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ON AXIAL EXTENSION AND TORSION OF HELICOIDAL SHELLS

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1. Introduction. We consider in what follows rotationally symmetric states of stress and strain in helicoidal thin elastic shells acted upon by axial forces and torques. Previous work in this area has been concerned with the special case of the problem when the shell is of the form of a pretwisted rectangular strip [2, 4, 7], and with the problem of the shallow helicoidal shell as a generalization of the corresponding problem for a circular ring plate [3].

With the middle surface equation of the shell given by $z = a\theta$ where r, θ, z are cylindrical coordinates, the work in [2, 4, 7] was for shells for which the edges were at $\theta = \pm \theta_0$ and $r = \pm r_0$. The principal generalization of the earlier work which is made here consists in assuming radial edges of the shell at $r = r_i$ and $r = r_0$. In so doing, the problem of the pretwisted strip becomes a special case of a class of problems which also includes the problem of the helicoidal spring, provided the cross section of the windings is such as to allow use of the theory of thin shells. The present generalization of the problem of the pretwisted strip is based in part on work included in the second named author's dissertation [8].

The earlier solution of the problem of the pretwisted strip [2, 4] was based on two types of relatively simple non-rotationally symmetric states of displacement associated with rotationally symmetric strains. Our present more general solution requires the introduction of a third, less simple, type of non-rotationally symmetric displacement state.

Principal elements of the earlier work were the reduction of the shell problem to a second order ordinary differential equation for a radial displacement function [2] with perturbation expansions of the solution [2, 4, 7], and the discovery of a simple closed form expression for the transverse bending portion of the solution [4]. For the present generalization of the earlier work, certain unexpected difficulties arose in connection with the use of the second order differential equation for the displacement function. While these difficulties could be overcome, it was found to be simpler and more instructive to use a stress function variable in place of the original radial displacement variable.

2. ***Differential equations and boundary conditions.*** *The set of differential* equations which governs the behavior of the shell is here taken in a form given in [5], specialized to the case of a helicoidal shell and cylindrical coordinates. The deformation of the shell is described in this formulation by three translational displacement components u_r, u_θ and w in radial, tangential and normal direction, and three rotational displacement components ϕ_r, ϕ_θ and ω turning about tangential, radial and normal axes. In terms of these six displacement components, we have four inplane strain measures ϵ_r, $\epsilon_{r\theta}$, $\epsilon_{\theta r}$ and ϵ_θ, two transverse shear strain measures γ_r and γ_θ, and four curvature change measures κ_r, $\kappa_{r\theta}$, $\kappa_{\theta r}$ and κ_θ in the form

$$
\begin{aligned}
\epsilon_r &= u_r', & \epsilon_{r\theta} &= u_\theta' + \frac{w}{R} - \omega, \\
\epsilon_\theta &= \frac{\dot{u}_\theta}{\alpha} + \frac{r u_r}{\alpha^2}, & \epsilon_{\theta r} &= \frac{\dot{u}_r}{\alpha} - \frac{r u_\theta}{\alpha^2} + \frac{w}{R} + \omega, \\
\gamma_r &= \phi_r + w' - \frac{u_\theta}{R}, & \gamma_\theta &= \phi_\theta + \frac{\dot{w}}{\alpha} - \frac{u_r}{R}, \\
\kappa_r &= \phi_r' + \frac{\omega}{R}, & \kappa_{r\theta} &= \phi_\theta', \\
\kappa_\theta &= \frac{r\phi_r}{\alpha^2} + \frac{\dot{\phi}_\theta}{\alpha} - \frac{\omega}{R}, & \kappa_{\theta r} &= \frac{\dot{\phi}_r}{\alpha} - \frac{r\phi_\theta}{\alpha^2},
\end{aligned} \tag{1}
$$

where primes and dots indicate differentiation with respect to r and θ, respectively, and where

$$
\alpha = \sqrt{a^2 + r^2}, \qquad \frac{1}{R} = \frac{a}{\alpha^2} \tag{2}
$$

The ten strain measures (1) are related to stress resultants and couples through a system of stress strain relations which is here taken in the form [6]

$$
\begin{gathered}
\epsilon_r = A(N_r - \nu_s N_\theta), \qquad \epsilon_\theta = A(N_\theta - \nu_s N_r), \\
\epsilon_{r\theta} = \epsilon_{\theta r} = \tfrac{1}{2}(1 + \nu_s) A (N_{r\theta} + N_{\theta r}), \\
M_r = D(\kappa_r + \nu_b \kappa_\theta), \qquad M_\theta = D(\kappa_\theta + \nu_b \kappa_r), \\
M_{r\theta} = M_{\theta r} = \tfrac{1}{2}(1 - \nu_b) D(\kappa_{r\theta} + \kappa_{\theta r}),
\end{gathered} \tag{3}
$$

and

$$
\gamma_r = \gamma_\theta = 0 \tag{4}
$$

where for a homogeneous and isotropic medium

$$
A = \frac{1}{Eh}, \qquad D = \frac{Eh^3}{12(1 - \nu^2)}, \qquad \nu_s = \nu_b = \nu \tag{5}
$$

In the absence of surface loads, the four inplane stress resultants N_r, $N_{r\theta}$, $N_{\theta r}$ and N_θ, the two transverse shear resultants Q_r and Q_θ, and the four stress couples, M_r, $M_{r\theta}$, $M_{\theta r}$ and M_θ satisfy the equilibrium conditions [5]

$$
\begin{gathered}
(\alpha N_r)' + \dot{N}_{\theta r} - \frac{r}{\alpha} N_\theta + \frac{a}{\alpha} Q_\theta = 0, \\
(\alpha N_{r\theta})' + \dot{N}_\theta + \frac{r}{\alpha} N_{\theta r} + \frac{a}{\alpha} Q_r = 0, \\
(\alpha Q_r)' + \dot{Q}_\theta - \frac{a}{\alpha}(N_{r\theta} + N_{\theta r}) = 0, \\
(\alpha M_r)' + \dot{M}_{\theta r} - \frac{r}{\alpha} M_\theta - \alpha Q_r = 0, \\
(\alpha M_{r\theta})' + \dot{M}_\theta + \frac{r}{\alpha} M_{\theta r} - \alpha Q_\theta = 0, \\
N_{r\theta} - N_{\theta r} + \frac{M_\theta - M_r}{R} = 0
\end{gathered} \tag{6}
$$

We consider a shell with edge coordinates $r = r_i$, $r = r_0$ (with $r_i < r_0$), and $\theta = \pm\theta_0$. The shell is acted upon by equal and opposite axial forces F and torques T at the axial edges $\theta = \pm\theta_0$, and is free of tractions otherwise (Fig. 1). The conditions of no stress at the radial edges requires satisfaction of the Kirchhoff-Basset conditions [5]

$$N_r + \frac{M_{r\theta}}{R} = N_{r\theta} = Q_r + \frac{\dot{M}_{r\theta}}{\alpha} = M_r = 0 \tag{7}$$

at $r = r_i$ and $r = r_0$.

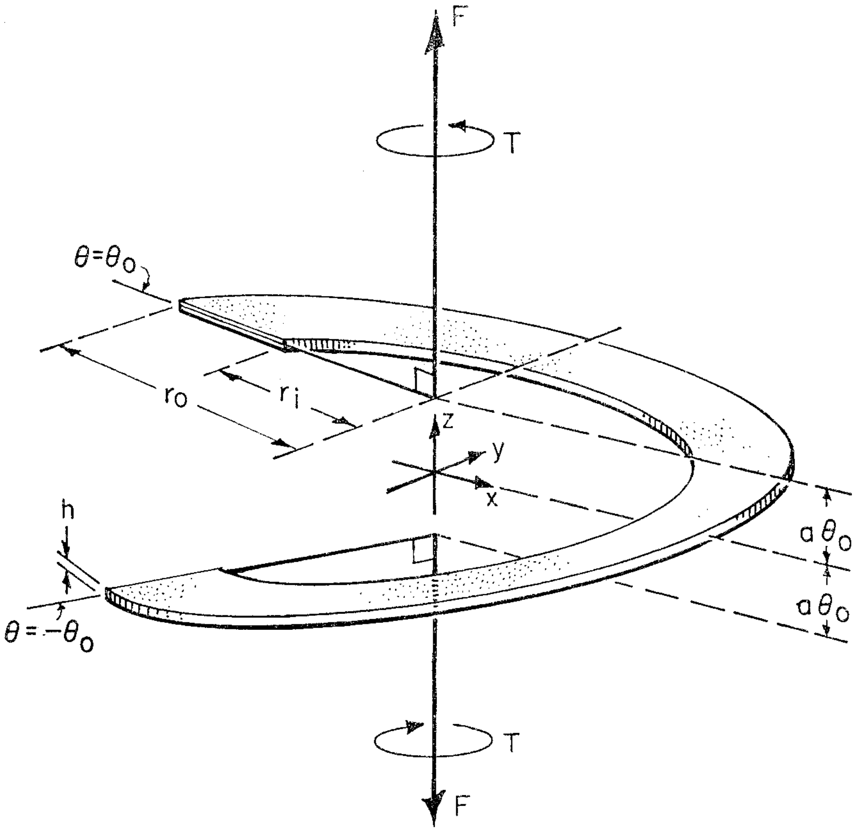

FIG. 1. Helicoidal shell acted upon by axial forces and torques

At the axial edges, instead of prescribing the effective resultants and couples, $N_\theta + M_{\theta r}/R$, $N_{\theta r}$, $Q_\theta + M'_{\theta r}$ and M_θ in detail, we limit ourselves to prescribing a resultant axial force F and a twisting moment T together with vanishing side forces and bending moments. Letting Z_θ, R_θ and Θ_θ be the effective stress resultants in axial, radial and circumferential directions, respectively,

$$\begin{aligned} Z_\theta &= \frac{a}{\alpha} N_\theta + \frac{r}{\alpha} Q_\theta + \left(\frac{r}{\alpha} M_{\theta r}\right)', \qquad R_\theta = N_{\theta r}, \\ \Theta_\theta &= \frac{r}{\alpha} N_\theta - \frac{a}{\alpha} Q_\theta - \left(\frac{a}{\alpha} M_{\theta r}\right)' \end{aligned} \tag{8}$$

we then have as boundary conditions for $\theta = \pm\theta_0$

$$\int_{r_i}^{r_0} Z_\theta \, dr - \left[\frac{r}{\alpha} (M_{r\theta} + M_{\theta r}) \right]_{r_i}^{r_0} = F,$$
$$\int_{r_i}^{r_0} r\Theta_\theta \, dr + \left[\frac{ar}{\alpha} (M_{r\theta} + M_{\theta r}) \right]_{r_i}^{r_0} = T \tag{9}$$

and

$$\int_{r_i}^{r_0} R_\theta \, dr = 0, \qquad \int_{r_i}^{r_0} \Theta_\theta \, dr + \left[\frac{a}{\alpha} (M_{r\theta} + M_{\theta r}) \right]_{r_i}^{r_0} = 0$$
$$\int_{r_i}^{r_0} M_\theta \, dr = 0, \qquad \int_{r_i}^{r_0} rZ_\theta \, dr - \left[\frac{r^2}{\alpha} (M_{r\theta} + M_{\theta r}) \right]_{r_i}^{r_0} = 0 \tag{10}$$

The terms on the left side of these equations outside the integrals are associated with the corner forces introduced by the assumption (4) of vanishing transverse shear strain. Equations (9) and (10) are generalizations of the corresponding conditions given in [2] and reduce to the latter if $r_i = -r_0$.

3. Non-axisymmetric displacement states. It was found in [2] that the displacement state of a pretwisted strip subject to equal and opposite axial forces and torques is of the form

$$u_r = u(r), \qquad u_\theta = \frac{ka + \psi r^2}{\alpha} a\theta, \qquad w = \frac{k - \psi a}{\alpha} ra\theta$$
$$\phi_r = \frac{\psi a^2 \theta}{\alpha}, \qquad \phi_\theta = \frac{u - kr + \psi ar}{R}, \qquad \omega = \frac{\psi ra\theta}{\alpha} \tag{11a}$$

where k and ψ are suitable constants the meaning of which becomes clear from the form of the axial and circumferential displacement components

$$u_z = \frac{au_\theta + rw}{\alpha} = ka\theta, \qquad v = \frac{ru_\theta - aw}{\alpha} = \psi ra\theta \tag{11b}$$

The results for the limiting case of a flat strip, given when $a \to \infty$ in $a\theta = z$, suggest that the displacement state for a general helicoidal shell subject to the same external loads cannot have as simple a dependence on θ; nevertheless, we may expect that the strain state continues to be axisymmetric. By a systematic consideration of the strain displacement relations (1) we find that the requirement of a suitably generalized axisymmetric state of strain leads us to the following generalization of the displacement state (11a),

$$u_r = u(r) + ca^2(1 - \cos\theta - \theta\sin\theta)$$
$$u_\theta = \frac{ka + \psi r^2}{\alpha} a\theta - \frac{ca^2 r}{\alpha} (\theta\cos\theta - 2\sin\theta)$$
$$w = \frac{kr - \psi ar}{\alpha} a\theta + \frac{ca}{\alpha} [a^2\theta\cos\theta - (a^2 - r^2)\sin\theta] \tag{12a}$$
$$\phi_\theta = \frac{u - kr + \psi ar}{R} + ca\left(\frac{a}{R} - \cos\theta\right)$$
$$\phi_r = \frac{\psi a^2\theta}{\alpha} - \frac{car\sin\theta}{\alpha}, \qquad \omega = \frac{\psi ra\theta}{\alpha} + \frac{ca^2\sin\theta}{\alpha}$$

The corresponding axial and circumferential displacement components are

$$u_z = ka\theta + car \sin\theta, \quad v = \psi ra\theta + ca^2 (\sin\theta - \theta\cos\theta) \tag{12b}$$

In this the new constant parameter c is to be related to k and ψ by a suitable condition which is discussed further on.

The strain measures corresponding to the displacement state (12) are

$$\epsilon_{r\theta} = \epsilon_{\theta r} = \kappa_r = \kappa_\theta = \gamma_r = \gamma_\theta = 0 \tag{13a}$$

and

$$\begin{aligned} \epsilon_r &= u', \qquad \epsilon_\theta = \frac{ru}{\alpha^2} + \frac{car + ka + \psi r^2}{R} \\ \kappa_{r\theta} &= \frac{1}{R}\left[u' - 2\,\frac{ru + ca^2 r}{\alpha^2} - (k - \psi a)\,\frac{a^2 - r^2}{\alpha^2}\right] \\ \kappa_{\theta r} &= \frac{1}{\alpha^2 R}(-ru - ca^2 r + kr^2 + \psi a^3) \end{aligned} \tag{13b}$$

For a reduction of the problem as stated, we first consider the transverse bending stress strain relation

$$M_{r\theta} = M_{\theta r} = \frac{(1 - \nu_b)D}{2R}\left(u' - 3\,\frac{ru}{\alpha^2} - 3ca^2\,\frac{r}{\alpha^2} + k\,\frac{2r^2 - a^2}{\alpha^2} + \psi a\,\frac{2a^2 - r^2}{\alpha^2}\right) \tag{14a}$$

Equation (14a) can be written in the form

$$M_{r\theta} = M_{\theta r} = (1 - \nu_b)D\left(\frac{\epsilon_r - 3\epsilon_\theta}{2R} + \frac{ka + \psi a^2}{\alpha^2}\right) \tag{14b}$$

Since witihn the range of applicability of the stress strain relations (3) terms of the order ϵ/R are considered negligible in the expressions for stress couples, it is consistent to replace the above expression for the twisting couples by the abbreviated explicit relation

$$M_{r\theta} = M_{\theta r} = (1 - \nu_b)D\,\frac{ka + \psi a^2}{\alpha^2} \tag{15}$$

which was first obtained in [4].

The remaining stress strain relations give

$$N_{r\theta} = N_{\theta r} = M_r = M_\theta = 0 \tag{16a}$$

and

$$\begin{aligned} N_r &= \frac{1}{(1 - \nu_s^2)A}\left(u' + \nu_s\,\frac{ru + ca^2 r + ka^2 + \psi ar^2}{\alpha^2}\right) \\ N_\theta &= \frac{1}{(1 - \nu_s^2)A}\left(\nu_s\,u' + \frac{ru + ca^2 r + ka^2 + \psi ar^2}{\alpha^2}\right) \end{aligned} \tag{16b}$$

With (15) and (16a) the equilibrium equations (6) reduce to

$$Q_r = Q_\theta = 0^* \tag{17a}$$

and

$$(\alpha N_r)' - r\alpha^{-1}N_\theta = 0 \tag{17b}$$

Introduction of (16b) into (17b) leads to the following second order linear ordinary differential equation for u,

$$u'' + \frac{r}{\alpha^2}u' + \frac{\nu_s a^2 - r^2}{\alpha^4}u$$
$$= \frac{1}{\alpha^4}\{(1+\nu_s)ka^2r + \psi ar[(1-\nu_s)r^2 - 2\nu_s a^2] - ca^2(\nu_s a^2 - r^2)\} \tag{18}$$

Equation (18) is considerably simpler than the corresponding equation which is obtained without the step from (14) to (15) [2].

Boundary conditions for the solution of (18) follow from the edge conditions (7) which reduce to the one condition

$$N_r + \frac{M_{r\theta}}{R} = \frac{\epsilon_r + \nu_s\epsilon_\theta}{(1-\nu_s^2)A} + \frac{1-\nu_b}{2}D\frac{\kappa_{r\theta}+\kappa_{\theta r}}{R} = 0 \tag{19}$$

Since all terms of the form $D\kappa/R$ have been neglected in the derivation of the normal in-plane stress strain relations, it is consistent to replace (19) by

$$N_r(r_i) = N_r(r_0) = 0 \tag{20a}$$

or

$$r = r_i, r_0: \quad u' + \nu_s\alpha^{-2}(ru + ca^2r + ka^2 + \psi ar^2) = 0 \tag{20b}$$

It is possible to reduce the problem formally in such a way that the constant c disappears. This is done by introducing a function $\hat{u} = u + ca^2$, and this was in fact the way in which the problem was originally considered. The difficulty with the use of the function $\hat{u}$ is that as $a \to \infty$, $\hat{u} \sim ca^2 \to \infty$, which for $c \neq 0$ leads to difficulties with numerical and perturbation procedures [8]. While the use of u eliminates these difficulties, it will be simpler yet to use a stress function approach for the explicit solution of the problem.

4. Stress function formulation. Choosing as our point of departure the simplified equilibrium equation (17b), we consider N_r as stress function, in terms of which

$$N_\theta = \frac{\alpha}{r}(\alpha N_r)' \tag{21}$$

* It is of interest to note that the useful relation $Q_\theta = 0$, which follows from the fifth equation in (6) depends on the step from (14b) to (15). Retention of the ϵ/R-terms in (14b) would mean that a small but non-vanishin Q_θ had to be carried along in (17b) and elsewhere.

The first two of the stress strain relations (3) are arranged to read

$$\epsilon_r = A\left[N_r - \nu_s \frac{\alpha}{r}(\alpha N_r)'\right], \qquad \epsilon_\theta = A\left[\frac{\alpha}{r}(\alpha N_r)' - \nu_s N_r\right] \tag{22}$$

The first two strain displacement relations in (13b) give rise to a compatibility equation

$$\epsilon_r = \left(\frac{\alpha^2}{r}\epsilon_\theta\right)' + k\frac{a^2}{r^2} - \psi a \tag{23}$$

Introduction of (22) into (23) leads to the stress function differential equation

$$N_r'' - \frac{2a^2 - 3r^2}{r(a^2 + r^2)}N_r' - \frac{(1 - \nu_s)a^2}{(a^2 + r^2)^2}N_r = \frac{\psi a r^2 - k a^2}{A(a^2 + r^2)^2} \tag{24}$$

The boundary conditions associated with (24) are given directly by (20a).

Of the four boundary conditions (10), the first and third are identically satisfied in view of (16a). By use of the first and of the fifth equilibrium equation in (6), in conjunction with the conditions (7), we can show that the remaining equations in (10) are also satisfied identically and the remaining two conditions (9) may be written in the form

$$\begin{aligned} F &= \int_{r_i}^{r_0} \frac{a}{\alpha} N_\theta \, dr - \left[\frac{r}{\alpha} M_{r\theta}\right]_{r_i}^{r_0} \\ T &= \int_{r_i}^{r_0} \frac{r^2}{\alpha} N_\theta \, dr + \left[\frac{r(2a^2 + r^2)}{a\alpha} M_{r\theta}\right]_{r_i}^{r_0} \end{aligned} \tag{25}$$

In this $N_\theta/\alpha = (\alpha N_r)'/r$ and $M_{r\theta}$ is given by (15).

Equations (25) will be used to obtain expressions for F and T in terms of the overall displacement variables k and ψ.

Finally, the displacement function u is obtained by writing equation (13) for ϵ_θ, in the form

$$u = A\left[\frac{\alpha^4}{r^2}N_r' + \frac{(1 - \nu_s)\alpha^2}{r}N_r\right] - \frac{ka^2}{r} - \psi a r - c a^2 \tag{26}$$

5. Non-dimensionalization. We set

$$\rho = \frac{r}{r_0}, \qquad \rho_i = \frac{r_i}{r_0}, \qquad \lambda = \frac{r_0}{a} \tag{27}$$

and

$$n = \frac{N}{Eh}, \qquad m = \frac{12(1 + \nu_b)r_0}{Eh^3} M \tag{28}$$

The nondimensionalization of N_r, N_θ and $M_{r\theta} = M_{\theta r}$ is arranged in such a way that n_θ and m come out to be unity for the limiting cases of extension and torsion of a flat strip (with $\lambda = 0$ and $\rho_i = -1$) when $k = 1$ and $\psi r_0 = 1$, respectively.

Introducing (27) and (28) into the differential equation (24), we have

$$n_r^{\bullet\bullet} - \frac{2 - 3\lambda^2\rho^2}{\rho(1 + \lambda^2\rho^2)} n_r^{\bullet} - \frac{\lambda^2(1 - \nu_s)}{(1 + \lambda^2\rho^2)^2} n_r = \frac{\psi r_0 \lambda^3\rho^2 - k\lambda^2}{(1 + \lambda^2\rho^2)^2} \tag{29a}$$

where now dots indicate differentiation with respect to ρ. The boundary conditions (20a) become

$$n_r(\rho_i) = n_r(1) = 0 \tag{29b}$$

In terms of ρ and λ, we have further

$$n_\theta = \frac{1 + \lambda^2\rho^2}{\lambda^2\rho} n_r^{\bullet} + n_r, \qquad m = \frac{k\lambda + \psi r_0}{1 + \lambda^2\rho^2}$$
$$\frac{u}{r_0} = \frac{(1 + \lambda^2\rho^2)^2}{\lambda^4\rho^2}\left[n_r^{\bullet} + \frac{(1 - \nu_s)\lambda^2\rho}{1 + \lambda^2\rho^2} n_r - \frac{c r_0 \lambda^2\rho^2 + k\lambda^2\rho + \psi r_0\lambda^3\rho^3}{(1 + \lambda^2\rho^2)^2}\right] \tag{30}$$

For a dimensionless representation of F and T, we introduce the additional parameter

$$\mu = \frac{h}{r_0} \tag{31}$$

Therewith,

$$\frac{F}{Ehr_0} = \int_{\rho_i}^1 \frac{n_\theta}{\sqrt{1 + \lambda^2\rho^2}}\, d\rho - \left[\frac{\mu^2\lambda\rho m}{12(1 + \nu_b)\sqrt{1 + \lambda^2\rho^2}}\right]_{\rho_i}^1$$
$$\frac{T}{Ehr_0^2} = \int_{\rho_i}^1 \frac{\lambda\rho^2 n_\theta}{\sqrt{1 + \lambda^2\rho^2}}\, d\rho + \left[\frac{\mu^2\rho(2 + \lambda^2\rho^2)m}{12(1 + \nu_b)\sqrt{1 + \lambda^2\rho^2}}\right]_{\rho_i}^1 \tag{32}$$

We can write (32) in the form

$$F = C_{Fk}k + C_{F\psi}\psi, \qquad T = C_{Tk}k + C_{T\psi}\psi \tag{33}$$

Expressing the integrands of the various integrals in (33) in terms of n_r by means of the first equation in (30), it is not difficult to see, through the use of the differential equation (29a) and the boundary conditions (29b), that the contribution of m to the coupling coefficients $C_{F\psi}$ and C_{Tk} is negligible because of the basic assumptions that terms of the order h/R and $h^2/(r_0 - r_i)^2$ are negligible compared to unity. With this observation, we have the following expressions for the stiffness coefficients C,

$$\frac{C_{Fk}}{2Ehr_0} = C_{Fk}^M + \mu^2 C_{Fk}^B = \left[\int_{\rho_i}^1 \frac{n_{\theta k}}{2\sqrt{1 + \lambda^2\rho^2}}\, d\rho\right] + \mu^2\left[\frac{-\lambda^2\rho}{24(1 + \nu_b)(1 + \lambda^2\rho^2)^{\frac{3}{2}}}\right]_{\rho_i}^1$$
$$\frac{C_{F\psi}}{2Ehr_0^2} = C_{F\psi}^M = \int_{\rho_i}^1 \frac{n_{\theta\psi}}{2\sqrt{1 + \lambda^2\rho^2}}\, d\rho = \int_{\rho_i}^1 \frac{\lambda\rho^2 n_{\theta k}}{2\sqrt{1 + \lambda^2\rho^2}}\, d\rho = C_{Tk}^M = \frac{C_{Tk}}{2Ehr_0^2} \tag{34}$$
$$\frac{C_{T\psi}}{2Ehr_0^3} = C_{T\psi}^M + \mu^2 C_{T\psi}^B = \left[\int_{\rho_i}^1 \frac{\lambda\rho^2 n_{\theta\psi}}{2\sqrt{1 + \lambda^2\rho^2}}\, d\rho\right] + \mu^2\left[\frac{\rho(2 + \lambda^2\rho^2)}{24(1 + \nu_b)(1 + \lambda^2\rho^2)^{\frac{3}{2}}}\right]_{\rho_i}^1$$

In this $n_{\theta k}$ and $n_{\theta\psi}$ stand for n_θ for the case of *no twist* ($\psi = 0, k = 1$), and for the case of *pure twist* ($k = 0, \psi r_0 = 1$), respectively, and C_{Fk} and $C_{T\psi}$ are written in a way to separate the effect of membrane and bending stresses. The relation

$C_{F\psi} = C_{Tk}$ is guaranteed by reciprocity and can be verified directly by means of Green's theorem.

Inverting equation (33), we further have

$$k = K_{kF}F + K_{kT}T, \qquad \psi = K_{\psi F}F + K_{\psi T}T \tag{35}$$

where the flexibility coefficients K are expressed in terms of the stiffness coefficients C as follows

$$\begin{aligned} K_{kF} &= \frac{C_{T\psi}^M + \mu^2 C_{T\psi}^B}{2Ehr_0\Delta}, \qquad K_{\psi T} = \frac{C_{Fk}^M + \mu^2 C_{Fk}^B}{2Ehr_0^3\Delta} \\ K_{kT} &= K_{\psi F} = -\frac{C_{F\psi}^M}{2Ehr_0^2\Delta} = -\frac{C_{Tk}^M}{2Ehr_0^2\Delta} \end{aligned} \tag{36}$$

with

$$\begin{aligned} \Delta &= \Delta_0 + \mu^2\Delta_1 + \mu^4\Delta_2 \\ &= (C_{Fk}^M C_{T\psi}^M - C_{F\psi}^M C_{Tk}^M) + \mu^2(C_{Fk}^M C_{T\psi}^B + C_{Fk}^B C_{T\psi}^M) + \mu^4 C_{Fk}^B C_{T\psi}^B \end{aligned} \tag{37}$$

6. Parametric expansion for small λ and determination of *c*. The form of the boundary value problem (29) indicates the possibility of a perturbation expansion

$$n_r = \sum_{m=0}^{\infty} n_m \lambda^{2m+p} \tag{38}$$

for values of λ smaller than unity. Taking $p = 2$ for the case of *no twist* and $p = 3$ for the case of *pure twist,* we obtain a sequence of boundary value problems for the coefficients of the expansion (38),

$$\text{(i)} \quad \rho \ddot{n}_0 - 2\dot{n}_0 = (-\rho, \rho^3), \qquad n_0(\rho_i) = n_0(1) = 0 \tag{39a}$$

$$\begin{aligned} \text{(ii)} \quad (\rho\ddot{n}_1 - 2\dot{n}_1) + [2\rho^3\ddot{n}_0 + \rho^2\dot{n}_0 - (1-\nu_s)\rho n_0] = 0,& \\ n_1(\rho_i) = n_1(1) = 0& \end{aligned} \tag{39b}$$

$$\begin{aligned} \text{(iii)} \quad (\rho\ddot{n}_m - 2\dot{n}_m) + [2\rho^3\ddot{n}_{m-1} + \rho^2\dot{n}_{m-1} - (1-\nu_s)\rho n_{m-1}]& \\ + (\rho^5\ddot{n}_{m-2} + 3\rho^4\dot{n}_{m-2}) = 0, \quad n_m(\rho_i) = n_m(1) = 0, \quad (m \geqq 2)& \end{aligned} \tag{39c}$$

We limit ourselves here to solving (39a) and (39b) and obtain in this way

$$\begin{aligned} n_r = k\lambda^2\{[a_{k0} + \tfrac{1}{2}\rho^2 + b_{k0}\rho^3] + \lambda^2[a_{k1} - \tfrac{1}{2}(1-\nu_s)a_{k0}\rho^2 + b_{k1}\rho^3& \\ - \tfrac{1}{8}(5+\nu_s)\rho^4 - \tfrac{1}{10}(14+\nu_s)b_{k0}\rho^5] + \cdots\}& \\ + \psi r_0\lambda^3\{[a_{\psi 0} + b_{\psi 0}\rho^3 + \tfrac{1}{4}\rho^4] + \lambda^2[a_{\psi 1} - \tfrac{1}{2}(1-\nu_s)a_{\psi 0}\rho^2 + b_{\psi 1}\rho^3& \\ - \tfrac{1}{10}(14+\nu_s)b_{\psi 0}\rho^5 - \tfrac{1}{72}(27+\nu_s)\rho^6] + \cdots\}& \end{aligned} \tag{40}$$

where

$$a_{k0} = -\frac{\rho_i^2(1-\rho_i)}{2(1-\rho_i^3)}, \quad b_{k0} = -\frac{1-\rho_i^2}{2(1-\rho_i^3)}, \quad a_{\psi 0} = \frac{\rho_i^3(1-\rho_i)}{4(1-\rho_i^3)}, \quad b_{\psi 0} = -\frac{1-\rho_i^4}{4(1-\rho_i^3)}$$

$$a_{k1} = \frac{\rho_i^2(1-\rho_i)}{2(1-\rho_i^3)}\left[(1-\nu_s)a_{k0} - \frac{1}{5}(14+\nu_s)b_{k0}\rho_i(1+\rho_i) - \frac{1}{4}(5+\nu_s)\rho_i\right]$$

$$b_{k1} = \frac{1}{2(1-\rho_i^3)}\left[(1-\nu_s)a_{k0}(1-\rho_i^2) + \frac{1}{5}(14+\nu_s)b_{k0}(1-\rho_i^5) + \frac{1}{4}(5+\nu_s)(1-\rho_i^4)\right] \tag{41}$$

$$a_{\psi 1} = \frac{\rho_i^2(1-\rho_i)}{2(1-\rho_i^3)}\left[(1-\nu_s)a_{\psi 0} - \frac{1}{5}(14+\nu_s)b_{\psi 0}\rho_i(1+\rho_i) - \frac{(27+\nu_s)\rho_i(1-\rho_i^3)}{36(1-\rho_i)}\right]$$

$$b_{\psi 1} = \frac{1}{2(1-\rho_i^3)}\left[(1-\nu_s)a_{\psi 0}(1-\rho_i^2) + \frac{1}{5}(14+\nu_s)b_{\psi 0}(1-\rho_i^5) + \frac{1}{18}(27+\nu_s)(1-\rho_i^6)\right]$$

Correspondingly, we have

$$n_\theta = k\{[1+3b_{k0}\rho] + \lambda^2[\nu_s a_{k0} + 3b_{k1}\rho - \tfrac{1}{2}(2+\nu_s)\rho^2 - \tfrac{1}{2}(6+\nu_s)b_{k0}\rho^3] + \cdots\} + \psi r_0\lambda\{[3b_{\psi 0}\rho + \rho^2] + \lambda^2[\nu_s a_{\psi 0} + 3b_{\psi 1}\rho - \tfrac{1}{2}(6+\nu_s)b_{\psi 0}\rho^3 - \tfrac{1}{12}(12+\nu_s)\rho^4] + \cdots\} \tag{42}$$

$$m = (k\lambda + \psi r_0)\{1 - \lambda^2\rho^2 + \cdots\} \tag{43}$$

To obtain the corresponding expression for u and for the as yet undetermined constant c, we write the last equation in (26) in the form

$$r_0^{-1}\lambda^4\rho^2 u = (1+\lambda^2\rho^2)^2 n_r^{\bullet} + (1-\nu_s)\lambda^2\rho(1+\lambda^2\rho^2)n_r - (cr_0\lambda^2\rho^2 + k\lambda^2\rho + \psi r_0\lambda^3\rho^3) \tag{44}$$

We now choose c to have a simple dependence on λ which still leads to the correct limiting result for $\lambda = 0$ (the flat strip) and for $\lambda = \infty$ (the circular ring plate sector). This is accomplished by writing c in the form

$$c = \frac{kc_k + \psi r_0\lambda c_\psi}{r_0(1+\lambda^2)} \tag{45}$$

where c_k and c_ψ are independent of λ. For $\lambda^2 \ll 1$, we substitute λ^2-expansions for c and for n_r into (44) to get

$$
\begin{aligned}
r_0^{-1}\lambda^2\rho^2 u = {} & k\{(\overset{\bullet}{n}_{k,0} - c_k\rho^2 - \rho) + \lambda^2[\overset{\bullet}{n}_{k,1} + 2\rho^2\overset{\bullet}{n}_{k,0} + (1 - \nu_s)\rho n_{k,0} + c_k\rho^2] \\
& + \textstyle\sum_{m=0}^{\infty} [(\rho^4\overset{\bullet}{n}_{k,m} + 2\rho^2\overset{\bullet}{n}_{k,m+1} + \overset{\bullet}{n}_{k,m+2}) \\
& + \rho(1 - \nu_s)(\rho^2 n_{k,m} + n_{k,m+1}) + (-1)^{m+1}c_k\rho^2]\lambda^{2m+4}\} \\
& + \psi r_0\lambda\{(\overset{\bullet}{n}_{\psi,0} - c_\psi\rho^2 - \rho^3) + \lambda^2[\overset{\bullet}{n}_{\psi,1} + 2\rho^2\overset{\bullet}{n}_{\psi,0} \\
& + (1 - \nu_s)\rho n_{\psi,0} + c_\psi\rho^2] \\
& + \textstyle\sum_{m=0}^{\infty} [(\rho^4\overset{\bullet}{n}_{\psi,m} + 2\rho^2\overset{\bullet}{n}_{\psi,m+1} + \overset{\bullet}{n}_{\psi,m+2}) \\
& + \rho(1 - \nu_s)(\rho^2 n_{\psi,m} + n_{\psi,m+1}) + (-1)^{m+1}c_\psi\rho^2]\lambda^{2m+4}\}
\end{aligned} \tag{46}
$$

where $n_{k,m}$ and $n_{\psi,m}$ are the mth term in the perturbation expansion (40) for the case of no twist and the case of pure twist respectively. We determine c_k and c_ψ, by considering the limiting case $\lambda = 0$. For this limiting case, u must remain finite. In order that this be so, the leading terms in both portions on the right hand side of (46) must vanish. In this way, we obtain

$$
\begin{aligned}
\overset{\bullet}{n}_{k,0} - c_k\rho^2 - \rho &= (3b_{k0} - c_k)\rho^2 = 0, \\
\overset{\bullet}{n}_{\psi,0} - c_\psi\rho^2 - \rho^3 &= (3b_{\psi 0} - c_\psi)\rho^2 = 0
\end{aligned} \tag{47}
$$

or

$$
c_k = 3b_{k0} = -\frac{3}{2}\frac{1 - \rho_i^2}{1 - \rho_i^3}, \qquad c_\psi = 3b_{\psi 0} = -\frac{3}{4}\frac{1 - \rho_i^4}{1 - \rho_i^3} \tag{48}
$$

and then

$$
\begin{aligned}
\frac{u}{r_0} = {} & k\left\{\left[3b_{k1} - \nu_s\rho + \frac{3}{2}(2 - \nu_s)b_{k0}\rho^2\right] + \lambda^2\left[\cdots\right] + \cdots\right\} \\
& + \psi r_0\lambda\left\{\left[3b_{\psi 1} + \frac{3}{2}(2 - \nu_s)b_{\psi 0}\,\rho^2 - \frac{\nu_s}{3}\rho^3\right] + \lambda^2\left[\cdots\right] + \cdots\right\}
\end{aligned} \tag{49}
$$

The determination of the higher order terms in u requires terms in the expansion for n_r beyond those given in (40). Note that when $\rho_i = -1$, $c_k = c_\psi = 0$ so that for this case $c = 0$, as it should be.

Having determined c, the axial and circumferential displacement components given by (12b) now take the form

$$
\begin{aligned}
u_z &= ka\left[\theta - \frac{3a^2}{2(a^2 + r_0^2)}\left(\frac{1 - \rho_i^2}{1 - \rho_i^3}\right)\rho\sin\theta\right] \\
&\qquad - \frac{3\psi a^3 r_0^2}{4(a^2 + r_0^2)}\left(\frac{1 - \rho_i^4}{1 - \rho_i^3}\right)\rho\sin\theta \\
v &= \psi a r_0\left[\rho\theta - \frac{3a^2}{4(a^2 + r_0^2)}\left(\frac{1 - \rho_i^4}{1 - \rho_i^3}\right)(\sin\theta - \theta\cos\theta)\right] \\
&\qquad - \frac{3ka^4}{2r_0(a^2 + r_0^2)}\left(\frac{1 - \rho_i^2}{1 - \rho_i^3}\right)(\sin\theta - \theta\cos\theta)
\end{aligned} \tag{50}
$$

When $\rho_i = -1$, equations (50) reduce as they should to equations (11b).

Corresponding expressions for the stiffness coefficients as defined in (34) are

$$C_{Fk}^{M} = \frac{1}{2}\left\{\left[(1-\rho_i) + \frac{3}{2}b_{k0}(1-\rho_i^2)\right] + \lambda^2\left[\nu_s a_{k0}(1-\rho_i) - \frac{9+\nu_s}{8}b_{k0}(1-\rho_i^4) - \frac{3}{4}b_{k1}(1-\rho_i^2) - \frac{3+\nu_s}{6}(1-\rho_i^3)\right] + 0(\lambda^4)\right\}$$

$$C_{Fk}^{B} = -\frac{\lambda^2}{24(1+\nu_b)}\left\{(1-\rho_i) - \frac{3}{2}\lambda^2(1-\rho_i^3) + 0(\lambda^4)\right\}$$

$$C_{F\psi}^{M} = C_{Tk}^{M} = \frac{\lambda}{2}\left\{\left[\frac{1}{3}(1-\rho_i^3) + \frac{3}{2}b_{\psi 0}(1-\rho_i^2)\right] + \lambda^2\left[\nu_s a_{\psi 0}(1-\rho_i) + \frac{3}{2}b_{\psi 1}(1-\rho_i^2) - \frac{1}{8}b_{\psi 0}(1-\rho_i^4) - \frac{18+\nu_s}{60}(1-\rho_i^5)\right] + 0(\lambda^4)\right\} \tag{51}$$

$$C_{T\psi}^{M} = \frac{\lambda^2}{2}\left\{\left[\frac{3}{4}b_{\psi 0}(1-\rho_i^4) + \frac{1}{5}(1-\rho_i^5)\right] + \lambda^2\left[\frac{\nu_s}{3}a_{\psi 0}(1-\rho_i^3) + \frac{3}{4}b_{\psi 1}(1-\rho_i^4) - \frac{9+\nu_s}{12}b_{\psi 0}(1-\rho_i^6) - \frac{18+\nu_s}{84}(1-\rho_i^7)\right] + 0(\lambda^4)\right\}$$

$$C_{T\psi}^{B} = \frac{1}{12(1+\nu_b)}\{(1-\rho_i) - \lambda^2(1-\rho_i^3) + 0(\lambda^4)\}$$

It is evident from this that the contribution of C_{Fk}^B to C_{Fk} is negligible for small λ. On the other hand, the contribution of $C_{T\psi}^B$ to $C_{T\psi}$ is important in the range $\lambda^2 = 0(\mu^2)$. Since μ^2 is always small compared to unity, we need the leading term only of the perturbation solution for $C_{T\psi}^B$.

It was found in [7], for the case $\rho_i = -1$ that the λ^2-expansion is practical for values of λ up to about 0.5. It was later shown in [4] that an alternate expansion is possible which is valid for all values of λ. This alternate expansion is in terms of a new parameter $\eta^2 = \lambda^2/(1+\lambda^2)$. We may note here that the two leading terms of this η^2-expansion can be obtained by merely replacing in equation (40), (42), (43), (49) and (51) all λ^2 factors inside braces by η^2.

When $\rho_i = -1$, we have $b_{km} = b_{\psi m} = 0$ for $m = 0, 1, \cdots$ and the above results reduce to those obtained in [2]. In addition, we have from (37) and (51)

$$\frac{\Delta_0}{\Delta_1} = \frac{C_{Fk}^M C_{T\psi}^M - C_{F\psi}^M C_{Tk}^M}{C_{Fk}^M C_{T\psi}^B} = 0(\lambda^2) \tag{52}$$

for $\lambda^2 \ll 1$. Therefore, the contribution of the Δ_1 term cannot be neglected in (36) and (37). In fact, for fixed values of μ and as $\lambda \to 0$, we have from (51)

$$K_{kF} \sim \frac{1}{2Ehr_0} \frac{1 + \frac{6 + \nu_b}{5} \frac{\lambda^2}{\mu^2}}{1 + \frac{8 + 8\nu_b}{15} \frac{\lambda^2}{\mu^2}} \rightarrow \frac{1}{2Ehr_0}$$

$$K_{kT} = K_{\psi F} \sim -\frac{1}{Eh^2 r_0} \frac{(1 + \nu_b) \frac{\lambda}{\mu}}{1 + \frac{8 + 8\nu_b}{15} \frac{\lambda^2}{\mu^2}} \rightarrow 0 \tag{53}$$

$$K_{\psi T} \sim \frac{3}{2Gh^3 r_0} \frac{1}{1 + \frac{8 + 8\nu_b}{15} \frac{\lambda^2}{\mu^2}} \rightarrow \frac{3}{2Gh^3 r_0}$$

the limiting values for $\lambda = 0$ being the known elementary results for the stretching and twisting of rectangular plates.

In regard to the approximate formulas for small λ in (53), we may note that the result for $K_{\psi T}$ was first obtained by Chen Chu [1] while the result for K_{kF} was first obtained in [2]. The approximate formulas for $K_{kT} = K_{\psi F}$ have not previously been given.

7. Parametric expansion for large λ. We introduce a new parameter $\epsilon = 1/\lambda$ and write (24) as

$$\ddot{n}_r + \frac{3\rho^2 - 2\epsilon^2}{\rho(\rho^2 + \epsilon^2)} \dot{n}_r - \frac{\epsilon^2(1 - \nu_s)}{(\rho^2 + \epsilon^2)^2} n_r = \frac{\psi r_0 \epsilon \rho^2 - k\epsilon^2}{(\rho^2 + \epsilon^2)^2} \tag{54}$$

The solution of (54) may be expanded in the form

$$n_r = \sum_{m=0}^{\infty} N_m \epsilon^{2m+p} \tag{55}$$

where the coefficients N_m are independent of ϵ, and where we take $p = 2$ for the case of no twist and $p = 1$ for the case of pure twist. The expansion (55) will be valid as long as $\epsilon^2 < \rho_{\min}^2$ which means that it is suitable only provided an appropriate neighborhood of $\rho = 0$ lies outside the shell surface.

Substitution of (55) into (54) and into the boundary conditions (29b) leads to the following sequence of boundary value problems

$$\text{(i)} \quad \rho^5 \ddot{N}_0 + 3\rho^4 \dot{N}_0 = (-\rho, \rho^3), \qquad N_0(\rho_i) = N_0(1) = 0 \tag{56a}$$

$$\text{(ii)} \quad (\rho^5 \ddot{N}_1 + 3\rho^4 \dot{N}_1) + [2\rho^3 \ddot{N}_0 + \rho^2 \dot{N}_0 - (1 - \nu_s)\rho N_0] = 0, \qquad N_1(\rho_i) = N_1(1) = 0 \tag{56b}$$

$$\text{(iii)} \quad (\rho^5 \ddot{N}_m + 3\rho^4 \dot{N}_m) + [2\rho^3 \ddot{N}_{m-1} + \rho^2 \dot{N}_{m-1} - (1 - \nu_s)\rho N_{m-1}] + (\rho \ddot{N}_{m-2} - 2\dot{N}_{m-2}) = 0, \qquad N_m(\rho_i) = N_m(1) = 0 \quad (m \geqq 2) \tag{56c}$$

We limit ourselves here to solving (56a) and (56b). With these we obtain

$$
\begin{aligned}
n_r = k\epsilon^2 \Big\{ & \Big[A_{k0} + B_{k0} \frac{1}{\rho^2} + \frac{1 + 2 \ln \rho}{4\rho^2} \Big] \\
& + \epsilon^2 \Big[A_{k1} + B_{k1} \frac{1}{\rho^2} - A_{k0} \frac{(1 - \nu_s)(1 + 2 \ln \rho)}{4\rho^2} \\
& \qquad - B_{ko} \frac{9 + \nu_s}{8\rho^4} - \frac{9 + 5\nu_s}{64\rho^4} - \frac{(9 + \nu_s) \ln \rho}{16\rho^4} \Big] + \cdots \Big\} \\
& + \psi r_0 \epsilon \Big\{ \Big[A_{\psi 0} + B_{\psi 0} \frac{1}{\rho^2} + \frac{\ln \rho}{2} \Big] + \epsilon^2 \Big[A_{\psi 1} + B_{\psi 1} \frac{1}{\rho^2} \\
& \qquad - A_{\psi 0} \frac{(1 - \nu_s)(1 + 2 \ln \rho)}{4\rho^2} - B_{\psi 0} \frac{9 + \nu_s}{8\rho^4} \\
& \qquad - \frac{1 + 2 \ln \rho}{8\rho^2} - \frac{(1 - \nu_s)\langle 1 + 2 \ln \rho + 2 (\ln \rho)^2 \rangle}{16\rho^2} \Big] + \cdots \Big\}
\end{aligned}
\tag{57}
$$

where

$$
\begin{aligned}
A_{k0} &= \frac{\ln \rho_i}{2(1 - \rho_i^2)}, \quad B_{k0} = -\frac{1}{4}\left(1 + \frac{2 \ln \rho_i}{1 - \rho_i^2}\right), \quad A_{\psi 0} = -B_{\psi 0} = \frac{\rho_i^2 \ln \rho_i}{2(1 - \rho_i^2)} \\
A_{k1} &= -\frac{1}{32\rho_i^2(1 - \rho_i^2)} \{4(1 - \nu_s)A_{k0}\, \rho_i^2 \ln \rho_i + (9 + \nu_s)B_{k0}(1 - \rho_i^2) \\
& \qquad + (9 + 5\nu_s)(1 - \rho_i^2) + 2(9 + \nu_s) \ln \rho_i\} \\
B_{k1} &= \frac{1}{64\rho_i^2(1 - \rho_i^2)} \{4(1 - \nu_s)A_{k0}\rho_i^2(1 - \rho_i^2 + 2 \ln \rho_i) \\
& \qquad + 2(9 + \nu_s)B_{k0}(1 - \rho_i^4) + (9 + 5\nu_s)(1 - \rho_i^4) + 4(9 + \nu_s) \ln \rho_i\} \\
A_{\psi 1} &= -\frac{1}{16\rho_i^2(1 - \rho_i^2)} \{4(1 - \nu_s)A_{\psi 0}\rho_i^2 \ln \rho_i + (9 + \nu_s)B_{\psi 0}(1 - \rho_i^2) \\
& \qquad + 2[(3 - \nu_s) + (1 - \nu_s) \ln \rho_i]\rho_i^2 \ln \rho_i\} \\
B_{\psi 1} &= \frac{1}{16\rho_i^2(1 - \rho_i^2)} \{2(1 - \nu_s)A_{\psi 0}\, \rho_i^2(1 - \rho_i^2 + 2 \ln \rho_i) \\
& \qquad + (9 + \nu_s)B_{\psi 0}(1 - \rho_i^4) + \rho_i^2[(3 - \nu_s)(1 - \rho_i^2) \\
& \qquad + 2(3 - \nu_s) \ln \rho_i + 2(1 - \nu_s)(\ln \rho_i)^2]\}
\end{aligned}
\tag{58}
$$

Corresponding formulas for n_θ, m and u are

$$
\begin{aligned}
n_\theta = k\epsilon^2 \Big\{ & \Big[A_{k0} - B_{k0} \frac{1}{\rho^2} + \frac{1 - 2 \ln \rho}{4\rho^2} \Big] \\
& + \epsilon^2 \Big[A_{k1} - B_{k1} \frac{1}{\rho^2} + A_{k0} \frac{(1 - \nu_s)(2 \ln \rho - 1)}{4\rho^2} \\
& \qquad + B_{k0} \frac{11 + 3\nu_s}{8\rho^4} - \frac{(9 - 11\nu_s) - 4(11 + 3\nu_s) \ln \rho}{64\rho^4} \Big] + \cdots \Big\} \\
& + \psi r_0 \epsilon \Big\{ \Big[A_{\psi 0} - B_{\psi 0} \frac{1}{\rho^2} + \frac{1 + \ln \rho}{2} \Big] \\
& + \epsilon^2 \Big[A_{\psi 1} - B_{\psi 1} \frac{1}{\rho^2} + A_{\psi 0} \frac{(1 - \nu_s)(2 \ln \rho - 1)}{4\rho^2} + B_{\psi 0} \frac{11 + 3\nu_s}{8\rho^4} \\
& + \frac{(5 + \nu_s) + 2(1 + \nu_s) \ln \rho + 2(1 - \nu_s)(\ln \rho)^2}{16\rho^2} \Big] + \cdots \Big\}
\end{aligned}
\tag{59}
$$

$$m = \frac{k\epsilon + \psi r_0 \epsilon^2}{\rho^2}\left\{1 - \frac{\epsilon^2}{\rho^2} + \frac{\epsilon^4}{\rho^4} + \cdots\right\} \tag{60}$$

and

$$\begin{aligned}
\frac{u}{r_0} = k\epsilon^2 \Bigg\{ & \left[A_{k0}(1 - \nu_s)\rho - B_{k0}\frac{1 + \nu_s}{\rho} - \frac{(3 + \nu_s) + 2(1 + \nu_s)\ln\rho}{4\rho}\right] \\
& + \epsilon^2 \Bigg[A_{k1}(1 - \nu_s)\rho - B_{k1}\frac{1 + \nu_s}{\rho} \\
& + A_{k0}\frac{2(1 - \nu_s^2)\ln\rho + (3 + \nu_s)(1 - \nu_s)}{4\rho} + B_{k0}\frac{(3 + \nu_s)(1 + \nu_s)}{8\rho^3} \\
& + \frac{(3 + \nu_s)(1 + \nu_s)\ln\rho}{16\rho^3} - 3b_{k0} + \frac{7 + 4\nu_s + 5\nu_s^2}{64\rho^3}\Bigg] + \cdots\Bigg\} \\
+ \psi r_0 \epsilon \Bigg\{ & \left[A_{\psi 0}(1 - \nu_s)\rho - B_{\psi 0}\frac{1 + \nu_s}{\rho} + \frac{(1 - \nu_s)\rho\ln\rho - \rho}{2}\right. \\
& + \epsilon^2 \Bigg[A_{\psi 1}(1 - \nu_s)\rho - B_{\psi 1}\frac{1 + \nu_s}{\rho} \\
& + A_{\psi 0}\frac{2(1 - \nu_s^2)\ln\rho + (3 + \nu_s)(1 - \nu_s)}{4\rho} + B_{\psi 0}\frac{(3 + \nu_s)(1 + \nu_s)}{8\rho^3} \\
& + \frac{(5 - \nu_s^2)\ln\rho}{\rho} + \frac{(1 - \nu_s^2)(\ln\rho)^2}{8\rho} - 3b_{\psi 0} + \frac{13 + 4\nu_s - \nu_s^2}{16\rho}\Bigg] + \cdots\Bigg\}
\end{aligned} \tag{61}$$

The corresponding two-term ϵ^2 expansions for the stiffness coefficients are

$$\begin{aligned}
C_{Fk}^M = -\frac{\epsilon^3}{2}\Bigg\{ & \left[A_{k0}\ln\rho_i + B_{k0}\frac{1 - \rho_i^2}{2\rho_i^2} + \frac{\ln\rho_i}{4\rho_i^2}\right] \\
& + \epsilon^2\Bigg[A_{k1}\ln\rho_i + B_{k1}\frac{1 - \rho_i^2}{2\rho_i^2} + A_{k0}\frac{(1 - \rho_i^2) - (1 - \nu_s)\ln\rho_i}{4\rho_i^2} \\
& - B_{k0}\frac{(15 + 3\nu_s)(1 - \rho_i^4)}{32\rho_i^4} \\
& + \frac{(1 - 7\nu_s)(1 - \rho_i^4) - (15 + 3\nu_s)\ln\rho_i}{128\rho_i^4}\Bigg] + \cdots\Bigg\} \\
C_{Fk}^B = \frac{\epsilon}{24(1 + \nu_b)\rho_i} & \left\{(1 - \rho_i^2) - \frac{3\epsilon^2}{2\rho_i^2}(1 - \rho_i^4) + \cdots\right\} \\
C_{F\psi}^M = C_{Tk}^M = \frac{\epsilon^2}{2}\Bigg\{ & \left[A_{\psi 0}\ln\rho_i + B_{\psi 0}\frac{1 - \rho_i^2}{2\rho_i^2} + \frac{2\ln\rho_i + (\ln\rho_i)^2}{4}\right] \\
& + \epsilon^2\Bigg[A_{\psi 1}\ln\rho_i + B_{\psi 1}\frac{1 - \rho_i^2}{2\rho_i^2} + A_{\psi 0}\frac{\nu_s(1 - \rho_i^2) - (1 - \nu_s)\ln\rho_i}{4\rho_i^2} \\
& - B_{\psi 0}\frac{(15 + 3\nu_s)(1 - \rho_i^2)}{32\rho_i^2} - \frac{(1 + \nu_s)(1 - \rho_i^2)}{32\rho_i^2} \\
& - \frac{(1 - \nu_s)(\ln\rho_i)^2}{16\rho_i^2}\Bigg] + \cdots\Bigg\}
\end{aligned} \tag{62}$$

$$C^M_{T\psi} = \frac{\epsilon}{2}\left\{\left[A_{\psi 0}\frac{1-\rho_i^2}{2} + B_{\psi 0}\ln\rho_i + \frac{1-\rho_i^2}{8} - \frac{\rho_i^2\ln\rho_i}{4}\right] + \epsilon^2\left[A_{\psi 1}\frac{1-\rho_i^2}{2}\right.\right.$$
$$+ B_{\psi 1}\ln\rho_i + A_{\psi 0}\frac{(3-\nu_s)\ln\rho_i - (1-\nu_s)(\ln\rho_i)^2}{4} + B_{\psi 0}\frac{3(5+\nu_s)(1-\rho_i^2)}{16\rho_i^2}$$
$$\left.\left. - \frac{3(1+\nu_s)\ln\rho_i - 3(1-\nu_s)(\ln\rho_i)^2 + 2(1-\nu_s)(\ln\rho_i)^3}{48}\right] + \cdots\right\}$$

$$C^B_{T\psi} = -\frac{\epsilon^3}{48(1+\nu_b)\rho_i^2}\{(1-\rho_i^2) + \cdots\}$$

We now have that the contribution of $C^B_{T\psi}$ to $C_{T\psi}$ is always negligible since the parametric expansion procedure is valid only for $\epsilon^2/\rho_i^2 < 1$. On the other hand, the contribution of C^B_{Fk} to C_{Fk} is important in the range $\epsilon^2 = 0(\mu^2)$. If in addition $\epsilon^2/\rho_i^2 = 0(\mu^2)$, then we need only to retain the leading term of the perturbation expansion for C^B_{Fk} in (62).

The leading term of the foregoing ϵ^2-expansion of the solution agrees with the corresponding result obtained earlier by means of shallow shell theory [3].

8. Power series solution. A power series solution which is convergent for all finite λ and for $-1 \leqq \rho \leqq 1$ may be obtained by introducing a new independent variable

$$x = \frac{r}{\alpha} = \frac{\lambda\rho}{\sqrt{1+\lambda^2\rho^2}} \tag{63}$$

Introducton of (63) into (29a) transforms the differential equation for n_r into

$$\frac{d^2n_r}{dx^2} - \frac{2-5x^2}{x(1-x^2)}\frac{dn_r}{dx} - \frac{1-\nu_s}{1-x^2}n_r = \frac{\psi r_0}{\lambda}\frac{x^2}{(1-x^2)^2} - \frac{k}{1-x^2} \tag{64}$$

A pair of fundamental solutions of the homogeneous equation are

$$\varphi_1 = 1 + \textstyle\sum_{m=1}^\infty a_m x^{2m} \qquad \varphi_2 = x^3[1 + \textstyle\sum_{m=1}^\infty b_m x^{2m}] \tag{65}$$

where

$$a_m = \prod_{p=1}^m \frac{4(p-1)(p-4) + (1-\nu_s)}{2p(2p-3)},$$
$$b_m = \prod_{p=1}^m \frac{(2p+1)(2p-5) + (1-\nu_s)}{2p(2p+3)} \tag{66}$$

An exact particular solution for the k-contribution of the right hand side of (64) is

$$n_{rpk} = k\varphi_k, \qquad \varphi_k = (1-\nu_s)^{-1} \tag{67}$$

The corresponding ψ-contribution is here obtained as

$$n_{rp\psi} = \psi r_0\lambda^{-1}\varphi_\psi, \qquad \varphi_\psi = \tfrac{1}{4}\textstyle\sum_{m=0}^\infty c_m x^{2m+4} \tag{68}$$

In this $c_0 = 1$, $c_1 = -\frac{1}{4}(3+\nu_s)$ and

$$c_m = \frac{[2(4m-5)(m+1) + (1-\nu_s)]c_{m-1} - [4m(m-3) + (1-\nu_s)]c_{m-2}}{2(m+2)(2m+1)}, \tag{69}$$

for $m \geqq 2$.

Altogether, we have as an exact solution to the differential equation (64)

$$n_r = C_1\varphi_1 + C_2\varphi_2 + k\varphi_k + \psi r_0\lambda^{-1}\varphi_\psi \tag{70}$$

where C_1 and C_2 are constants to be determined by the boundary conditions (29b). We note that for reasons of symmetry, $C_2 = 0$ when $\rho_i = -1$. For this case the above results reduce to those obtained in [7].

The power series solution (70) is used within the context of the present work to establish the analyticity of n_r in the interval $-1 \leqq \rho \leqq 1$ for all finite λ. Because of the regular singular point at $x = 0$, this analyticity is not obvious from the form of the differential equation. Having n_r analytic for the range of values of ρ and λ under consideration, we will be in a position to make a definite statement on the truncation error in a solution of the boundary value problem by means of finite difference procedures.

9. Finite difference solution. We choose a set of M equally spaced tabular points $\rho_1, \rho_2, \cdots, \rho_M$ in the interval $(\rho_i, 1)$ with $\rho_1 = \rho_i$ and $\rho_M = 1$. The spacing t between two consecutive tabular points is given by

$$t = \frac{1 - \rho_i}{M - 1} \tag{71}$$

We add to this set two other points, $\rho_0 = \rho_i - t$ and $\rho_{M+1} = 1 + t$. Writing

$$f_j = n_r(\rho_j) \qquad j = 0, 1, \cdots, M, M + 1 \tag{72}$$

we approximate first and second derivatives by

$$\dot{n}_r(\rho_j) = \frac{1}{2t}(f_{j+1} - f_{j-1}), \qquad \ddot{n}_r(\rho_j) = \frac{1}{t^2}(f_{j+1} - 2f_j + f_{j-1}) \tag{73}$$

for $j = 1, 2, 3, \cdots, M$. Furthermore, we require f to satisfy the differential equation (29a) at $\rho = \rho_1, \cdots, \rho_M$ and the boundary conditions (29b) at ρ_1 and ρ_M.

For $j = 2, 3, \cdots, M - 1$, the differential equation (29a) now becomes the finite difference equation

$$A_jf_{j-1} + B_jf_j + C_jf_{j+1} = kD_{kj} + \psi r_0D_{\psi j} \tag{74}$$

where

$$\begin{aligned} A_j &= (1 + \lambda^2\rho_j^2)[\rho_j(1 + \lambda^2\rho_j^2) + \tfrac{1}{2}t(2 - 3\lambda^2\rho_j^2)] \\ B_j &= -\rho_j[2(1 + \lambda^2\rho_j^2)^2 - (1 - \nu_s)t^2\lambda^2] \\ C_j &= (1 + \lambda^2\rho_j^2)[\rho_j(1 + \lambda^2\rho_j^2) - \tfrac{1}{2}t(2 - 3\lambda^2\rho_j^2)] \\ D_{kj} &= -t^2\lambda^2\rho_j\,. \qquad D_{\psi j} = t^2\lambda^3\rho_j^3 \end{aligned} \tag{75}$$

The boundary conditions at ρ_1 and $\rho_M = 1$ become

$$f_1 = f_M = 0 \tag{76}$$

Introduction of (76) into (74) leaves as system of $M - 2$ linear simultaneous algebraic equations for the $M - 2$ unknowns $f_2, f_3, \cdots, f_{M-1}$,

$$\begin{aligned} B_2 f_2 + C_2 f_3 &= kD_{k2} + \psi r_0 D_{\psi 2} \\ A_j f_{j-1} + B_j f_j + C_j f_{j+1} &= kD_{kj} + \psi r_0 D_{\psi j} \\ A_{M-1} f_{M-2} + B_{M-1} f_{M-1} &= kD_{kM-1} + \psi r_0 D_{\psi M-1} \end{aligned} \tag{77}$$

The solution of this system for a given set of values of λ, ν_s, k and ψr_0 is possible by various known techniques.*

The system (77) can be solved very efficiently by the following algorithm. Let

$$\bar{C}_2 = -B_2^{-1} C_2, \qquad \bar{D}_{k2} = B_2^{-1} D_{k2}, \qquad \bar{D}_{\psi 2} = B_2^{-1} D_{\psi 2} \tag{78}$$

The first equation of (77) can be written as

$$f_2 = \bar{C}_2 f_3 + \bar{D}_{k2} k + \bar{D}_{\psi 2} \psi r_0 \tag{79}$$

Using this result, we can eliminate f_2 from equation (77) for $j = 3$. The resulting equation will then be used to express f_3 in terms of f_4 so that

$$f_3 = \bar{C}_3 f_4 + \bar{D}_{k3} k + \bar{D}_{\psi 3} \psi r_0 \tag{80}$$

where

$$\begin{aligned} \bar{C}_3 &= -(B_3 + A_3 \bar{C}_2)^{-1} C_3 \\ \bar{D}_{k3} &= (B_3 + A_3 \bar{C}_2)^{-1} (D_{k3} - A_3 \bar{D}_{k2}) \\ \bar{D}_{\psi 3} &= (B_3 + A_3 \bar{C}_2)^{-1} (D_{\psi 3} - A_3 \bar{D}_{\psi 2}) \end{aligned} \tag{81}$$

Repeating this process for the succeeding difference equations, we get for $j = 3, 4, \cdots, M - 2)$,

$$f_j = \bar{C}_j f_{j+1} + \bar{D}_{kj} k + \bar{D}_{\psi j} \psi r_0 \tag{82}$$

where

$$\begin{aligned} \bar{C}_j &= -(B_j + A_j \bar{C}_{j-1})^{-1} C_j \\ \bar{D}_{kj} &= (B_j + A_j \bar{C}_{j-1})^{-1} (D_{kj} - A_j \bar{D}_{kj-1}) \\ \bar{D}_{\psi j} &= (B_j + A_j \bar{C}_{j-1})^{-1} (D_{\psi j} - A_j \bar{D}_{\psi j-1}) \end{aligned} \tag{83}$$

Finally, elimination of f_{M-2} from the last equation in (77) yields

$$f_{M-1} = \bar{D}_{kM-1} k + \bar{D}_{\psi M-1} \psi r_0 \tag{84}$$

* It seems appropriate at this point to mention that the expression for the first and second derivatives of n_r given by (73) are the Stirling formulas for finite (central) difference interpolation truncated after the first three terms with the system (77) being the finite difference analogue of the boundary value problem (29). It was shown in the last section that n_r is analytic in $-1 \leqq \rho \leqq 1$ for all finite λ; therefore $n_r^{\cdots}$ is bounded there. It follows that the truncation error associated with the discrete approximation is $0(t^2)$. In view of this, we expect f_j to approach $n_r(\rho_j)$ as $t \to 0$, as long as round-off errors do not play a significant role in our problem.

with

$$\bar{D}_{kM-1} = (B_{M-1} + A_{M-1}\bar{C}_{M-2})^{-1}(D_{kM-1} - A_{M-1}\bar{D}_{kM-2}) \tag{85}$$

$$D_{\psi M-1} = (B_{M-1} + A_{M-1}\bar{C}_{M-2})^{-1}(D_{\psi M-2} - A_{M-1}\bar{D}_{\psi M-2}) \tag{86}$$

Having obtained the value of f_{M-1} at the end of our forward elimination procedure, we obtain the value of f_{M-2} by substituting the known f_{M-1} into (82) with $j = M - 2$. In general, having f_{j+1}, we can use (82) to get f_j. Finally, equation (83) is then used to give f_2.

Having f_j for all interior points $\rho_1, \rho_2, \cdots, \rho_M$, we can compute the value of f_0 and f_{M+1} by means of the difference equation (74) for $j = 1$, and $j = M$. The values of f_0 and f_{M+1} are needed to calculate the quantities n_θ and u/r_0.

To evaluate the integrals which appear in the expression for the stiffness coefficients, we use the simple trapezoidal rule

$$\int_{\rho_i}^{1} g(\rho)\, d\rho = t\left\{\sum_{j=1}^{M} g(\rho_j) - \frac{1}{2}[g(\rho_1) + g(\rho_M)]\right\} - \frac{1}{12}t^2 g^{\cdot\cdot}(\xi) \tag{87}$$

As all the relevant integrands have bounded second derivatives in the interval $\rho_i \leqq \rho \leqq 1$, we have that the error term on the right goes to zero as $t \to 0$.

The accuracy of our finite difference solution has been tested against the two-term λ^2-perturbation solution obtained in section 6. The discrepancy between the two solutions is less than 1 per cent for $\lambda \leqq 0.3$ and is generally $0(\lambda^4)$ using only fifty tabular points. Doubling the number of tabular points changes the finite difference solution by less than 0.01 percent. A corresponding comparison with the two-term η^2-perturbation solution shows still closer agreement with the finite difference solution, the two solutions coinciding to at least four significant figures for $\lambda \leqq 0.75$.

For large values of λ, the results of the finite difference solution for $\rho_i > 0$ have been compared with the results of the two-term ϵ^2-perturbation solution obtained in section 7. The discrepancy between the two solutions is generally of the order ϵ^4/ρ_i^4. For $\rho_i = 0.5$ and $\lambda = 5$, for instance, this discrepancy is less than 3 per cent using again only fifty tabular points.

10. Numerical results for pretwisted strips. The behavior of the quantities C_{Fk}^M, $C_{F\psi}^M = C_{Tk}^M$, $C_{T\psi}^M$ and $C_{T\psi}^B$ which contribute to the influence coefficients is shown in Figure 2 as function of λ in the interval (0,100) for $\nu = 0.3$. As first observed in [7], the bending contribution represented by C_{Fk}^B to the influence coefficients is $0(\lambda^2\mu^2)$ for $\lambda < 1$ and $0(\mu^2)$ for $\lambda \geqq 1$ relative to the membrane contribution. We therefore do not present the corresponding plot for C_{Fk}^B. The bending contribution represented by $C_{T\psi}^B$ is significant only in the range $\lambda^2 = 0(\mu^2)$. This fact is emphasized by the dotted portion of $C_{T\psi}^B$ in Figure 2. Altogether, the results of Figure 2 in part extend and in part confirm previous calculations [2, 3, 7] which were for the range (0, 5).

Figures 3, 4, 5 and 6 contain graphs of the coefficients K_{kF}, $K_{kT} = K_{\psi F}$ and $K_{\psi T}$ for a range of λ beyond that previously considered, and for representative values of μ. They show some qualitative features of the behavior of the shell. The following observations may be made.

Away from $\lambda = 0$ (say $\lambda^2 \gg \mu^2$) the flexibility coefficients are practically inde-

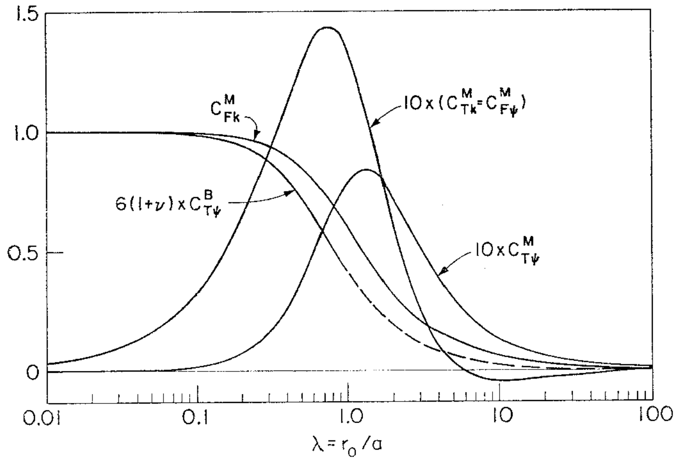

FIG. 2. Stiffness coefficients for a pretwisted strip with $\nu = 0.3$

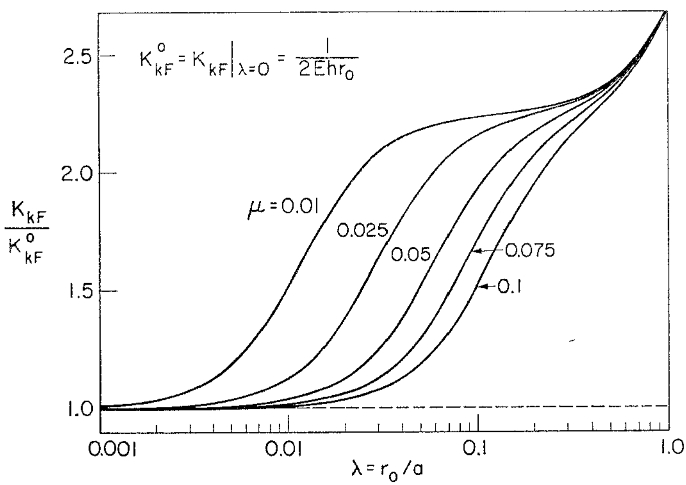

FIG. 3. Extensional flexibility for a pretwisted strip with $\nu = 0.3$ for $\lambda \leqq 1$

pendent of μ. Therefore, the shell reacts as a membrane to both stretching and twisting in this range of values of λ.

The torsional rigidity $I = 1/K_{\psi T}$ decreases by a factor of the order of μ^2 in going from $\lambda = 0(1)$ to $\lambda = 0(\mu)$. Evidently, for $\lambda = 0(\mu)$, the bending action in the shell dominates the membrane action (as should be the case for very small values of λ) in the effort to resist twisting by axial torques. In contrast, the axial extensional stiffness $K = 1/K_{kF}$ does not change order of magnitude as we enter the range of λ in which bending is important. In fact, the order of magnitude of K_{kF}^0 is independent of the thickness parameter μ. We may therefore say that the shell always responds as a membrane in regard to stretching by axial forces.

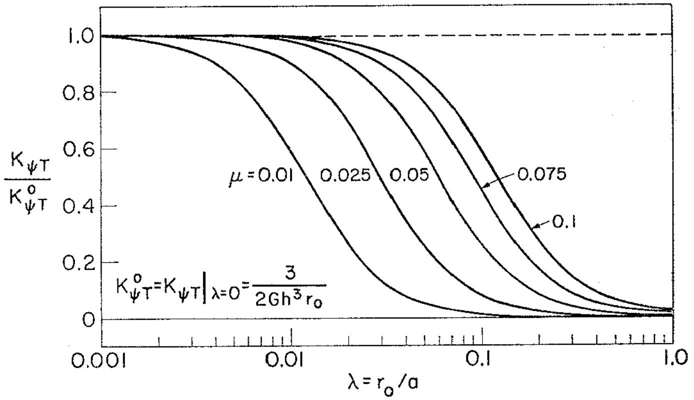

FIG. 4. Torsional flexibility for a pretwisted strip with $\nu = 0.3$ for $\lambda \leqq 1$

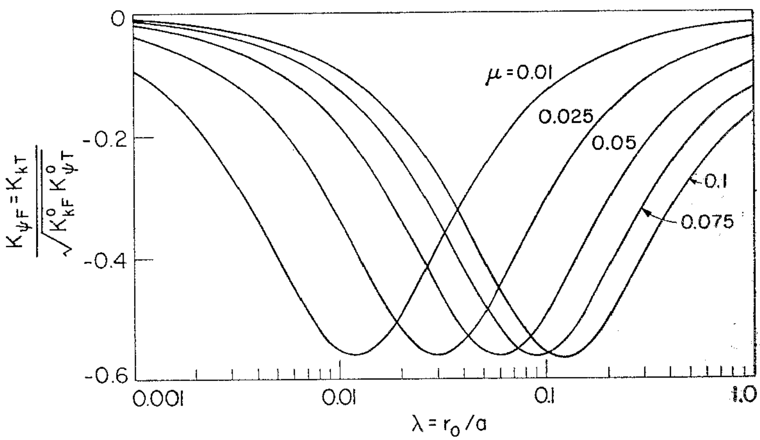

FIG. 5. Coupling coefficient in flexibility relations for a pretwisted strip with $\nu = 0.3$ for $\lambda \leqq 1$.

Our data confirm the expected qualitative fact that a pretwisted strip is stronger than a flat strip insofar as resistance to twisting by axial torques is concerned. In contrast, the strip is weaker insofar as resistance to stretching by axial forces is concerned.

The coupling coefficients $K_{kT} = K_{\psi F}$ reach a local maximum in absolute value in the neighborhood of $\lambda = \mu$. This maximum is proportional to $1/\mu$. Evidently, the bending action of the shell again dominates in this range of values of λ.

We briefly consider the details of the stress distributions in the pretwisted strip. The stresses of interest are the direct and twisting stresses given by

$$\sigma_\theta^M = \frac{N_\theta}{h}, \qquad \tau^B = \frac{6M}{h^2} \tag{88}$$

For the case of no-twist which we denote by a subscript k we show in Figures 7 and 8 the dimensionless stress quantities

$$\left(\frac{\sigma_\theta^M}{E}\right)_k = n_{\theta k} , \qquad \left(\frac{\tau^B}{E}\right)_k = \frac{\mu}{2(1+\nu)} m_k \tag{89}$$

for representative values of λ. We note that $n_{\theta k}$ and m_k are both independent of μ and that, by symmetry, we need only to show graphs for $0 \leqq \rho \leqq 1$. It is seen from these graphs that the bending stresses in the shell are $0(\mu)$ for $\lambda \leqq 1$ and $0(\lambda\mu)$ for $\lambda > 1$ compared to the direct stresses. In view of the fact that $h/R =$

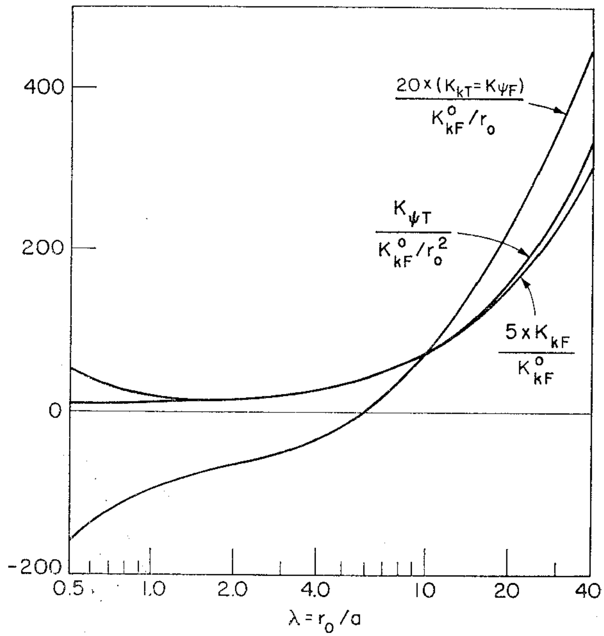

FIG. 6. Flexibility coefficients for a pretwisted strip with $\nu = 0.3$ for $\lambda \geqq 0.5$

$0(\lambda\mu) \ll 1$, they are therefore an order of magnitude smaller than the direct stresses. Note that as $\lambda \to 0$, we recover the results for the axial extension of a rectangular plate.

For $\lambda^2 \gg 1$, our results show a narrow-layer phenomenon near $\rho = 0$ with a layer width of the order $1/\lambda$. For fixed values of λ and μ (thereby fixing the overall stiffness of the shell), the shell is stiffer at $\rho = 0$ than at $\rho = 1$ in resisting stretching without twist. Therefore it develops larger stresses near the axis of symmetry. This difference in stiffness increases as λ increases and thus accounts for the stress pattern and the boundary layer.

For $\lambda \gtrsim 4$, the applied loads are absorbed almost entirely by the central portion of the shell. The small amount of negative $\sigma_{\theta k}^M$ away from the axis of symmetry is a Poisson's ratio effect as it becomes positive for all λ and $0 \leqq \rho \leqq 1$ if $\nu = 0$.

For the case of pure twist which we denote by a subscript ψ, we show in Figures 9 and 10 the representative stress quantities

$$\left(\frac{\tau^B}{G\mu}\right)_\psi = m_\psi\,, \qquad \left(\frac{\sigma_\theta^M}{G\mu}\right)_\psi = \frac{2(1+\nu)}{\mu}\, n_{\theta\psi} \tag{90}$$

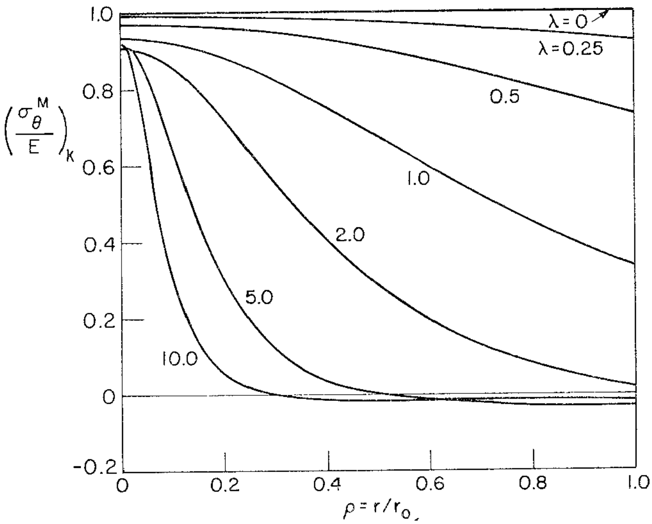

FIG. 7. Distribution of direct stresses σ_θ^M for a pretwisted strip for the case of *No Twist* with $\nu = 0.3$.

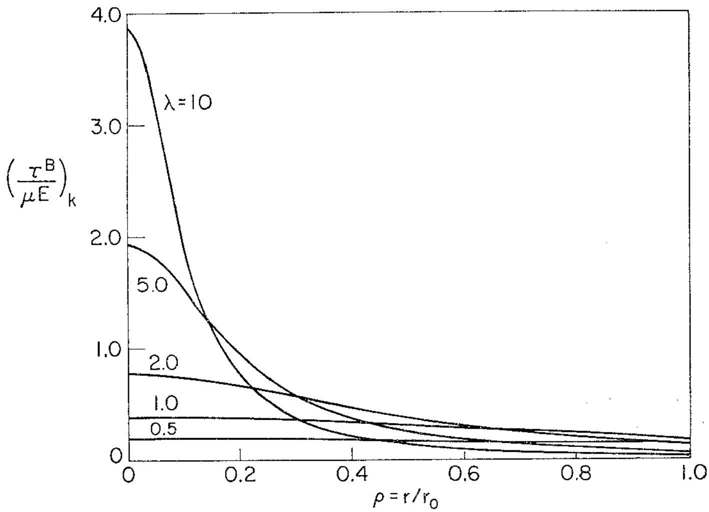

FIG. 8. Distribution of twisting stresses τ^B for a pretwisted strip for the case of *No Twist* with $\nu = 0.3$.

for representative values of λ. The quantities $n_{\theta\psi}$ and m_ψ are again independent of μ. We see from the graphs that the bending stress is an order of magnitude

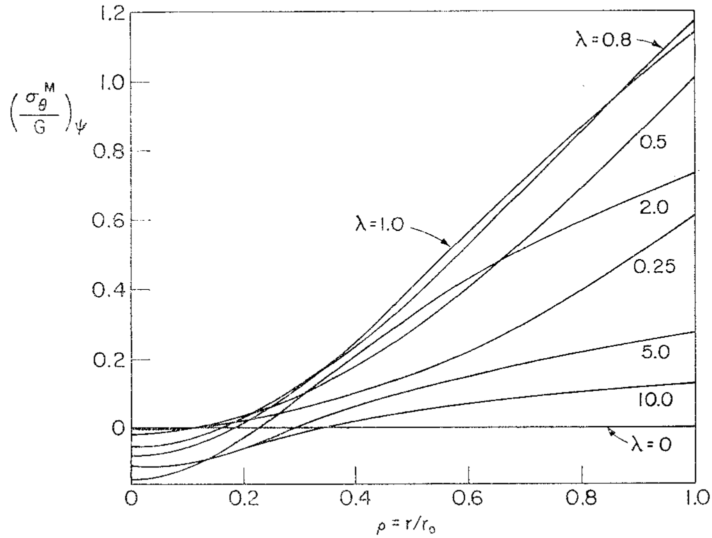

FIG. 9. Distribution of direct stresses σ_θ^M for a pretwisted strip for the case of *Pure Twist* with $\nu = 0.3$.

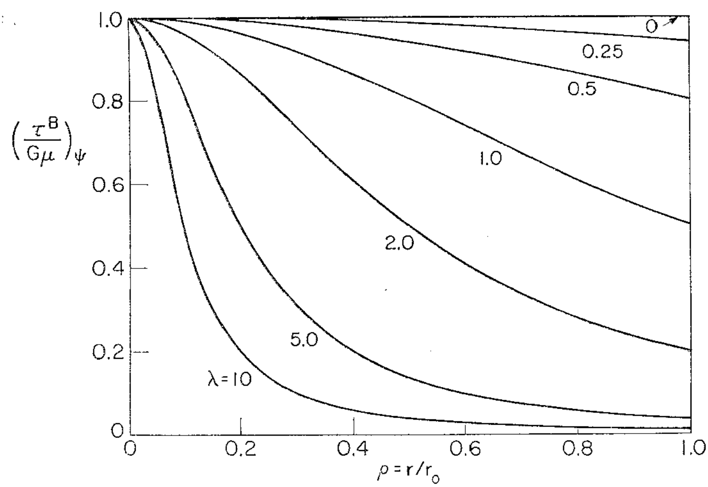

FIG. 10. Distribution of twisting stresses τ^B for a pretwisted strip for the case of *Pure Twist* with $\nu = 0.3$.

smaller than the direct stress, except for $\lambda^2 \ll 1$. The perturbation solution obtained in section 6 shows more precisely that λ must be $O(\mu)$ for the bending stress to be important. In the limit as $\lambda \to 0$, the stresses reduce, as they should, to those for the torsion of a rectangular plate.

For $\lambda \geqq 5.0$, we have again a boundary layer forming near the axis of symmetry.

11. Stiffness coefficients and stresses for coreless helicoidal shells. In this section, we present results of calculations for stresses and stiffness coefficients for various values of ρ_i in the range $0 < \rho_i < 1$. We note that these results for $\rho_i > 0$ may be considered as results for helicoidal springs with wide rectangular cross section acted upon by axial forces and torques.

We begin with the results for an eccentric flat strip as shown in Figure 11a. For this problem, the stretching and twisting are uncoupled. We expect the de-

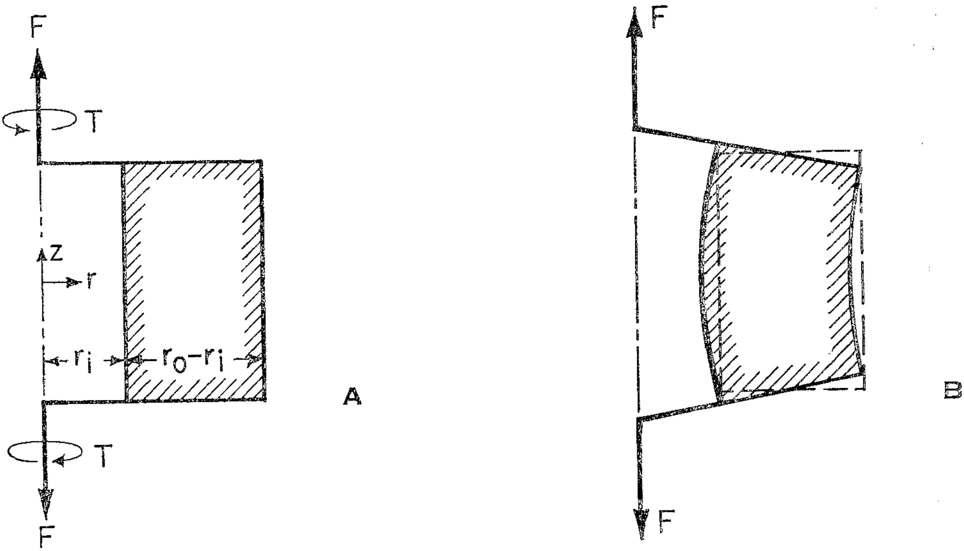

FIG. 11a. Eccentric flat strip acted upon by axial forces and torques
FIG. 11b. Deformation of an eccentric flat strip under axial forces

formation of the strip to be quite different from that for the symmetrically placed strip. In particular, the deformation of the strip acted upon only by axial forces is qualitatively described in Figure 11b. The solution for this problem of generalized plane stress corresponds to the results of section 6 in the limit as $a \to \infty$ and $\theta \to 0$ with $z = a\theta$ kept finite. We have

$$u_z = kz(1 + 3b_{k0}\rho), \qquad u_r = kr_0[3b_{k1} - \nu_s\rho + \tfrac{3}{2}(2 - \nu_s)b_{k0}\rho^2], \tag{91}$$
$$v = 0, \qquad n_r = 0, \qquad n_\theta = k(1 + 3b_{k0}\rho), \qquad m = 0$$

and

$$F = Ehr_0k[(1 - \rho_i) + \tfrac{3}{2}b_{k0}(1 - \rho_i^2)]$$

The corresponding solution for the flat strip subject to axial torques is

$$v = \psi rz, \qquad u_r = u_z = 0, \qquad m = \psi r_0, \qquad n_r = n_\theta = 0 \tag{92}$$

and

$$T = 2(1 - \nu_b)D\psi r_0(1 - \rho_i)$$

A small amount of pretwist in the strip will give rise to a coupling between stretching and twisting of the structure. The perturbation solution of section 6 delineates the effect of this small pretwist on the deformation, stress distribution, and the overall stiffness of the shell.

At the other end of the spectrum, we have for $a = 0$ the problem of the circular ring plate sector subject to axial forces and torques as shown in Figure 12. The solution for the uncoupled problem of stretching and twisting corresponds to terms linear in ϵ in the ϵ^2-perturbation solution of section 7. In the limit as $a \to 0$, we must keep $\psi_0 = \psi a$ and $k_0 = ka$ finite in order that the assumed displacement state be meaningful. We have then

$$u_z = k_0\theta, \qquad u_r = v = 0, \qquad m = k_0 r_0^{-1}\rho^{-2}, \qquad n_r = n_\theta = 0 \tag{93}$$

and

$$F = -\frac{(1 - \nu_b)Dk_0}{r_0^2}\left(1 - \frac{1}{\rho_i^2}\right)$$

for the plate twisted by axial forces, and

$$\begin{aligned} n_\theta &= \frac{\psi_0}{2}\left\{\left(1 + \frac{\rho_i^2 \ln \rho_i}{1 - \rho_i^2}\right) + \ln \rho + \frac{\ln \rho_i}{1 - \rho_i^2}\left(\frac{\rho_i}{\rho}\right)^2\right\} \\ n_r &= \frac{\psi_0}{2}\left\{\ln \rho - \frac{\rho_i^2}{\rho^2}\frac{1 - \rho^2}{1 - \rho_i^2}\ln \rho_i\right\}, \qquad m = 0, \\ u_r &= \frac{\psi_0 r_0}{2}\left\{\frac{\rho_i^2 \ln \rho_i}{1 - \rho_i^2}\left[(1 - \nu_s)\rho + \frac{1 + \nu_s}{\rho}\right] + \rho\left[(1 - \nu_s)\ln \rho - 1\right]\right\}, \\ v &= \psi_0 r\theta, \quad u_z = 0, \qquad T = \frac{\psi_0 E h r_0^3}{8}(1 - \rho_i^2)\left\{1 - \left(\frac{2\rho_i \ln \rho_i}{1 - \rho_i^2}\right)^2\right\} \end{aligned} \tag{94}$$

for the classical plane stress problem of the plate subject to axial torques.

Similarly, it can be verified that retaining terms both linear and quadratic in ϵ corresponds to the solution by means of shallow shell theory [3].

Having studied the problem for shells with large and small pitch, we now investigate the in between range of pitch values by means of the finite difference solution of section 9. Figures 13, 14 and 15 contain a complete set of data for C_{Fk}^M, $C_{F\psi}^M = C_{Tk}^M$ and $C_{T\psi}^M$ for $0 < \lambda \leqq 100$ for representative values of $\rho_i > 0$, as obtained by numerical integration.

To examine the stress distribution across the width of the shell, we consider again the representative direct and bending stress quantities defined in (88). The perturbation solution obtained in section 6 shows that for $\lambda^2 \ll 1$, the bending stresses are significant only in the case of pure twist and only in the range $\lambda = 0(\mu)$. For this range of values of λ, we have effectively a uniformly distributed twisting stress couple so that $\tau^B = \psi r_0 G\mu$.

For $\lambda^2 \gg 1$, the ϵ^2-perturbation solution of section 7 shows that the bending stresses are significant only for the case of no twist and only if $\epsilon = 0(\mu)$. If

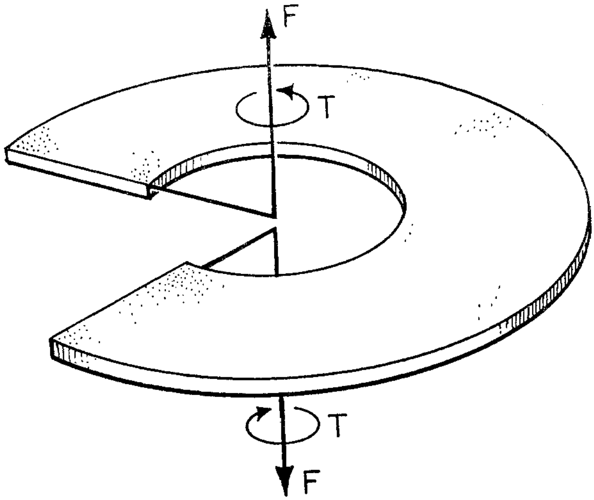

FIG. 12. Circular ring plate sector acted upon by axial forces and torques

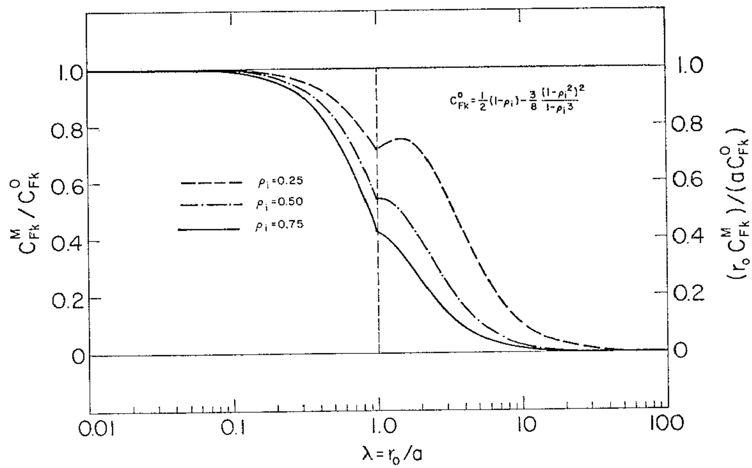

FIG. 13. Extensional stiffness for coreless helicoidal shells with $\nu = 0.3$

$\epsilon/\rho_i = 0(\mu)$ also, then we have effectively

$$\tau^B = k_0 G \mu \rho^{-2}$$

which is just the bending stress distribution of the ring plate sector.

Our finite difference solution shows that $\lambda = 0(\mu)$ and $1/\lambda = 0(\mu)$ are in fact the only two ranges of values for λ for which the bending stresses are significant compared to the direct stresses.

Motivated by the results for $\lambda = 0$ and $\lambda = \infty$, we present in figures (16) through (19) plots of the relevant dimensionless direct stresses $k\sigma_{\theta k}^{M}/\sigma_0$, $\psi r_0 \mu \sigma_{\theta\psi}^{M}/\tau_0$, $k\mu\sigma_{\theta k}^{M}/\tau_\infty$ and $\psi r_0 \sigma_{\theta\psi}^{M}/\sigma_\infty$ for $\rho_i = 0.5$, where the subscripts k and ψ

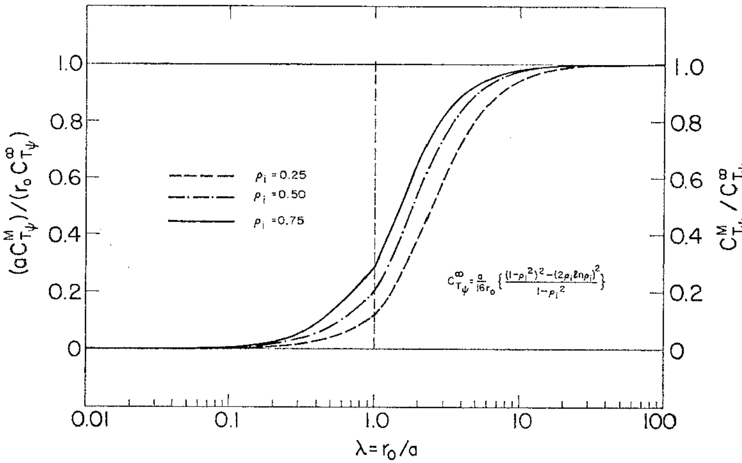

FIG. 14. Torsional rigidity for coreless helicoidal shells with $\nu = 0.3$

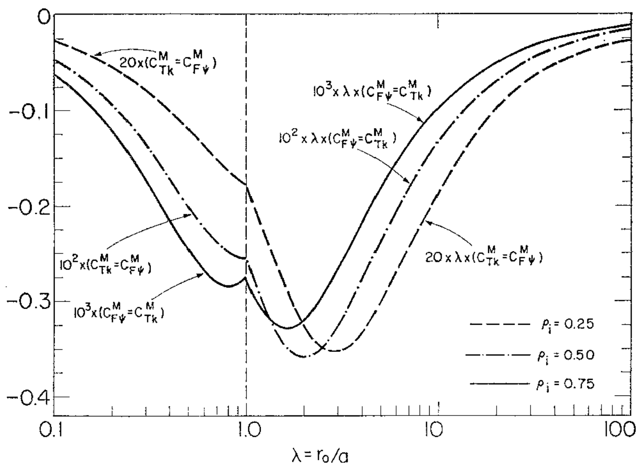

FIG. 15. Coupling coefficient in stiffness relations for coreless helicoidal shells with $\nu = 0.3$

are as usual designating no twist and pure twist, respectively, and where

$$\tau_0 = \tau^B|_{\lambda=0} = \psi r_0 G\mu, \qquad \sigma_0 = \sigma_\theta^M|_{\lambda=0} = \frac{kE(1-\rho_i)(2+\rho_i)}{2(1+\rho_i+\rho_i^2)}$$

$$\tau_\infty = \tau^B|_{\substack{\lambda=\infty\\ \rho=\rho_i}} = \frac{k_0 G\mu}{\rho_i^2}, \qquad \sigma_\infty = \sigma_\theta^M|_{\substack{\lambda=\infty\\ \rho=\rho_i}} = \frac{\psi_0 r_0 E}{2}\left[1 + \ln\rho_i + \frac{(1+\rho_i^2)\ln\rho_i}{1-\rho_i^2}\right] \tag{95}$$

Clearly, the quantities plotted are independent of μ.

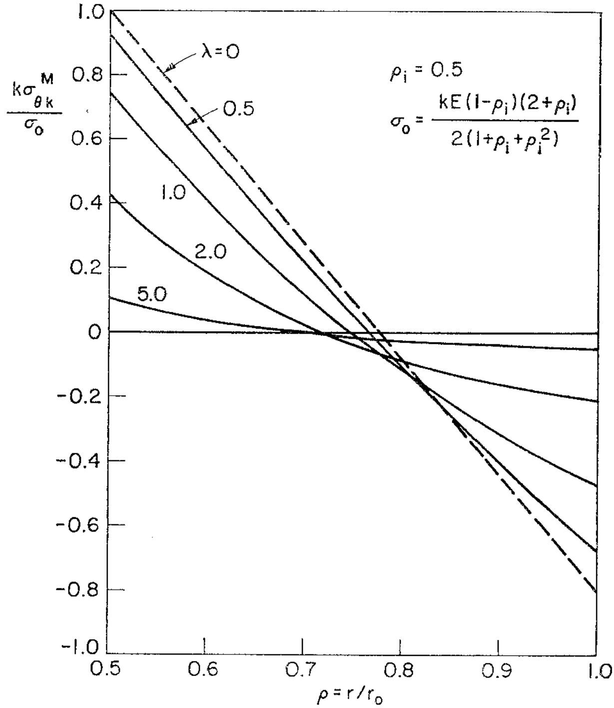

FIG. 16. Distribution of direct stresses σ_θ^M for a coreless helicoidal shell for the case of *No Twist* with $\rho_i = 0.5$, $\nu = 0.3$ and $\lambda \leqq 1$.

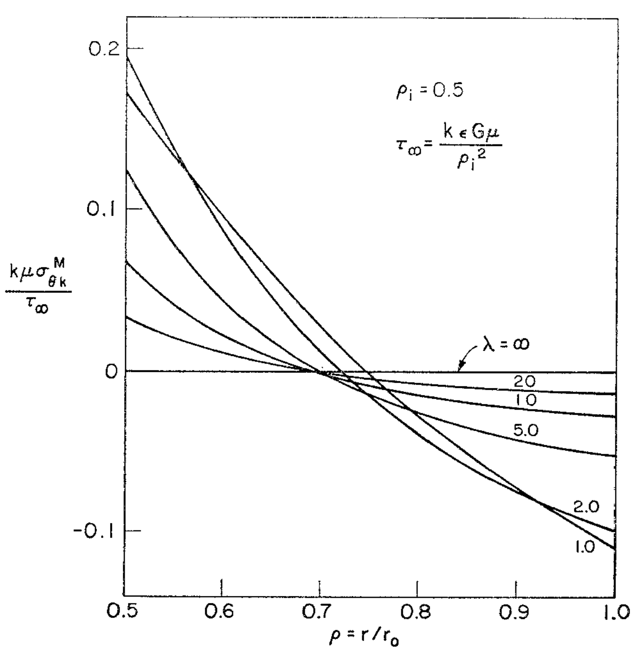

FIG. 17. Distribution of direct stresses σ_θ^M for a coreless helicoidal shell for the case of *No Twist* with $\rho_i = 0.5$, $\nu = 0.3$, and $\lambda \geqq 1$.

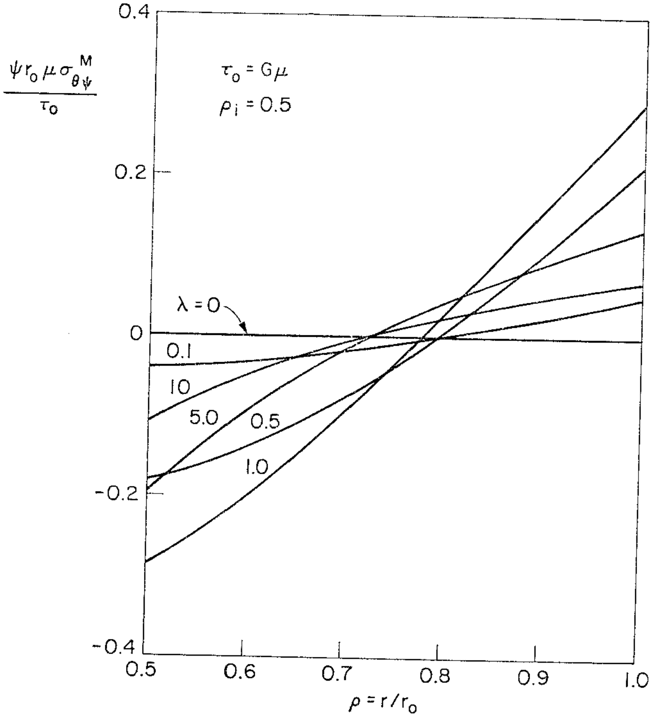

FIG. 18. Distribution of direct stresses σ_θ^M for a coreless helicoidal shell for the case of *Pure Twist* with $\rho_i = 0.5$ ν, $= 0.3$ and $\lambda \leqq 1$.

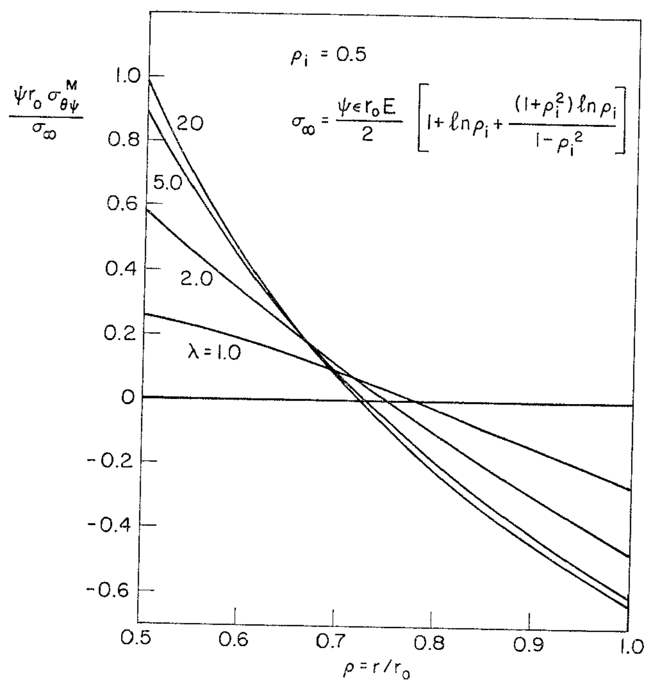

FIG. 19. Distribution of direct stresses σ_θ^M for a coreless helicoidal shell for the case of *Pure Twist* with $\rho_i = 0.5$, $\nu = 0.3$ and $\lambda \geqq 1$.

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