The Axisymmetric Deformation of a Thin, or Moderately Thick, Elastic Spherical Cap

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A refined shell theory is developed for the elastostatics of a moderately thick spherical cap in axisymmetric deformation. This is a *two-term* asymptotic theory, valid as the dimensionless shell thickness tends to zero. The theory is more accurate than "thin shell" theory, but is still much more tractable than the full three-dimensional theory. A fundamental difficulty encountered in the formulation of shell (and plate) theories is the determination of correct two-dimensional boundary conditions, applicable to the shell solution, from edge data prescribed for the three-dimensional problem. A major contribution of this article is the derivation of such boundary conditions for our refined theory of the spherical cap. These conditions are more difficult to obtain than those already known for the semi-infinite cylindrical shell, since they depend on the cap angle as well as the dimensionless thickness. For the stress boundary value problem, we find that a Saint-Venant-type principle does not apply in the refined theory, although it does hold in thin shell theory. We also obtain correct boundary conditions for pure displacement and mixed boundary data. In these cases, conventional formulations do not generally provide even the first approximation solution correctly.

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As an illustration of the refined theory, we obtain two-term asymptotic solutions to two problems, (i) a complete spherical shell subjected to a normally directed equatorial line loading and (ii) an unloaded spherical cap rotating about its axis of symmetry.

1. Introduction

The elastostatic theory of thin shells is a two-dimensional system of differential equations and boundary conditions that determines a first approximation to the behavior of three-dimensional shell structures subject only to distributed edge loads. Thin shell theory is essentially identical to the leading term (as $\epsilon \to 0$) of the interior (outer) asymptotic solution of the corresponding three-dimensional problem; here $\epsilon = h/R$, where $2h$ is the thickness and R the midsurface radius of the shell. (See [1] for a fuller explanation and original references.) When applied to the axisymmetric deformation of a spherical shell, thin shell theory yields particularly simple and elegant results for the stress boundary value problem but does not provide an estimate of the error incurred by neglecting higher-order contributions. One reason for developing a (properly formulated) refined theory in this article is to improve on the accuracy of thin shell theory away from the shell edges. This gives an estimate of the errors involved in the thin shell approximation and provides a theory of thicker shells. A more important purpose of our work, however, is to remedy a serious deficiency of the conventional shell theory for thin or thick shells. These conventional formulations generally do not even provide a *first* approximation solution for boundary value problems involving displacement edge data. This has previously been shown to be the case for flat plates and circular cylindrical shells [1–5].

In the present article we develop a two-term asymptotic theory for a spherical cap in axisymmetric deformation.[1] We begin with the eigenvalues and eigenfunctions of the exact three-dimensional theory, first obtained by Lur'e [6]. The determinantal equation satisfied by these eigenvalues and the radial variation of the eigenfunctions are complicated but explicit combinations of elementary functions. The angular variation of the eigenfunctions involves Legendre functions of complex order. Of the eigenvalues lying in the positive quadrant, just one (the shell eigenvalue) varies like $\epsilon^{-1/2}$ as $\epsilon \to 0$, while all the others vary like ϵ^{-1}. These latter eigenvalues are associated with a boundary layer of thickness $O(h)$ adjacent to the edge of the cap, and the corresponding eigenfunctions are related to the Papkovich–Fadle (PF) eigenfunctions for a semi-infinite strip in plane strain.

[1]See also [7] for further details.

On the other hand, the shell eigenvalue is associated with a boundary layer of thickness $O(\sqrt{hR})$, which is large compared to $O(h)$ when ϵ is small. The leading terms (as $\epsilon \to 0$) of the shell eigenvalue and eigenfunction coincide, as expected, with those predicted by thin shell theory. To construct our refined theory, we determine the shell eigenvalue, and eigenfunction, correct to two terms. The expressions involved are obviously more complicated than those of the (one term) thin shell theory, but still have a vastly simpler form than their exact counterparts. For even an approximate theory to be *uniformly* valid throughout the shell, the PF-eigenfunctions must be included. However, their influence is significant only in a narrow boundary layer of thickness $O(h)$ adjacent to the edge of the cap. Outside this boundary layer, they can be disregarded, leaving only the shell eigenfunction and its conjugate.[2] Even though the shell eigenfunction is itself significant only in its own (thicker) boundary layer, we often refer to it as the "interior" solution.

A fundamental difficulty encountered in the formulation of any shell or plate theory lies in the determination of correct two-dimensional boundary conditions, applicable to the interior shell solution, from edge data prescribed for the three-dimensional problem. A major contribution of this article is the derivation of such boundary conditions for our refined theory of the spherical cap. These provide the appropriate modification to the corresponding boundary conditions of thin shell theory. It is achieved in Section 3 by a method, based on the elastic reciprocal theorem, which has previously been used in thick plate theory (see [2–5] and [8]) and in the axisymmetric deformation of a moderately thick *cylindrical* shell [1]. This method requires the construction of certain auxiliary elastic states in the shell that satisfy appropriate homogeneous boundary conditions at the edge. In the present case, for pure stress or mixed data at the cap edge, these auxiliary states are constructed correct to relative order $O(\epsilon)$. The elastic reciprocal theorem is then used to deduce $O(\epsilon)$-corrected boundary conditions that must be satisfied by the *interior* solution. These refined boundary conditions for the spherical cap are more difficult to obtain than those found [1] for the semi-infinite cylindrical shell, since they depend on the cap angle as well as the dimensionless thickness. For the case of pure displacement data we obtain only leading-order boundary conditions; however, even these have not previously been formulated correctly in the thin shell literature.

When pure stress data are prescribed, the boundary conditions take a particularly simple and elegant form. To leading order (as $\epsilon \to 0$), they are equivalent to the requirement that the interior solution have the same stress and couple resultants (integrated across the shell thickness) as those of the data; these are the conditions usually assumed in thin shell theory. However,

[2]This applies equally to thin shell theory, which is not valid inside the PF boundary layer.

the $O(\epsilon)$-correction terms show that one cannot simply equate these stress and couple resultants in the refined theory since the pointwise distribution of the stress data across the shell thickness now matters. Thus an appeal to Saint Venant's principle to determine the interior solution is not only unjustified, but is actually wrong in the $O(\epsilon)$-corrected theory.

As an illustration of our refined theory, we obtain two-term asymptotic solutions to two problems. These are (i) a complete spherical shell subjected to a normally directed equatorial line loading and (ii) an unloaded spherical cap rotating about its axis of symmetry. In the course of solving the second example, we expand (in powers of ϵ) the solution due to Goldberg et al. [9] for the *complete* spinning spherical shell (see Appendix D). This expansion, which is new to the literature, is more useful than the exact solution to this classical problem when the spherical shell is thin, or moderately thick.

2. The exact eigenfunctions and their asymptotic approximations

Consider the spherical cap R_α: $a \le r \le b$, $0 \le \phi \le \alpha$, $0 \le \theta < 2\pi$, where r, ϕ, θ are a set of spherical polar coordinates and $\alpha (0 < \alpha < \pi)$ is a constant. The cap is composed of homogeneous, isotropic, elastic material with Young's modulus E and Poisson's ratio ν. The small displacement linear theory of elasticity is assumed throughout. Body forces are absent and the spherical surfaces of the cap (on $r = a$, b) are traction free.[3] The cap is "loaded" by prescribing boundary data on its edge $\mathscr{E}_\alpha$ lying in $\phi = \alpha$, which generates an axisymmetric[4] deformation of the cap. The boundary data may actually be prescribed tractions, but displacements, or mixed data, may be prescribed instead. In all cases, the problem is to determine the resulting elastic field $\{\mathbf{u}(r,\phi), \boldsymbol{\tau}(r,\phi)\}$ in the cap.

Exact eigenfunctions

Following Lur'e [6] and Vilenskaia and Vorovich [10], we first determine the eigenfunctions for the spherical cap. These are elastic states regular in R_α that satisfy traction-free conditions on the surfaces $r = a, b$.

Let the Cartesian axis Oz point along $\phi = 0$, the axis of symmetry of the cap. Then (see [11]) any axisymmetric elastic field can be expressed in the form

$$\mathbf{u} = \operatorname{grad}(\Phi + z\Psi) - 4(1-\nu)\Psi\mathbf{k}, \tag{2.1}$$

[3]Other loadings can be reduced to this case. See Section 4 for an example with a nonzero body force.

[4]In this article, "axisymmetric" always refers to torsionless axisymmetry.

where Φ, Ψ are axisymmetric scalar potentials satisfying

$$\nabla^2\Phi = \nabla^2\Psi = 0, \tag{2.2}$$

and $\mathbf{k}$ is the unit vector in the direction Oz. Consider the potential pairs

$$\begin{aligned} \Phi &= r^{\eta}P_{\eta}(\cos\phi) \\ \Psi &= 0 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \Phi &= -(\eta-2+4\nu)r^{\eta+2}P_{\eta}(\cos\phi) \\ \Psi &= (2\eta+3)r^{\eta+1}P_{\eta+1}(\cos\phi), \end{aligned} \tag{2.4}$$

where η is any complex constant and $P_{\eta}(z)$ is the Legendre function of degree η satisfying $P_{\eta}(1)=1$. The axisymmetric elastic fields derived from (2.3) or (2.4) (via (2.1)) are regular in the whole of space except along the *negative* z-axis; in particular, they are regular in the cap R_{α} for any $\alpha(0<\alpha<\pi)$. It may also be verified that the stresses τ_{rr} and $\tau_{r\phi}$ derived from either (2.3) or (2.4) vary with ϕ as $P_{\eta}(\cos\phi)$ and $(\sin\phi)P'_{\eta}(\cos\phi)$ respectively. Similar remarks apply to the two further elastic fields obtained by replacing η by $-\eta-1$ in (2.3), (2.4). We now form a general linear combination of these four elastic fields with coefficients A, B, C, D. Because of the common angular dependence of the constituent terms, it is possible to construct nontrivial elastic states that satisfy[5]

$$\tau_{rr} = \tau_{r\phi} = 0 \tag{2.5}$$

on $r=a,b,\ 0\le\phi<\pi,\ 0\le\theta<2\pi$ provided that A, B, C, D satisfy a certain 4×4 system of *homogeneous* linear equations; the matrix elements are functions of η, a, b, ν. For a nontrivial solution, the determinant of this matrix must vanish, which leads to the eigenvalue equation

$$\left(\frac{\sinh(\gamma\beta)}{\beta\sinh\gamma}\right)^2 = \frac{\beta^4-\frac{5}{2}\beta^2+\frac{73}{16}-4\nu^2}{\beta^4+\beta^2\left[4(1-\nu^2)-\frac{5}{2}\right]+\frac{9}{16}}, \tag{2.6}$$

where

$$\beta = \eta+\tfrac{1}{2}, \tag{2.7}$$

$$\gamma = \ln(b/a). \tag{2.8}$$

[5]The assumed axisymmetry implies that $\tau_{r\theta}\equiv 0$.

With a, b, ν fixed, each eigenvalue β generates an eigenfunction,[6] that is, a nontrivial elastic state satisfying the boundary conditions (2.5). The analytic forms of the components of displacement and stress for a typical eigenfunction are given by Lur'e [6]. They have a complicated dependence on the parameters β, a, b, ν. Levine and Klosner [12, p. 199] state that Lur'e's formulae contain errors that are corrected by them in [13], a source not readily accessible. However, in this article, we use only asymptotic expansions of the eigenfunction field components, and we obtain these directly from the relatively simple 4×4 linear system. Our results therefore do not depend on the correctness (or otherwise) of the formulae given by either of the above mentioned sources.

Asymptotic approximations

Let the midradius R and dimensionless thickness ϵ of the cap be defined by

$$R = \frac{1}{2}(a+b), \tag{2.9}$$

$$\epsilon = \frac{b-a}{b+a} = \frac{h}{R}, \tag{2.10}$$

where $2h$ is the thickness of the cap. The distribution of the eigenvalues of (2.6) as $\epsilon \to 0$ is described in [10]. Disregarding the roots $\beta = \pm 1$, all the eigenvalues are complex and appear in symmetrical sets of four; we may therefore concentrate on those in the first quadrant. There is a countable infinity of eigenvalues of the form

$$\beta_n = \frac{w_n}{\epsilon}\left(1 + O(\epsilon^2)\right) \tag{2.11}$$

as $\epsilon \to 0$, where $\{w_n\}$ are the nonzero roots of

$$\sinh^2 2w + (2w)^2 = 0 \tag{2.12}$$

lying in the first quadrant. We call these the *PF-eigenvalues* because of their relationship to the Papkovich–Fadle eigenvalues for the elastic strip. On applying the principle of the argument to (2.6), we find that there is exactly one eigenvalue lying in the first quadrant unaccounted for. This is the *shell eigenvalue*, which has the form

[6]The real roots $\beta = \pm 1$ are exceptional in that they have no corresponding nontrivial eigenfunctions (see [10, p. 347]). We therefore do not regard the roots $\beta = \pm 1$ as eigenvalues.

$$\beta = \frac{(1+i)\,\omega}{\epsilon^{1/2}}\left[1 - i\left(\frac{1+24\nu^2}{80\,\omega^2}\right)\epsilon + O(\epsilon^2)\right] \tag{2.13}$$

as $\epsilon \to 0$, where $\omega(\nu)$ is the "shell constant" defined by

$$\omega = \left(\frac{3}{4}(1-\nu^2)\right)^{1/4}. \tag{2.14}$$

The leading term in (2.13) is the shell eigenvalue predicted by *thin* shell theory. In our refined theory, the full expression must be used.

Corresponding to each of these eigenvalues is an eigenfunction, unique to within normalization. For reasons of accuracy, we obtained the asymptotic forms of these eigenfunctions directly from the relatively simple 4×4 linear system for A, B, C, D; Mathematica [14] was used to perform these computations, correct to *two* terms as $\epsilon \to 0$. The method was similar to that used by Gregory and Wan [1] for the cylindrical shell. The resulting field components for the shell eigenfunction are given in Appendix A. The leading terms in these expressions are, as expected, the values predicted by thin shell theory. The correction terms, which are new to the literature, are an essential part of our refined theory. These fields are renormalized in Appendix B to display more conveniently their values on the edge $\phi = \alpha$.

We also computed two term approximations[7] (as $\epsilon \to 0$) for the PF-eigenfunctions [7, Appendix B]. We do not reproduce them here since they do not appear explicitly in our refined *shell* theory. However, we did use them (see [7]) to check the conditions for rapid decay derived in Section 3.

3. Conditions for rapid decay

Once the eigenvalues β and eigenfunctions $\{\mathbf{u}^\beta, \boldsymbol{\tau}^\beta\}$ are known, one may, in principle, construct the elastic fields generated by the prescribed edge data in the form

$$\{\mathbf{u}, \boldsymbol{\tau}\} = \sum_{\beta \in B} c_\beta \{\mathbf{u}^\beta, \boldsymbol{\tau}^\beta\}, \tag{3.1}$$

where B is the set of eigenvalues with $\Re(\beta) > 0$ (β and $-\beta$ yield the same eigenfunction). The expression (3.1) satisfies the governing equations in R_α and traction-free conditions on $r = a, b$. It remains to choose the coefficients $\{c_\beta\}$ so as to satisfy[8] the prescribed data on the edge $\mathscr{E}_\alpha$.

[7]Although the PF-eigenvalues (2.11) have no correction term of relative order $O(\epsilon)$ as $\epsilon \to 0$, their corresponding eigenfunctions do.

[8]We assume that this set of eigenfunctions is *complete* for the expansion of standard boundary data on $\mathscr{E}_\alpha$. This has never been proved, but it is encouraging to note that the leading terms of $\{\mathbf{u}^\beta, \boldsymbol{\tau}^\beta\}$ (as $\epsilon \to 0$) are known to be complete for the expansion of traction or mixed data on $\mathscr{E}_\alpha$ (see [15, 16]).

Levine and Klosner [12] have used this expansion to obtain numerical solutions to three boundary value problems for a spherical cap. They satisfied the edge conditions approximately by overdetermined boundary collocation. The object of the present work, however, is to solve such problems analytically in the context of a *refined shell theory*. This theory disregards the contribution of the PF-eigenfunctions to (3.1) and approximates the shell eigenfunction by its two-term asymptotic approximation (as $\epsilon \to 0$). A fundamental problem encountered in this approach is to establish the connection between the prescribed data on $\mathscr{E}_\alpha$ and the values of the corresponding field components of the refined shell solution at the boundary. For, on $\mathscr{E}_\alpha$, the PF-eigenfunctions are *not* negligible, even when the cap is thin. This problem is equivalent to finding the conditions that the data (prescribed on $\mathscr{E}_\alpha$) should satisfy in order to generate an elastic field in the cap that *decays rapidly*[9] with increasing distance from the edge. Gregory and Wan [1–5] and Lin and Wan [8] have developed methods, for finding conditions on the prescribed data that are necessary for this rapid decay. In particular, in [1], Gregory and Wan developed a refined theory for the axisymmetric deformation of a *cylindrical* shell based on elastic reciprocity.

Case A: Traction data $\tau_{\phi\phi}(r,\alpha)=\bar{\tau}_{\phi\phi}(r)$, $\tau_{r\phi}(r,\alpha)=\bar{\tau}_{r\phi}(r)$. Suppose that the prescribed data on $\mathscr{E}_\alpha$ are

$$\tau_{\phi\phi}(r,\alpha) = \bar{\tau}_{\phi\phi}(r), \tag{3.2}$$

$$\tau_{r\phi}(r,\alpha) = \bar{\tau}_{r\phi}(r). \tag{3.3}$$

It is tempting to suppose that, for these data to generate a rapidly decaying state, their *stress resultants* and *stress couple* must be zero. The analysis below proves that such an appeal to the two-dimensional analogue of "Saint-Venant's principle" is not only unjustified, but actually incorrect in the refined theory. However, just as in the case of cylindrical shells, our results do confirm that these conditions are correct in classical thin shell theory.

Since the cap is closed at the pole, the data must satisfy

$$\int_a^b r\left[\cos\alpha\,\bar{\tau}_{r\phi}(r) - \sin\alpha\,\bar{\tau}_{\phi\phi}(r)\right] dr = 0 \tag{3.4}$$

[9]That is, its expansion series (3.1) consists *only* of PF-eigenfunctions.

in order that the whole cap be in equilibrium. In terms of the dimensionless radial coordinate, ξ, defined by

$$r = R(1+\xi\epsilon), \tag{3.5}$$

(3.4) becomes

$$\int_{-1}^{1}(1+\xi\epsilon)\left[\cos\alpha\,\bar{\tau}_{r\phi}(\xi)-\sin\alpha\,\bar{\tau}_{\phi\phi}(\xi)\right]d\xi = 0. \tag{3.6}$$

(Our theory also applies to the spherical segment $a \le r \le b$, $0 < \alpha_1 \le \phi \le \alpha_2 < \pi$, $0 \le \theta < 2\pi$, having two edges on which data may be prescribed. In that case (3.4) is no longer necessary for overall equilibrium but *is* a necessary condition for rapid decay. This can be shown by using the general method set out below, taking $\{\mathbf{U},\mathbf{T}\}$ to be a rigid translation in the z-direction. Indeed, we could have obtained the condition (3.4) in this way.)

By using the elastic reciprocal theorem as in [3], it follows that, to generate a rapidly decaying state, the data $\bar{\tau}_{\phi\phi}$, $\bar{\tau}_{r\phi}$ must satisfy necessary conditions of the form

$$\int_{a}^{b}\left[U_r\bar{\tau}_{r\phi}+U_\phi\bar{\tau}_{\phi\phi}\right]r\,dr = 0. \tag{3.7}$$

In (3.7), U_r, U_ϕ are the displacements (on $\phi=\alpha$) of any axisymmetric elastic field $\{\mathbf{U},\mathbf{T}\}$ that is regular in the cap (except on $\phi=0$) and satisfies

$$T_{rr} = T_{r\phi} = 0 \qquad \text{on } r = a,b, \tag{3.8}$$

$$T_{r\phi} = T_{\phi\phi} = 0 \qquad \text{on } \phi = \alpha, \tag{3.9}$$

and is *not rapidly increasing*[10] as ϕ decreases. We will now construct two suitable states $\{\mathbf{U},\mathbf{T}\}$ correct to relative order $O(\epsilon)$ as $\epsilon \to 0$. Let $\boldsymbol{\tau}^R$, $\boldsymbol{\tau}^I$ denote the real and imaginary parts of the shell eigenfunction, normalized as in Appendix B. Also appearing in our construction is the shell eigenfunction for the "complementary shell" $a \le r \le b$, $\alpha \le \phi \le \pi$, $0 \le \theta < 2\pi$, similarly normalized. This can be obtained from the expressions in Appendix B by making the substitutions $\alpha \to \pi-\alpha$, $\phi \to \pi-\phi$, and by reversing the signs of u_ϕ, $\tau_{r\phi}$. We denote the real and imaginary parts of this eigenfunction by $\boldsymbol{\tau}^{CR}, \boldsymbol{\tau}^{CI}$.

[10]That is, not increasing as rapidly as the PF-eigenfunctions regular at the opposite "pole."

Consider the combination

$$\boldsymbol{\tau}^{CR} + \boldsymbol{\tau}^{R}. \tag{3.10}$$

The first term in (3.10) *grows* exponentially as ϕ decreases, but is not "rapidly increasing." On $r = a, b$, the expression (3.10) satisfies (3.8) exactly. On the edge $\phi = \alpha$, its $r\phi$-component of stress is $O(\epsilon^2)$ and its $\phi\phi$-component is

$$-\frac{2E}{5(1-\nu^2)}\left[3\nu^2 - 5\nu - 18 + 5(1-\nu)\cot^2\alpha + 10(1-\nu^2)\xi^2\right]\xi\epsilon + O(\epsilon^2) \tag{3.11}$$

as $\epsilon \to 0$. The stress field (3.10) is thus an approximation, correct to order $O(1)$, to a possible $\mathbf{T}$ for (3.7). To obtain a better approximation consider

$$\boldsymbol{\tau}^{CR} + \boldsymbol{\tau}^{R} + a_1(\boldsymbol{\tau}^{R} - \boldsymbol{\tau}^{I})\epsilon, \tag{3.12}$$

where

$$a_1 = \frac{1}{10\omega^2}\left[3\nu^2 + 5\nu + 12 - 5(1-\nu)\cot^2\alpha\right]. \tag{3.13}$$

The $\phi\phi$-component of (3.12) on $\phi = \alpha$ is

$$\frac{4E}{5}(3 - 5\xi^2)\xi\epsilon + O(\epsilon^{3/2}) \tag{3.14}$$

while the $r\phi$-component is still $O(\epsilon^2)$.

Now imagine that

$$\tau_{\zeta\zeta} = \frac{4}{5}E(3 - 5\xi^2)\xi, \tag{3.15}$$

$$\tau_{\xi\zeta} = 0 \tag{3.16}$$

is the prescribed end-data for the plane semi-infinite elastic strip $-1 \le \xi \le 1$, $\zeta \ge 0$, whose long sides are traction free. *This end-data is self-equilibrating in* (ξ, ζ)*-space* and so (see [16]) can be expanded as a series of the Papkovich–Fadle eigenfunctions *for this strip*. Using this set of expansion coefficients, replace the strip PF-eigenfunctions by those for the spherical cap. The leading term of these eigenfunctions coincides with the strip eigenfunctions, and so the new expansion will generate a rapidly decaying state $\boldsymbol{\tau}^{PF}$ in the cap. The $\phi\phi$- and $r\phi$-components of this state on $\phi=\alpha$ are given by $\frac{4}{5}E(3-5\xi^2)\xi+O(\epsilon)$ and $O(\epsilon)$ respectively. It follows that the combination

$$\boldsymbol{\tau}^{CR}+\boldsymbol{\tau}^{R}+a_1(\boldsymbol{\tau}^{R}-\boldsymbol{\tau}^{I})\epsilon-\boldsymbol{\tau}^{PF}\epsilon \tag{3.17}$$

has $\phi\phi$- and $r\phi$-components on $\phi=\alpha$ of orders $O(\epsilon^{3/2})$ and $O(\epsilon^2)$ respectively. Thus (3.17) is a refinement of (3.10) in the sense that the conditions (3.9) are more accurately satisfied as $\epsilon\to 0$.

In principle, this process of refinement could proceed indefinitely to yield a $\mathbf{T}$ that satisfies (3.9) exactly. The corresponding (infinite) expression for $\mathbf{U}$ could then be substituted into (3.7) to yield a condition for rapid decay that was also exact. In practice, this cannot be achieved, since (i) the asymptotic expansions of $\boldsymbol{\tau}^R$ and $\boldsymbol{\tau}^I$ would have to be found to an arbitrary number of terms and (ii) the stress field, $\boldsymbol{\tau}^{PF}$, and its associated displacement field, $\mathbf{u}^{PF}$, are not actually known.[11] However, it follows from (3.17) that, on $\phi=\alpha$,

$$R^{-1}U_r = 4+2(a_1-2\nu\xi)\epsilon+a_2\epsilon^{3/2}+O(\epsilon^2), \tag{3.18}$$

$$R^{-1}U_\phi = 2\cot\alpha\,\xi\epsilon-4\omega a_1\,\xi\epsilon^{3/2}+O(\epsilon^2) \tag{3.19}$$

as $\epsilon\to 0$, where a_2 is a constant independent of ϵ. On substituting (3.18), (3.19) into (3.7), and multiplying by $\frac{1}{4}(1-\frac{1}{2}a_1\epsilon-\frac{1}{4}a_2\epsilon^{3/2})$, we obtain

$$\int_{-1}^{1}(1+\xi\epsilon)\Big[\big(1-\nu\xi\epsilon+O(\epsilon^2)\big)\bar{\tau}_{r\phi} + \big(\tfrac{1}{2}\cot\alpha\epsilon-\omega a_1\epsilon^{3/2}+O(\epsilon^2)\big)\,\xi\,\bar{\tau}_{\phi\phi}\Big]\,d\xi=0 \tag{3.20}$$

(as $\epsilon\to 0$) as a necessary condition for rapid decay.

A second condition for rapid decay can be obtained in a similar way by starting with the combination

$$(\boldsymbol{\tau}^{CR}-\boldsymbol{\tau}^{CI})-(\boldsymbol{\tau}^{R}-\boldsymbol{\tau}^{I}). \tag{3.21}$$

[11]The determination of even the leading term of $\mathbf{u}^{PF}$ requires the solution of the boundary value problem (3.15), (3.16) for the semi-infinite strip. This solution, which is independent of ϵ, is not available in closed form. Fortunately however, since $R^{-1}\mathbf{u}^{PF}$ is of order $O(\epsilon)$ as $\epsilon\to 0$, it follows that $\mathbf{u}^{PF}$ does not appear in the expansion of $\mathbf{U}$ correct to order $O(\epsilon)$.

After mutual simplification, these two conditions become

$$\int_{-1}^{1} (1+\xi\epsilon)(1-\nu\xi\epsilon)\bar{\tau}_{r\phi}(\xi)\, d\xi = O(\epsilon^2), \tag{3.22}$$

$$\int_{-1}^{1} (1+\xi\epsilon)\left[(1-\tfrac{1}{2}\nu\xi\epsilon)\xi\bar{\tau}_{\phi\phi}(\xi) - \tfrac{1}{2}(\nu\cot\alpha)\xi^2\epsilon\bar{\tau}_{r\phi}(\xi)\right] d\xi = O(\epsilon^2) \tag{3.23}$$

as $\epsilon \to 0$. To simplify the writing, we have assumed that the data $\bar{\tau}_{\phi\phi}$, $\bar{\tau}_{r\phi}$ are of order $O(1)$ as $\epsilon \to 0$; if this is not so, then the error terms $O(\epsilon^2)$ may be modified.

The refined necessary conditions for rapid decay with Case A boundary data are (3.6), (3.22), (3.23). These conditions are similar, but not identical, to the conditions obtained by Gregory and Wan [1] for the corresponding cylindrical shell problem.

Notes: (i) The limiting forms of the conditions (3.4), (3.22), (3.23) as $\epsilon \to 0$ can be written (on restoring the radial variable r)

$$\int_a^b \bar{\tau}_{r\phi}(r)\, r\, dr = 0, \tag{3.24}$$

$$\int_a^b \bar{\tau}_{\phi\phi}(r)\, r\, dr = 0, \tag{3.25}$$

$$\int_a^b (r-R)\bar{\tau}_{\phi\phi}(r)\, r\, dr = 0, \tag{3.26}$$

which are equivalent to Saint-Venant's principle. However, Saint-Venant's principle is not equivalent to the conditions for rapid decay in the refined theory unless $\nu = 0$.

(ii) A valuable check on *any* necessary condition for rapid decay is that it must be satisfied by the edge values of the individual PF-eigenfunctions. All our conditions, including (3.22), (3.23), were checked in this way.

Case B: Mixed data $u_r(r, \alpha) = \bar{u}_r(r)$, $\tau_{\phi\phi}(r, \alpha) = \bar{\tau}_{\phi\phi}(r)$. To generate a rapidly decaying state in this case, the data $\bar{u}_r$, $\bar{\tau}_{\phi\phi}$ must satisfy necessary conditions of the form

$$\int_a^b \left[T_{r\phi}\bar{u}_r - U_\phi \bar{\tau}_{\phi\phi}\right] r\, dr = 0, \tag{3.27}$$

where {**U**,**T**} satisfies the same conditions as in Case A, except that (3.9) is replaced by

$$U_r = T_{\phi\phi} = 0 \qquad \text{on } \phi = \alpha. \tag{3.28}$$

Hemispherical cap: The situation is simplified when $\alpha = \pi/2$, that is, when the cap is hemispherical. In this case, the equilibrium condition is

$$\int_{-1}^{1} (1+\xi\epsilon)\bar{\tau}_{\phi\phi}(\xi)\,d\xi = 0. \tag{3.29}$$

Two more necessary conditions for rapid decay can be found by following the same procedure as that used by Gregory and Wan [1] for the cylindrical shell. The resulting refined necessary conditions are

$$\int_{-1}^{1} (1+\xi\epsilon)\Big[\xi\big(1-\tfrac{1}{2}\nu\xi\epsilon\big)E^{-1}\bar{\tau}_{\phi\phi}(\xi) - \tfrac{1}{4}\xi^2(1-\xi^2)\epsilon R^{-1}\bar{u}_r(\xi)\Big]\,d\xi$$

$$= O(\epsilon^2), \tag{3.30}$$

$$\int_{-1}^{1} (1+\xi\epsilon)\Big[\big(3(1-\xi^2)-(7+\nu)\xi(1-\xi^2)\epsilon\big)R^{-1}\bar{u}_r(\xi)$$

$$+(1+\nu)(2-\nu)\xi^3\epsilon E^{-1}\bar{\tau}_{\phi\phi}(\xi)\Big]\,d\xi = O(\epsilon^2). \tag{3.31}$$

[Here we have assumed that $R^{-1}\bar{u}_r$ and $E^{-1}\bar{\tau}_{\phi\phi}$ are of order $O(1)$ as $\epsilon \to 0$.]

Cap of general angle. For the cap of general angle α, it does not seem possible to obtain *refined* states {**U**,**T**} in an explicit form. This is because their PF-part is present at relative order $O(\epsilon)$, unlike Case A (and Case C, as it transpires), where the PF-part first appears at relative order $O(\epsilon^2)$. As a result, we can only obtain the leading-order conditions

$$\int_{-1}^{1} (1+\xi\epsilon)E^{-1}\bar{\tau}_{\phi\phi}(\xi)\,d\xi = O(\epsilon), \tag{3.32}$$

$$\int_{-1}^{1} (1+\xi\epsilon)\xi E^{-1}\bar{\tau}_{\phi\phi}(\xi)\,d\xi = O(\epsilon), \tag{3.33}$$

$$\int_{-1}^{1} (1+\xi\epsilon)(1-\xi^2)R^{-1}\bar{u}_r(\xi)\,d\xi = O(\epsilon). \tag{3.34}$$

Note that (3.44) differs significantly from the conventional condition on u_r for thin shells.

Case C: Mixed data $u_\phi(r,\alpha) = \bar{u}_\phi(r)$, $\tau_{r\phi}(r,\alpha) = \bar{\tau}_{r\phi}(r)$. To generate a rapidly decaying state in this case, the data $\bar{u}_\phi$, $\bar{\tau}_{r\phi}$ must satisfy necessary conditions of the form

$$\int_b^a \left[T_{\phi\phi}\bar{u}_\phi - U_r \bar{\tau}_{r\phi} \right] r\,dr = 0, \tag{3.35}$$

where $\{\mathbf{U},\mathbf{T}\}$ satisfies the same conditions as in Case A, except that (3.9) is replaced by

$$U_\phi = T_{r\phi} = 0 \qquad \text{on} \quad \phi = \alpha. \tag{3.36}$$

One possible choice for $\{\mathbf{U},\mathbf{T}\}$, which satisfies (3.36) exactly, can be obtained from the membrane state $\{\mathbf{u}^M, \boldsymbol{\tau}^M\}$, given in Appendix C. In this state, $\tau_{r\phi}^M \equiv 0$ and u_ϕ^M = constant on $\phi = \alpha$; this constant may be annihilated by adding to $\{\mathbf{u}^M, \boldsymbol{\tau}^M\}$ an appropriate rigid body translation in the z-direction. The corresponding condition for rapid decay[12] is

$$\int_{-1}^{1} (1+\xi\epsilon)\left[(1+\xi\epsilon)^{-1} R^{-1} \bar{u}_\phi(\xi) + (1+\nu) E^{-1} \bar{\tau}_{r\phi}(\xi) \right] d\xi = 0. \tag{3.37}$$

Like the equilibrium condition (3.6), this condition is *exact*. Unlike Case B, we *are* able to find two further states $\{\mathbf{U},\mathbf{T}\}$ that satisfy (3.36) with errors of order $O(\epsilon^2)$ even at general cap angle α. The combinations we use are

$$\boldsymbol{\tau}^{CR} + \boldsymbol{\tau}^R + \frac{\cot\alpha}{2\omega}(\boldsymbol{\tau}^R - \boldsymbol{\tau}^I)\epsilon^{1/2} - \frac{\cot^2\alpha}{4\omega^2}\boldsymbol{\tau}^I\epsilon \tag{3.38}$$

and

$$\boldsymbol{\tau}^{CI} + \boldsymbol{\tau}^I + \frac{\cot\alpha}{2\omega}(\boldsymbol{\tau}^R + \boldsymbol{\tau}^I)\epsilon^{1/2} + \frac{\cot^2\alpha}{4\omega^2}\boldsymbol{\tau}^R\epsilon. \tag{3.39}$$

[12]Actually, this condition is necessary for any type of exponential decay and so must also be satisfied by the *shell* eigenfunction. This is a useful check on the formulae for u_ϕ, $\tau_{r\phi}$ in Appendices A and B.

The resulting conditions for rapid decay have a complicated α dependence, but simplify remarkably to give

$$\int_{-1}^{1}(1+\xi\epsilon)\left[(1-\nu\xi\epsilon)E^{-1}\bar{\tau}_{r\phi}(\xi)+\xi^{3}\epsilon R^{-1}\bar{u}_{\phi}(\xi)\right]d\xi = O(\epsilon^{2}), \quad (3.40)$$

$$\int_{-1}^{1}(1+\xi\epsilon)\left[\xi(1-\tfrac{1}{2}(2+\nu)\xi\epsilon)R^{-1}\bar{u}_{\phi}(\xi)\right.$$
$$\left.+\tfrac{1}{2}\nu(1+\nu)\xi^{2}\epsilon E^{-1}\bar{\tau}_{r\phi}(\xi)\right]d\xi = O(\epsilon^{2}) \quad (3.41)$$

as $\epsilon \to 0$. (Here we have assumed that $R^{-1}\bar{u}_{\phi}$ and $E^{-1}\bar{\tau}_{r\phi}$ are of order $O(1)$ as $\epsilon \to 0$.)

The refined necessary conditions for rapid decay with Case C boundary data are (3.37), (3.40), (3.41); these are valid for general cap angle α.

Case D: Displacement data $u_r(r,\alpha)=\bar{u}_r(r)$, $u_{\phi}(r,\alpha)=\bar{u}_{\phi}(r)$. In this case, even the conditions for rapid decay in classical thin shell theory are unavailable in explicit form. In fact one may argue as for the case of the cylindrical shell [1] that these conditions have the form

$$\int_{-1}^{1}\left[\tau_{\xi\zeta}^{X}(\xi,0)\bar{u}_{r}(\xi)+\tau_{\zeta\zeta}^{X}(\xi,0)\bar{u}_{\phi}(\xi)\right]d\xi = O(\epsilon) \quad (3.42)$$

as $\epsilon \to 0$. Here $\tau^{X}(\xi,\zeta)$ ($X=T$, B, or F) denote the stress fields of three canonical problems for the plane semi-infinite strip $-1\le\xi\le 1$, $\zeta\ge 0$, whose end $\zeta=0$ is clamped and which is subject to tension (T), bending (B) or flexure (F) at $\zeta=+\infty$.

Since the functions $\tau_{\xi\zeta}^{X}(\xi,0)$, $\tau_{\zeta\zeta}^{X}(\xi,0)$ are *not* polynomials in ξ, it follows that the plausible conditions

$$\int_{-1}^{1}\bar{u}_{r}\,d\xi = \int_{-1}^{1}\bar{u}_{\phi}(\xi)\,d\xi = \int_{-1}^{1}\xi\bar{u}_{\phi}(\xi)\,d\xi = 0 \quad (3.43)$$

are generally *not* the conditions for rapid decay, even in thin shell theory.

Examples

EXAMPLE 1: A HOLLOW SPHERE COMPRESSED BY AN EQUATORIAL LINE LOAD. Consider the problem of a hollow sphere compressed along its outer equator by a normal line load of magnitude P per unit length. We apply our theory to the *upper half* of this complete spherical shell. By the symmetry of the sphere and the loading, it follows that, on the edge $\phi = \pi/2$,

$$\int_0^{2\pi} \int_{R-h}^{R+h} r\tau_{r\phi} \, dr d\theta = -\pi(R+h)P. \tag{3.44}$$

Note that we have assigned *half* the line load P (applied to the full hollow sphere) to the upper hollow hemisphere. If we now write

$$r = R + h\xi = R(1+\epsilon\xi), \tag{3.45}$$

equation (3.44) becomes

$$\int_{-1}^{1} (1+\epsilon\xi)\tau_{r\phi}(\xi) \, d\xi = -\frac{(1+\epsilon)P}{2R\epsilon}. \tag{3.46}$$

By symmetry, on the edge $\phi = \pi/2$,

$$u_\phi(\xi) = 0, \tag{3.47}$$

and

$$\tau_{r\phi}(\xi) = 0 \qquad (\xi \neq 1). \tag{3.48}$$

Equations (3.46), (3.48) imply that $\tau_{r\phi}(\xi)$ must be a delta function of the form

$$\tau_{r\phi} = K\delta(\xi - 1), \tag{3.49}$$

where K is a constant. On substituting (3.49) into (3.46), we find that

$$\tau_{r\phi} = -\frac{P}{2\epsilon R}\delta(\xi - 1) \tag{3.50}$$

on $\phi = \pi/2$.

Equations (3.47), (3.50) are a particular example of Case C data for the upper hollow hemisphere. We now ask: What is the refined shell solution arising from these data? This is determined from the fact that the *difference* between the boundary values of the refined shell solution on $\phi = \pi/2$ and the prescribed values (3.47), (3.50) must generate a rapidly decaying state in the hollow hemisphere. This difference must therefore satisfy the necessary conditions (3.37), (3.40), (3.41). We write the displacement field of the shell solution in the form

$$A(\mathbf{u}^R + \mathbf{u}^I) + B(\mathbf{u}^R - \mathbf{u}^I) + C(R\mathbf{k}), \tag{3.51}$$

where A, B, C are constants and $\mathbf{k}$ is the unit vector in the direction Oz. Here $\mathbf{u}^R$, $\mathbf{u}^I$ are the real and imaginary parts of the "edge-normalized" shell eigenfunction of Appendix B, with the edge angle taking the value[13] $\alpha = \pi/2$. On applying the condition for rapid decay (3.37), we obtain

$$C = \frac{(1+\nu)(1+\epsilon)P}{4ER\epsilon}, \tag{3.52}$$

while from the conditions (3.40), (3.41) we obtain

$$A = -\frac{\omega P}{8ER\epsilon^{3/2}}\left[1+(1-\nu)\epsilon + O(\epsilon^2)\right], \tag{3.53}$$

$$B = -\frac{P}{320ER\omega\epsilon^{1/2}} \times\left[(31+10\nu-6\nu^2)+(31-21\nu-56\nu^2-34\nu^3)\epsilon+O(\epsilon^2)\right]. \tag{3.54}$$

The above expression for C is "exact" in the sense that it has exponentially small error as $\epsilon \to 0$, while the expressions for A, B are correct to relative order $O(\epsilon)$.

The term $C(R\mathbf{k})$ in (3.51) represents a rigid body translation CR in the direction Oz. When $\mathbf{u}^R$, $\mathbf{u}^I$ are negligible (near the poles for example), this is the only significant part of the solution. It follows that, under the action of this compressive line load, the polar diameter of the sphere expands by $2CR$. In terms of the inner and outer radii a, b, this expansion is given by

$$\frac{2(1+\nu)bRP}{E(b^2-a^2)}, \tag{3.55}$$

a formula correct to within exponentially small error as $\epsilon \to 0$.

As a measure of the influence of the correction terms appearing in the refined theory, we calculate[14] the *stress resultant* $N_{\theta\theta}$ at $\phi = \pi/2$. This is

$$N_{\theta\theta} = -\frac{\omega P}{2\epsilon^{1/2}}\left[1+\frac{31-40\nu-36\nu^2+40\omega^2(1-\nu)}{40\omega^2}\epsilon+O(\epsilon^2)\right]. \tag{3.56}$$

[13] The choice $\alpha = \pi/2$ leads to a considerably simpler form of the edge-normalized shell eigenfunction. In particular, all correction terms are in integer powers of ϵ relative to their leading terms.

[14] We are aware that the PF-part of the solution cannot be neglected when $\phi = \pi/2$. However, this is still a convenient measure of the difference between the refined and thin shell theories.

The coefficient of ϵ in (3.56) depends strongly upon Poisson's ratio ν, decreasing from about 1.89 when $\nu = 0$ to about 0.57 when $\nu = 0.5$.

EXAMPLE 2: THE SPINNING SPHERICAL CAP. Consider the problem of a spherical cap of semi-angle α rotating with constant angular speed Ω about its axis of symmetry. Surface loadings, edge loadings, and body forces are absent. By taking a frame of reference rotating with the cap, this becomes an *equilibrium* problem for a cap acted upon by the usual "centrifugal body force."

Goldberg et al. [9] have solved the special case of a *complete* spinning spherical shell, for which they presented some numerical results for Poisson's ratio equal to 0.3 and a selection of shell thicknesses. We repeated the analysis in [9] and expanded the solution in powers of ϵ, correct to relative order $O(\epsilon^2)$; the results are given in Appendix D. Let this "sphere" solution be called $\boldsymbol{\tau}^S$, and let the required solution for the spinning *cap* be called $\boldsymbol{\tau}^C$. Then the difference $\boldsymbol{\tau}^D$, given by

$$\boldsymbol{\tau}^D = \boldsymbol{\tau}^C - \boldsymbol{\tau}^S, \tag{3.57}$$

satisfies Case A (traction)-type edge data for a cap with no surface loading and no centrifugal body force. The prescribed edge values of $\tau^D_{r\phi}$, $\tau^D_{\phi\phi}$ are just the negatives of the expressions (D.4), (D.5) with ϕ replaced by α.

One may now solve for the refined shell solution corresponding to $\boldsymbol{\tau}^D$ by using the Case A conditions for rapid decay. Write

$$\boldsymbol{\tau}^D = A^{\mathrm{R}}\boldsymbol{\tau}^R + A^I\boldsymbol{\tau}^I + \boldsymbol{\tau}^{PF}, \tag{3.58}$$

where $\boldsymbol{\tau}^R$, $\boldsymbol{\tau}^I$ are the real and imaginary parts of the shell eigenfunction (normalized as in Appendix B), and $\boldsymbol{\tau}^{PF}$ is the PF-part of $\boldsymbol{\tau}^D$. The equilibrium condition (3.4) is satisfied identically while the conditions (3.22), (3.23) give A^R, A^I to be

$$\frac{4E\omega^2}{\rho\Omega^2R^2}A^R = -\kappa\epsilon + \frac{\cot\alpha}{2\omega}\left[5\kappa - 2(1+\nu)(9-\nu^2)\right]\epsilon^{3/2} + O(\epsilon^2), \tag{3.59}$$

$$\frac{4E\omega^2}{\rho\Omega^2R^2}A^I = \kappa\epsilon + \frac{(1-\nu)\kappa\cot\alpha}{\omega}\epsilon^{3/2} + O(\epsilon^2), \tag{3.60}$$

where the dimensionless constant κ is given by

$$\kappa = \nu(2+\nu) + (3+2\nu)\cos 2\alpha. \tag{3.61}$$

The solution to the spinning cap problem, given by our refined shell theory, is thus

$$\boldsymbol{\tau}^S + A^R \boldsymbol{\tau}^R + A^I \boldsymbol{\tau}^I. \tag{3.62}$$

An interesting feature of this problem is that the difference between the complete spinning spherical shell solution and that for the spinning cap is rather small, a fact only evident from an asymptotic method of solution. This is because the residual tractions at the cap edge are, at worst, of order $O(\epsilon)$ as $\epsilon \to 0$. This in turn causes the coefficients A^R, A^I to be of order $O(\epsilon)$.

Appendix A. The shell eigenfunction

The shell eigenvalue (lying in the first quadrant) is given by (2.13) to be

$$\beta = \frac{(1+i)\,\omega}{\epsilon^{1/2}}\left[1 - i\left(\frac{1+24\nu^2}{80\,\omega^2}\right)\epsilon + O(\epsilon^2)\right] \tag{A.1}$$

as $\epsilon \to 0$, where the constant ω is given by (2.14). Let the corresponding shell eigenfunction be denoted by $\{\mathbf{u}^{\beta}, \boldsymbol{\tau}^{\beta}\}$, and define the dimensionless radial coordinate ξ by

$$r = R(1 + \xi\epsilon). \tag{A.2}$$

Then, as $\epsilon \to 0$, the components of $\{\mathbf{u}^{\beta}, \boldsymbol{\tau}^{\beta}\}$ are given as follows.

Displacements

$$u_r^{\beta}(\xi, \phi) = v_r(\xi)\, P_{\beta - 1/2}(\cos\phi), \tag{A.3}$$

where

$$R^{-1}\Re(v_r) = 2 - 2\nu\xi\epsilon + O(\epsilon^2), \tag{A.4}$$

$$R^{-1}\Im(v_r) = -\frac{2\nu\omega^2\xi^2}{1-\nu}\epsilon + O(\epsilon^2). \tag{A.5}$$

and

$$u_{\phi}^{\beta}(\xi, \phi) = v_{\phi}^{(1)}(\xi)\,\frac{d}{d\phi}P_{\beta - 1/2}(\cos\phi), \tag{A.6}$$

where

$$R^{-1}\Re\left(v_{\phi}^{(1)}\right) = -2\xi\epsilon + \frac{\left(2\nu^{2}-20\nu-2+15\nu(1-\nu)\xi^{2}\right)}{15(1-\nu)}\epsilon^{2} + O(\epsilon^{3}), \tag{A.7}$$

$$R^{-1}\Im\left(v_{\phi}^{(1)}\right) = -\frac{4\,\omega^{2}}{3(1-\nu)}\epsilon + \frac{2\,\omega^{2}\xi\left(4-(2-\nu)\xi^{2}\right)}{3(1-\nu)}\epsilon^{2} + O(\epsilon^{3}). \tag{A.8}$$

Stresses

$$\tau_{rr}^{\beta}(\xi,\phi) = \sigma_{rr}(\xi)\,P_{\beta-1/2}(\cos\phi), \tag{A.9}$$

where

$$E^{-1}\Re(\sigma_{rr}) = -\xi\left(1-\xi^{2}\right)\epsilon + O(\epsilon^{2}), \tag{A.10}$$

$$E^{-1}\Im(\sigma_{rr}) = -\frac{2\,\omega^{2}\left(1-\xi^{2}\right)}{1-\nu}\epsilon + O(\epsilon^{2}). \tag{A.11}$$

$$\tau_{r\phi}^{\beta}(\xi,\phi) = \sigma_{r\phi}^{(1)}(\xi)\,\frac{d}{d\phi}P_{\beta-1/2}(\cos\phi), \tag{A.12}$$

where

$$E^{-1}\Re\left(\sigma_{r\phi}^{(1)}\right) = \frac{\left[3+15\nu+2\nu^{2}-5(1-\nu^{2})\xi^{2}\right]\left(1-\xi^{2}\right)}{10(1-\nu^{2})}\epsilon^{2} + O(\epsilon^{3}), \tag{A.13}$$

$$E^{-1}\Im\left(\sigma_{r\phi}^{(1)}\right) = \frac{2\,\omega^{2}\left(1-\xi^{2}\right)}{1-\nu^{2}}\epsilon - \frac{2\,\omega^{2}(7+\nu)\,\xi\left(1-\xi^{2}\right)}{3(1-\nu^{2})}\epsilon^{2} + O(\epsilon^{3}). \tag{A.14}$$

$$\tau_{\phi\phi}^{\beta}(\xi,\phi) = \sigma_{\phi\phi}^{(1)}(\xi)\,P_{\beta-1/2}(\cos\phi) + \sigma_{\phi\phi}^{(2)}(\xi)\cot\phi\,\frac{d}{d\phi}P_{\beta-1/2}(\cos\phi), \tag{A.15}$$

where

$$E^{-1}\Re\left(\sigma_{\phi\phi}^{(1)}\right) = \frac{\xi\left[18+5\nu-3\nu^2-10(1-\nu^2)\xi^2\right]}{5(1-\nu^2)}\epsilon + O(\epsilon^2), \quad \text{(A.16)}$$

$$E^{-1}\Im\left(\sigma_{\phi\phi}^{(1)}\right) = \frac{4\omega^2\xi}{1-\nu^2} + \frac{2\omega^2\left[\nu-3(2+\nu)\xi^2\right]}{3(1-\nu^2)}\epsilon + O(\epsilon^2), \quad \text{(A.17)}$$

$$E^{-1}\Re\left(\sigma_{\phi\phi}^{(2)}\right) = \frac{2\xi}{1+\nu}\epsilon + \frac{2+20\nu-2\nu^2-15(2+\nu)(1-\nu)\xi^2}{15(1-\nu^2)}\epsilon^2 + O(\epsilon^3), \quad \text{(A.18)}$$

$$E^{-1}\Im\left(\sigma_{\phi\phi}^{(2)}\right) = \frac{4\omega^2}{3(1-\nu^2)}\epsilon + \frac{2\omega^2\xi\left[-6+(2-\nu)\xi^2\right]}{3(1-\nu^2)}\epsilon^2 + O(\epsilon^3). \quad \text{(A.19)}$$

$$\tau_{\theta\theta}^{\beta}(\xi,\phi) = \sigma_{\theta\theta}^{(1)}(\xi)P_{\beta-1/2}(\cos\phi) + \sigma_{\theta\theta}^{(2)}(\xi)\cot\phi\frac{d}{d\phi}P_{\beta-1/2}(\cos\phi), \quad \text{(A.20)}$$

where

$$E^{-1}\Re\left(\sigma_{\theta\theta}^{(1)}\right) = 2 + \frac{\xi\left[-10+3\nu+15\nu^2+12\nu^3-5\nu(1-\nu^2)\xi^2\right]}{5(1-\nu^2)}\epsilon + O(\epsilon^2), \quad \text{(A.21)}$$

$$E^{-1}\Im\left(\sigma_{\theta\theta}^{(1)}\right) = \frac{4\nu\omega^2\xi}{1-\nu^2} - \frac{2\nu\omega^2\left[3+2\nu+3(2+\nu)\xi^2\right]}{3(1-\nu^2)}\epsilon + O(\epsilon^2), \quad \text{(A.22)}$$

(A.23)

and

$$E^{-1}\sigma_{\theta\theta}^{(2)} = -E^{-1}\sigma_{\phi\phi}^{(2)}. \quad \text{(A.24)}$$

Note: When $|\beta|$ is large (in accordance with (A.1) as $\epsilon \to 0$), and when ϕ is bounded away from 0 and π,

$$P_{\beta-1/2}(\cos\phi) \sim \left(\frac{i}{2\pi\beta\sin\phi}\right)^{1/2} e^{-i\beta\phi}, \quad \text{(A.25)}$$

$$\frac{d}{d\phi}P_{\beta-1/2}(\cos\phi) \sim -i\beta P_{\beta-1/2}(\cos\phi), \quad \text{(A.26)}$$

which displays the exponential variation of $\{\mathbf{u}^{\beta}, \boldsymbol{\tau}^{\beta}\}$ with ϕ.

Appendix B. The edge-normalized shell eigenfunction

The shell eigenfunction given in Appendix A is not most appropriately normalized for problems that directly, or indirectly, involve the prescription of boundary data at an edge $\mathscr{E}_\alpha$ of a spherical cap. In such cases, it is convenient to divide the eigenfunction of Appendix A by $P_{\beta-1/2}(\cos\alpha)$, so that $\{\mathbf{u}^\beta, \boldsymbol{\tau}^\beta\}$ is not exponentially large, or small, on the edge $\mathscr{E}_\alpha$. This makes no essential difference to the formulae for those components of $\{\mathbf{u}^\beta, \boldsymbol{\tau}^\beta\}$ that involve $P_{\beta-1/2}(\cos\phi)$, but those that involve $(d/d\phi)P_{\beta-1/2}(\cos\phi)$ now effectively acquire the multiplier $-(\sin\alpha)P'_{\beta-1/2}(\cos\alpha)/P_{\beta-1/2}(\cos\alpha)$. This may be replaced by its asymptotic expansion as $|\beta|\to\infty$ in the right-hand half-plane, namely,

$$\frac{-(\sin\alpha)P'_{\beta-1/2}(\cos\alpha)}{P_{\beta-1/2}(\cos\alpha)} \sim -i\beta - \frac{1}{2}\cot\alpha - \frac{i\,\mathrm{cosec}^2\alpha}{8\beta} + O(\beta^{-2}), \quad \text{(B.1)}$$

where β is given by (A.1). This multiplier may then be incorporated into the relevant field components to give the following.

Displacements

$$u_r^\beta(\xi,\phi) = v_r(\xi)\left(\frac{P_{\beta-1/2}(\cos\phi)}{P_{\beta-1/2}(\cos\alpha)}\right), \quad \text{(B.2)}$$

where $v_r(\xi)$ is given by (A.4), (A.5).

$$u_\phi^\beta(\xi,\phi) = v_\phi^{(2)}(\xi)\left(\frac{\sin\phi\, P'_{\beta-1/2}(\cos\phi)}{\sin\alpha\, P'_{\beta-1/2}(\cos\alpha)}\right), \quad \text{(B.3)}$$

where

$$\begin{aligned} R^{-1}\Re\left(v_\phi^{(2)}\right) = &-\omega^{-1}(1+\nu+2\omega^2\xi)\epsilon^{1/2} + (\cot\alpha)\xi\epsilon \\ &+ \frac{\omega^{-3}}{80}\Big[-(1+\nu)(16\nu^2+80\nu+14+5\cot^2\alpha) \\ &+2\omega^2(24\nu^2+80\nu+86+5\cot^2\alpha)\xi \\ &+60\nu(1-\nu^2)\xi^2 - 40\omega^2(1+\nu)(2-\nu)\xi^3\Big]\epsilon^{3/2} + O(\epsilon^2), \quad \text{(B.4)} \end{aligned}$$

$$R^{-1}\Im\left(v_{\phi}^{(2)}\right) = -\,\omega^{-1}(1+\nu-2\,\omega^2\,\xi\,)\epsilon^{1/2} + \frac{1}{2}\,\omega^{-2}(1+\nu)(\cot\alpha\,)\,\epsilon$$
$$+\,\frac{\omega^{-3}}{80}\Big[(1+\nu)(16\nu^2+80\nu+14+5\cot^2\alpha\,)$$
$$+2\,\omega^2(24\nu^2+80\nu+86+5\cot^2\alpha\,)\,\xi$$
$$-60\nu(1-\nu^2)\,\xi^{\,2}-40\,\omega^2(1+\nu)(2-\nu)\,\xi^{\,3}\Big]\,\epsilon^{3/2} + O(\,\epsilon^2\,). \quad \text{(B.5)}$$

Stresses

$$\tau_{rr}^{\beta}(\,\xi,\phi) = \sigma_{rr}(\,\xi\,)\left(\frac{P_{\beta-1/2}(\cos\phi)}{P_{\beta-1/2}(\cos\alpha\,)}\right), \quad \text{(B.6)}$$

where $\sigma_{rr}(\,\xi\,)$ is given by (A.10), (A.11).

$$\tau_{r\phi}^{\beta}(\,\xi,\phi) = \sigma_{r\phi}^{(2)}(\,\xi\,)\left(\frac{\sin\phi P'_{\beta-1/2}(\cos\phi)}{\sin\alpha P'_{\beta-1/2}(\cos\alpha\,)}\right), \quad \text{(B.7)}$$

where

$$E^{-1}\Re\left(\sigma_{r\phi}^{(2)}\right) = \frac{3}{2}\,\omega^{-1}(1-\xi^{\,2})\epsilon^{1/2}$$
$$+\,\frac{\omega(1-\xi^{\,2})}{120(1-\nu^2)}\Big[3(32\,\nu^2+60\nu+18+5\cot^2\alpha\,)$$
$$-80\,\omega^2(7+\nu)\,\xi-60(1-\nu^2)\,\xi^{\,2}\Big]\,\epsilon^{3/2} + O(\,\epsilon^2\,), \quad \text{(B.8)}$$

$$E^{-1}\Im\left(\sigma_{r\phi}^{(2)}\right) = \frac{3}{2}\,\omega^{-1}(1-\xi^{\,2})\epsilon^{1/2} - \frac{3}{4}\,\omega^{-2}(1-\xi^{\,2})(\cot\alpha\,)\,\epsilon$$
$$+\,\frac{\omega(1-\xi^{\,2})}{120(1-\nu^2)}\Big[-3(32\nu^2+60\nu+18+5\cot^2\alpha\,)$$
$$-80\,\omega^2(7+\nu)\,\xi+60(1-\nu^2)\,\xi^{\,2}\Big]\,\epsilon^{3/2} + O(\,\epsilon^2\,), \quad \text{(B.9)}$$

$$\tau_{\phi\phi}^{\beta}(\,\xi,\phi) = \sigma_{\phi\phi}^{(1)}(\,\xi\,)\left(\frac{P_{\beta-1/2}(\cos\phi)}{P_{\beta-1/2}(\cos\alpha\,)}\right) + \sigma_{\phi\phi}^{(3)}(\,\xi\,)\left(\frac{\cos\phi P'_{\beta-1/2}(\cos\phi)}{\sin\alpha P'_{\beta-1/2}(\cos\alpha\,)}\right), \quad \text{(B.10)}$$

where $\sigma_{\phi\phi}^{(1)}(\xi)$ is given by (A.16), (A.17), and

$$E^{-1}\Re\left(\sigma_{\phi\phi}^{(3)}\right) = \frac{\omega^{-1}}{1+\nu}(1+\nu+2\omega^2\xi)\epsilon^{1/2} - \frac{\cot\alpha}{1+\nu}\xi\epsilon$$

$$+\frac{\omega^{-3}}{80}\Big[(16\nu^2+80\nu+14+5\cot^2\alpha)$$

$$-2\omega^2(1+\nu)^{-1}(24\nu^2+120\nu+126+5\cot^2\alpha)\xi$$

$$-60(1-\nu)(2+\nu)\xi^2+40\omega^2(2-\nu)\xi^3\Big]\epsilon^{3/2}+O(\epsilon^2), \quad \text{(B.11)}$$

$$E^{-1}\Im\left(\sigma_{\phi\phi}^{(3)}\right) = \frac{\omega^{-1}}{1+\nu}(1+\nu-2\omega^2\xi)\epsilon^{1/2} - \frac{1}{2}\omega^{-2}(\cot\alpha)\epsilon$$

$$+\frac{\omega^{-3}}{80}\Big[-(16\nu^2+80\nu+14+5\cot^2\alpha)$$

$$-2\omega^2(1+\nu)^{-1}(24\nu^2+120\nu+126+5\cot^2\alpha)\xi$$

$$+60(1-\nu)(2+\nu)\xi^2+40\omega^2(2-\nu)\xi^3\Big]\epsilon^{3/2}+O(\epsilon^2). \quad \text{(B.12)}$$

$$\tau_{\theta\theta}^{\beta}(\xi,\phi) = \sigma_{\theta\theta}^{(1)}(\xi)\left(\frac{P_{\beta-1/2}(\cos\phi)}{P_{\beta-1/2}(\cos\alpha)}\right) + \sigma_{\theta\theta}^{(3)}(\xi)\left(\frac{\cos\phi P'_{\beta-1/2}(\cos\phi)}{\sin\alpha P'_{\beta-1/2}(\cos\alpha)}\right), \quad \text{(B.13)}$$

where $\sigma_{\theta\theta}^{(1)}$ is given by (A.21), (A.22), and

$$E^{-1}\sigma_{\theta\theta}^{(3)} = -E^{-1}\sigma_{\phi\phi}^{(3)}. \quad \text{(B.14)}$$

Appendix C. An exact three-dimensional membrane state

The exact eigenfunctions obtained in Section 2 are believed complete for the problem of the unpunctured spherical cap R_α, loaded around its edge $\mathscr{E}_\alpha$. However, if the cap is open at the North pole $\phi = 0$, so that it has a second edge which can be loaded, additional expansion states are needed. First there is the corresponding set of eigenfunctions that increase exponentially away from the South pole rather than the North; these make up the boundary layer near the second edge. However, it is evident that these are still not sufficient in general, since all of these eigenfunctions correspond to zero resultant axial force at each edge of the cap. With two edges available, this resultant force need not be zero and so we need one additional expansion state that has a nonzero value of this resultant. Such a "membrane" state, which is evidently nonunique, can be obtained in different ways. We first obtained the state $\{\mathbf{u}^M, \boldsymbol{\tau}^M\}$ below by guessing its form from the thin shell membrane solution of Reissner and Wan [17] (see [7]). However, it can be more quickly obtained as follows.

In (2.3), taking $\eta = 1$ gives

$$\Phi = rP_1(\cos\phi) = z, \qquad \Psi = 0. \tag{C.1}$$

This choice of potentials generates a trivial rigid body displacement in the z-direction. However, consider instead the field generated by

$$\Phi = rQ_1(\cos\phi), \qquad \Psi = 0, \tag{C.2}$$

where Q_1 is the companion Legendre function to P_1 (see [18, Chap. 8] given by

$$Q_1(\cos\phi) = \cos\phi \ln\left(\cot\tfrac{1}{2}\phi\right) - 1. \tag{C.3}$$

Since $rQ_1(\cos\phi)$ is a harmonic function, (C.2) will also generate an elastic field via (2.1). This elastic field is, after a simple normalization, given by

$$R^{-1}u_r^M = \cos\phi \ln\left(\cot\tfrac{1}{2}\phi\right) - 1, \tag{C.4}$$

$$R^{-1}u_\phi^M = -\sin\phi \ln\left(\cot\tfrac{1}{2}\phi\right) - \cot\phi, \tag{C.5}$$

$$E^{-1}\tau_{rr}^M = 0, \tag{C.6}$$

$$E^{-1}\tau_{r\phi}^M = 0, \tag{C.7}$$

$$E^{-1}\tau_{\phi\phi}^M = (1+\nu)^{-1}(1+\xi\epsilon)^{-1}\operatorname{cosec}^2\phi, \tag{C.8}$$

$$E^{-1}\tau_{\theta\theta}^M = -(1+\nu)^{-1}(1+\xi\epsilon)^{-1}\operatorname{cosec}^2\phi. \tag{C.9}$$

As usual, R is the midsurface radius of the cap, and the dimensionless radial coordinate ξ is defined by (A.2).

The above state is an *exact* three-dimensional state, regular except along the polar axes $\phi = 0$, π, and (trivially) satisfying the boundary conditions $\tau_{rr} = \tau_{r\phi} = 0$ on the surfaces $r = a, b$. It is easily verified that it has the resultant axial tension

$$\frac{2\pi R(b-a)E}{1+\nu}. \tag{C.10}$$

Surprisingly, the exact form of this state seems to be new to the literature. In particular, it does not appear in the book on three-dimensional elasticity problems by Lur'e [19]. In the present article, the membrane state is used in section 3 to derive conditions for rapid decay.

Appendix D. The complete spinning shell solution

The problem of a complete spherical shell, rotating with angular speed Ω about the z-axis, was solved by the method described in [9] and then expanded in powers of ϵ. The results follow.

Displacements

$$\begin{aligned}\frac{Eu_r}{\rho\Omega^2R^3} = &-\frac{1}{2}[\nu+(2+\nu)\cos 2\phi] - [\nu\xi\sin^2\phi]\epsilon \\ &+\frac{1}{6}(1-\nu^2)^{-1}\big[(2\nu^3-13\nu^2-5\nu+18)+3\nu(1+\nu)(2+\nu)\xi^2 \\ &-(3+2\nu)(\nu^2+9\nu-16)\cos 2\phi+3\nu(1+\nu)(4+\nu)\xi^2\cos 2\phi\big]\epsilon^2 + O(\epsilon^3).\end{aligned} \tag{D.1}$$

$$\begin{aligned}\frac{Eu_\phi}{\rho\Omega^2R^3} = &\frac{1}{2}(1+\nu)\sin 2\phi - \frac{1}{2}[(3+\nu)\xi\sin 2\phi]\epsilon \\ &+\frac{1}{6}(1-\nu)^{-1}\big[-(3+2\nu)(5-\nu)+3\nu(1-\nu)\xi^2\big]\sin 2\phi\,\epsilon^2 \\ &+O(\epsilon^3).\end{aligned} \tag{D.2}$$

Stresses

$$\frac{\tau_{rr}}{\rho\Omega^2R^2} = \frac{1}{2}(1-\nu)^{-1}[3+(3+2\nu)\cos 2\phi](1-\xi^2)\epsilon^2 + O(\epsilon^3). \tag{D.3}$$

$$\frac{\tau_{r\phi}}{\rho\Omega^2R^2} = -\frac{1}{2}(1-\nu^2)^{-1}(2\nu^2-3\nu-9)(1-\xi^2)\sin 2\phi\,\epsilon^2 + O(\epsilon^3). \tag{D.4}$$

$$\frac{\tau_{\phi\phi}}{\rho\Omega^2R^2} = (1-\nu^2)^{-1}\left[3-\nu^2-2(3+2\nu)\cos^2\phi\right]\xi\epsilon$$
$$+\frac{1}{3}(1-\nu^2)^{-1}\left[\nu(2+\nu)-3(3+2\nu)\xi^2+3(2-\nu)(3+2\nu)\cos^2\phi\right.$$
$$\left.+3(2+\nu)(3+2\nu)\xi^2\cos^2\phi\right]\epsilon^2+O(\epsilon^3). \quad \text{(D.5)}$$

$$\frac{\tau_{\theta\theta}}{\rho\Omega^2R^2} = \sin^2\phi-(1-\nu^2)^{-1}\left[2+\nu+(\nu^2+3\nu+1)\cos 2\phi\right]\xi\epsilon$$
$$+\frac{1}{6}(1-\nu^2)^{-1}\left[3-2\nu^2-3(\nu^2-2\nu-4)\xi^2\right.$$
$$\left.+(11-4\nu)(3+2\nu)\cos 2\phi+3(5\nu^2+8\nu+2)\xi^2\cos 2\phi\right]\epsilon^2$$
$$+O(\epsilon^3). \quad \text{(D.6)}$$

Note

This expansion, which is new to the literature, is more useful than the exact solution to this classical problem when the spherical shell is thin, or moderately thick. In particular, we see that the exterior equatorial diameter of the shell expands by

$$\frac{\rho\Omega^2R^3}{E}\left[2-2\nu\epsilon-\frac{30+16\nu-2\nu^2-4\nu^3}{3(1-\nu^2)}\epsilon^2+O(\epsilon^3)\right], \quad \text{(D.7)}$$

while the exterior polar diameter contracts by

$$\frac{\rho\Omega^2R^3}{E}\left[2(1+\nu)-\frac{66+18\nu-10\nu^2+6\nu^3}{3(1-\nu^2)}\epsilon^2+O(\epsilon^3)\right]. \quad \text{(D.8)}$$

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