)\nabla^2 w \tag{25a}$$

$$N_{\theta\theta}(1) = N_{\theta\theta}(1,\pm\pi/4) + N_{rr}(1,\pm\pi/4) = \frac{1}{a^2}\nabla^2 F \tag{25b}$$

where we have made immediate use of the relations (24a,b). It is understood that the right hand sides of (25) are evaluated at $\rho = 1$ and $\theta = \pm\pi/4$. Similarly, we have along the edge of a rigid insert

$$M_{rr}(1) = M_{rr}(1,\pm/4) = -\frac{D}{a^2}\nabla^2 w \tag{26a}$$

$$N_{rr}(1) = N_{rr}(1,\pm\pi/4) = \frac{1}{a^2(1+\nu_N)}\nabla^2 F \tag{26b}$$

where we have made use of (5a,b) in (26a) and (5c) in (26b). The usual edge evaluation applies to the right hand sides of (26). For a roller support, we will make use of (24b) and (25a) but no new relations.

We now apply the relations (25) and (26) to the six pairs stress concentration factors in (21)–(23) to express them in terms of χ or its Laplacian. To illustrate, we use (25a) to write $M_\theta^{\mathrm{TH}}(1)$ as

$$M_\theta^{\mathrm{TH}}(1) = -\frac{D}{a^2}(1+\nu_M)\nabla^2 w = -\frac{T}{2}\frac{1+\nu_m}{1-\nu_M}\nabla^2\chi^{\mathrm{TH}}$$

so that

$$k_b^{\mathrm{TH}} = \frac{M_\theta^{\mathrm{TH}}(1)}{M_\infty^T} = -\frac{1+\nu_M}{1-\nu_M}\nabla^2\chi^{\mathrm{TH}} \tag{27a}$$

It is understood that $\nabla^2\chi^{\mathrm{TH}}$ is evaluated at $\rho = 1$ and $\theta = \pm\pi/4$. Similarly, we use (25b) to write $N_\theta^{\mathrm{TH}}(1)$ as

$$N_\theta^{\mathrm{TH}}(1) = \frac{T}{2Da^2(1-\nu_M)}\nabla^2 F = -\frac{TR\mu^4}{2a^2(1-\nu_M)}\chi^{\mathrm{TH}}$$

so that

$$k_m^{\mathrm{TH}} = \frac{h}{6}\frac{N_\theta^{\mathrm{TH}}(1)}{M_\infty^T} = \frac{\mu^2}{\sqrt{3}}\sqrt{\frac{1+\nu_M}{1-\nu_M}}\chi^{\mathrm{TH}} \tag{27b}$$

where χ^{TH} is evaluated appropriately at the edge $\rho = 1$. The simplified expressions for the remaining five pairs of factors given below can be obtained in a similar way (omitting all negative signs with the understanding that all stress concentration factor are taken to be positive):

$$k_b^{\mathrm{TI}} = \frac{1}{1-\nu_M}\nabla^2\chi^{\mathrm{TI}}, \qquad k_m^{\mathrm{TI}} = \frac{\mu^2}{\sqrt{3}(1+\nu_N)}\sqrt{\frac{1+\nu_M}{1-\nu_M}}\chi^{\mathrm{TI}} \tag{27c,d}$$

$$k_b^{\mathrm{SH}} = \sqrt{3}\mu^2\sqrt{\frac{1+\nu_M}{1-\nu_M}}\chi^{\mathrm{SH}}, \quad k_m^{\mathrm{SH}} = \nabla^2\chi^{\mathrm{SH}} \tag{28a,b}$$

$$k_b^{\mathrm{SI}} = \frac{\sqrt{3}\mu^2}{\sqrt{1-\nu_M^2}}\chi^{\mathrm{SI}}, \quad k_m^{\mathrm{SI}} = \frac{\nabla^2\chi^{\mathrm{SI}}}{1+\nu_N} \tag{28c,d}$$

$$k_b^{\mathrm{TR}} = \frac{1}{1-\nu_M}\nabla^2\chi^R, \quad k_m^{\mathrm{TR}} = \frac{\mu^2}{\sqrt{3}}\sqrt{\frac{1+\nu_M}{1-\nu_M}}\chi^R \tag{29a,b}$$

$$k_b^{\mathrm{SR}} = \frac{\sqrt{3}\mu^2}{\sqrt{1-\nu_M^2}}\chi^R, \quad k_m^{\mathrm{SR}} = \nabla^2\chi^R \tag{29c,d}$$

It is clear from these transformed expressions for the stress concentration factors that we do not need to solve the original physical problems to learn about their stress concentration phenomenon, only the boundary value problem for χ. In fact, because of the static-geometric duality among the BVP, we only need to solve for the χ functions in three of the six cases as we explain in the next section.

7
Relations among the stress concentration factors

The forms of the relations in (29) imply, without invoking static-geometric duality, the relations

$$k_m^{SR} = (1 - \nu_M) k_b^{TR}, \quad (1 + \nu_M) k_b^{SR} = 3 k_m^{TR} \qquad \text{(30a,b)}$$

with these becoming consistent with equations (59c,d) and (59a,b) in (Reissner & Wan, 1994) upon changing ν_N into ν_M in (54d) and ν into ν_M in (59b) (to correct previous transcription errors).

With reference to equations (27)–(28), the static-geometric duality relations established in section (5) indicate that χ^{TH} and χ^{SH} coincide with χ^{SI} and χ^{TI}, respectively, upon changing ν_M into $-\nu_N$. Therefore, we have the following four static-geometric dual relations (indicated by $\longleftrightarrow$) involving the eight stress concentration factors in (27)–(28):

$$k_m^{SH} \longleftrightarrow (1 - \nu_M) k_b^{TI} \qquad \text{(31a)}$$

$$\frac{1}{\sqrt{3}} \sqrt{\frac{1 - \nu_M}{1 + \nu_M}} k_b^{SH} \longleftrightarrow \sqrt{3}(1 + \nu_N) \sqrt{\frac{1 - \nu_M}{1 + \nu_M}} k_m^{TI} \qquad \text{(31b)}$$

$$\sqrt{\frac{1 - \nu_M^2}{3}} k_b^{SI} \longleftrightarrow \sqrt{3} \sqrt{\frac{1 - \nu_M}{1 + \nu_M}} k_m^{TH} \qquad \text{(31c)}$$

$$(1 + \nu_N) k_m^{SI} \longleftrightarrow \frac{1 - \nu_M}{1 + \nu_M} k_b^{TH} \qquad \text{(31d)}$$

While (31a) and (31d) are the same as the corresponding results in (Reissner & Wan, 1994), equations (31b) and (31c) represent a modification of the corresponding earlier results which are valid only for the special case $\nu_M = \nu_N = \nu$.

To appreciate the significance of the relations (31a–d), we restate in Tables 1 and 2 earlier results for the special case $\nu_M = \nu_N = \nu$ which had in earlier work been obtained without the concept of the static-geometric duality (see E. Reissner (1980a, 1980b, 1980c), E. Reissner and J.E. Reissner (1982), and J.E. Reissner (1981)). With (31a–d) the results of Table 2 can now be obtained from those of Table 1 without solving four new BVP (keeping in mind that θ is chosen so that the stress concentration factors are all non-negative)!

For the more general cases with $\nu_M \neq \nu_N$, it is now found that the results in Table 1 are equivalent to the more general results upon replacing ν by ν_M. With this, we can then use (31a–d) to generalize the results in Table 2 to the following Table 3. With the results in Table 3 reducing to those in Table 2 upon setting $\nu_M = \nu_N = \nu$.

Table 2

μ	k_m^{SI}	k_b^{SI}	k_m^{TI}	k_b^{TI}
0	$\frac{4}{3-\nu}$	0	0	$\frac{4}{1-\nu}$
>>1	1	$\sqrt{\frac{3+3\nu}{1-\nu}}$	$\frac{\mu^2}{2\sqrt{3-3\nu^2}}$	$\frac{\mu^2}{2(1-\nu)}$

Table 3

μ	k_m^{SI}	k_b^{SI}	k_m^{TI}	k_b^{TI}
0	$\frac{4}{3-\nu_N}$	0	0	$\frac{4}{1-\nu_M}$
>>1	1	$(1+\nu_N)\sqrt{\frac{3}{1-\nu_M^2}}$	$\frac{\mu^2}{2(1+\nu_N)}\sqrt{\frac{1+\nu_M}{3-3\nu_M}}$	$\frac{\mu^2}{2(1-\nu_M)}$

8
Exact and asymptotic solutions for problems with a roller supported edge

We now complete the solutions for our six BVP by obtaining the stress concentration factors for the two physical problems for shells with a roller-supported edge. The exact solution for the function χ^R which has the desired properties at infinity (15a,b) is given by

$$\chi^R = [c_r \operatorname{ker}_2(\mu\rho) + c_i \operatorname{kei}_2(\mu\rho)] \sin 2\theta \equiv g(\mu\rho) \sin 2\theta \qquad (32)$$

The two unknown constants c_r and c_i are determined by the two edge conditions (17a,b) to be

$$c_r = -\frac{2}{\Delta}[\mu \operatorname{ker}_2'(\mu) + 2\operatorname{ker}_2(\mu)], \quad c_i = -\frac{2}{\Delta}[\mu \operatorname{ker}_2'(\mu) + 2\operatorname{kei}_2(\mu)] \qquad \text{(33a,b)}$$

with $f'(x) = \mathrm{d}[f(x)]/\mathrm{d}x$ and

$$\Delta = [\mu \operatorname{ker}_2'(\mu) + 2\operatorname{ker}_2(\mu)]^2 + [\mu \operatorname{kei}_2'(\mu) + 2\operatorname{kei}_2(\mu)]^2 \qquad \text{(33c)}$$

The maximum value of χ^R along the edge is given by

$$g(\mu) = -\frac{2}{\Delta}\{\mu[\operatorname{ker}_2'(\mu)\operatorname{ker}_2(\mu) + \operatorname{kei}_2'(\mu)\operatorname{kei}_2(\mu)] + 2[\operatorname{ker}_2^2(\mu) + \operatorname{kei}_2^2(\mu)]\} \qquad \text{(33d)}$$

with a corresponding expression for the maximum value of $\nabla^2\chi^R$. From these expressions, we obtain an exact solution for the two stress concentration factors k_m^{SR} and k_b^{SR} which will not be listed here. Instead, we give below the corresponding three term asymptotic expansions for these factors obtained with the help of the various asymptotic expansions for Kelvin functions (M. Abramowitz & I. Stegun, 1964);

$$k_m^{SR} \sim \sqrt{2}\mu\left[1 + \frac{3}{\sqrt{2}\mu} + \frac{3}{8\mu^2} + 0\left(\frac{1}{\mu^3}\right)\right] \qquad \text{(34a)}$$

$$k_b^{SR} \sim \sqrt{\frac{6}{1 - \nu_M^2}}\mu\left[1 - \frac{3}{8\mu^2} + 0\left(\frac{1}{\mu^3}\right)\right] \quad . \qquad \text{(34b)}$$

Plots of the one-, two- and three-term expansions of the two stress concentration factors as functions of the large parameter μ are given along with the corresponding exact solutions in Figures (1) and (2) as well as in Table 4. The three term asymptotic solutions are seen to be accurate to within 3% of the exact solutions for $\mu \geq 2$ and still within

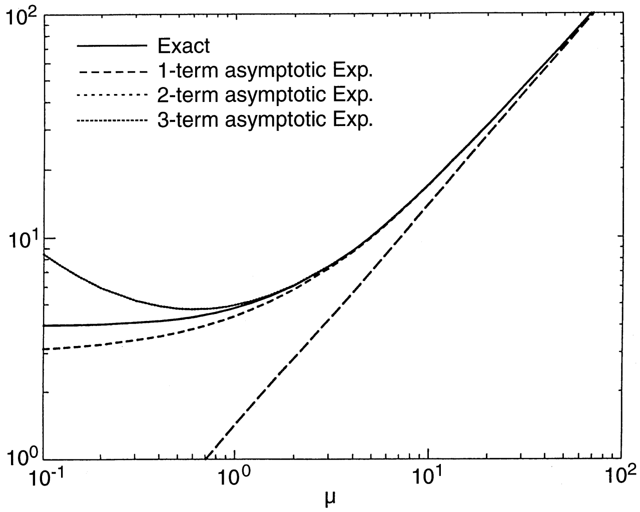

Fig. 1. One-, two- and three-term asymptotic expansion (see equation (34a)) as well as the exact value of k_m^{SR} for large μ

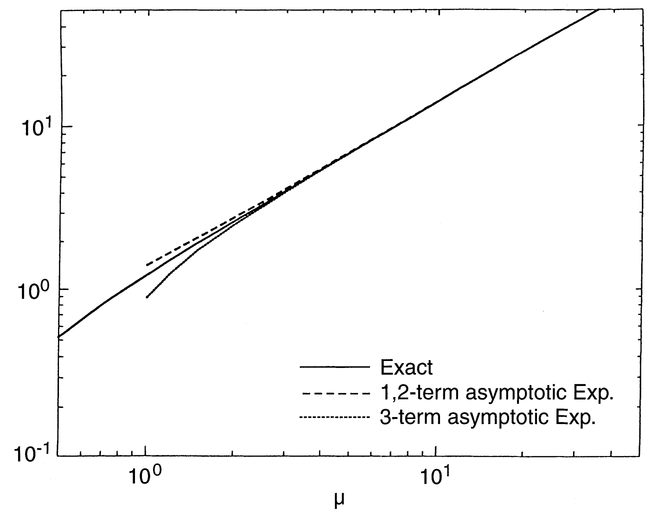

Fig. 2. One- (=two-) and three-term asymptotic expansion (see equation (34b)) as well as the exact value of $k_b^{SR}[(1-\nu)/3]$ for large μ

Table 4

μ	s_m^{SR}/μ		$\sqrt{\frac{1-\nu_M^2}{3}}k_b^{SR}/\mu$	
	Exact	Asymptotic (34a)	Exact	Asymptotic (34b)
1.0	4.800471	4.944544	1.231600	0.883884
2.0	3.029980	3.046796	1.342096	1.281631
5.0	2.034709	2.035427	1.397972	1.393000
7.5	1.823479	1.823642	1.406352	1.404786
10.0	1.719462	1.719517	1.409593	1.408910
20.0	1.565536	1.565539	1.412977	1.412888
50.0	1.474426	1.474426	1.414007	1.414001
75.0	1.454308	1.454308	1.414121	1.414119
100.0	1.444267	1.444267	1.414161	1.414161

that range of accuracy for $\mu \geq 1$ in the case of k_m^{SR}. It is also evident from (34a) and (34b) that the magnitude of the stress concentration for a thin shell is intermediate (of order μ) to the two extreme case of a rigid insert where both factors are $0(1)$ and the stress free hole case where the both factors are $0(\mu^2)$.

By (30a,b), the corresponding asymptotic behaviour of the stress concentration factors for the transverse twisting problem are given by

$$k_b^{TR} \sim \frac{\sqrt{2}\mu}{1-\nu_M}\left[1+\frac{3}{\sqrt{2}\mu}+\frac{3}{9\mu^2}+0\left(\frac{1}{\mu^3}\right)\right] \qquad (35a)$$

$$k_b^{TR} \sim \sqrt{\frac{2+2\nu_M}{3-3\nu_M}}\mu\left[1-\frac{3}{8\mu^2}+0\left(\frac{1}{\mu^3}\right)\right] \qquad (35b)$$

It is evident that the concentration factors are again intermediate in order of magnitude to the two extreme case of a rigid insert and a stress free hole.

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