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FURTHER CONSIDERATIONS OF STRESS CONCENTRATION PROBLEMS FOR TWISTED OR SHEARED SHALLOW SPHERICAL SHELLS

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Abstract—We extend earlier studies of closed-form and asymptotic solutions of the problems of transverse twisting and membrane shearing of shallow spherical shells with small circular traction-free holes or rigid inserts. Our first extension concerns a modified form of the relevant exact solution of the shell equations, so as to take advantage of the static–geometric duality property of linear shell theory. As a consequence we observe that the results for two of the four previously considered problems can be seen as the static geometric duals of the other two. The second extension concerns the formulation of a general boundary value problem, with the problems of the hole and the rigid insert as special cases. Furthermore, on the basis of this formulation we find, as a particularly simple boundary value problem of physical interest, the problem of a hole the edge of which is transversely fixed. The asymptotic properties of the twisting and shearing solutions for this case are shown to be intermediate, in a specific sense, to the previously established order of magnitude properties for the free hole or rigid insert cases.

1. INTRODUCTION

We are concerned in what follows with a generalization of known results, as well as with an improvement of the solution technique, for the four problems of transverse twisting and membrane shearing of an isotropic homogeneous shallow spherical shell having either a small circular hole or rigid insert (Reissner, 1980a,b,c, 1981, 1986; Reissner and Reissner, 1982).

We recall that the earlier analysis, prompted by the possibility of formal exact solutions for manageable, physically interesting illustrations of the applications of the theory of shallow shells, has led to results which were, in part, unexpected. While the effects of a circular hole for the problem of transverse twisting, and of a rigid insert for the problem of membrane shearing, were found to be similar to those for the corresponding problems of a flat plate, this turned out not to be the case for the membrane shear problem for shells with a hole and for the transverse twisting problem for shells with a rigid insert. For these latter problems, unexpectedly large stress concentrations were found along with unexpected novel effects concerning interior membrane and inextensional bending solution contributions. These effects were shown to depend on sufficiently large values of a dimensionless parameter μ involving the shell radius R, the hole or insert radius a, the bending stiffness coefficient D and the stretching compliance coefficient B, in the form

$$\mu^4 = \frac{a^4}{R^2 DB}. \tag{1}$$

In this paper, we consider a more general problem with boundary conditions which contain edge conditions for the problems of the hole and for the problems of the rigid insert as special cases. In addition, we improve upon the solution technique used in Reissner

(1980a,b,c, 1981, 1986) and Reissner and Reissner (1982) by using a modified form of the general solution of the shallow spherical shell equations, such that the two basic variables, the normal displacement w and the stress function K, are treated in a symmetrical fashion. The advantage of this symmetrical treatment is the possibility to realize additional symmetrization associated with the static–geometric (s–g) duality of linear shell theory, as first observed in Gol'denveizer (1940) and extended in Lur'e (1961) and Wan (1968). A specific consequence of this duality is a formulation of the displacement conditions at the edge of a rigid insert as the s–g duals of the traction conditions at the edge of a hole. This, in turn, leads to a recognition of the possibility of deducing two of the four earlier solutions for the shell with hole or rigid insert directly as the s–g duals of the other two solutions, thereby making it possible to simplify the discussion of the four problems treated previously as well as the related problems for shells with an elastically supported edge to be analyzed herein.

The new problem treated in this paper makes possible a study of the transition from problems with the usual magnitude of stress concentration for a flat plate to those with much higher magnitude for shells in the range $\mu \gg 1$.

2. THE DIFFERENTIAL EQUATIONS

With (r, θ) as polar coordinates in the base plane of the shallow spherical shell of radius R, we have, in terms of tangential and normal displacement components $\{u, w\}$, stress resultants N and stress couples M, the following constitutive equations, essentially as in Reissner (1946),

$$\varepsilon_{rr} = B(N_{rr} - \nu_N N_{\theta\theta}), \quad \varepsilon_{\theta\theta} = B(N_{\theta\theta} - \nu_N N_{rr}), \quad \varepsilon_{r\theta} = \varepsilon_{\theta r} = B(1+\nu_N)N_{r\theta} \tag{2a}$$

with

$$\varepsilon_{rr} = u_{r,r} + \frac{w}{R}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}(u_{\theta,\theta} + u_r) + \frac{w}{R}, \quad \varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2r}(u_{r,\theta} - u_\theta + r u_{\theta,r}) \tag{2b}$$

and

$$M_{\theta\theta} = D(\kappa_{\theta\theta} + \nu_M \kappa_{rr}), \quad M_{rr} = D(\kappa_{rr} + \nu_M \kappa_{\theta\theta}), \quad M_{r\theta} = M_{\theta r} = D(1-\nu_M)\kappa_{r\theta}, \tag{3a}$$

with

$$\kappa_{\theta\theta} = -\left(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta}\right), \quad \kappa_{rr} = -w_{,rr}, \quad \kappa_{r\theta} = \kappa_{\theta r} = -\left(\frac{w_{,\theta}}{r}\right)_{,r}. \tag{3b}$$

In these, ν_N and ν_M are the effective Poisson's ratios for stretching and bending actions, with $\nu_N = \nu_M = \nu$, $B = 1/Eh$, and $D = Eh^3/12(1-\nu^2)$ for homogeneous isotropic shells. *We will continue to distinguish ν_N from ν_M only when the distinction facilitates applications of the s–g duality.*

In the absence of surface loads, satisfaction of the equations of tangential force equilibrium is accomplished in terms of a stress function K upon setting,

$$N_{rr} = \frac{1}{r}K_{,r} + \frac{1}{r^2}K_{,\theta\theta}, \quad N_{\theta\theta} = K_{,rr}, \quad N_{r\theta} = N_{\theta r} = -\left(\frac{K_{,\theta}}{r}\right)_{,r}. \tag{4}$$

The two scalar moment equilibrium equations without surface loads give as expressions for the transverse shear resultants

$$Q_r = -D(\nabla^2 w)_{,r}, \quad Q_\theta = -\frac{D}{r}(\nabla^2 w)_{,\theta}. \tag{5}$$

The normal force equilibrium equation without surface loads and a compatibility equation for strains based on the strain displacement relations in eqns (2a,b) imply the satisfaction of the following system of two simultaneous differential equations for the stress function K and the normal displacement component w (Reissner, 1946)

$$-D\nabla^4 w + \frac{1}{R}\nabla^2 K = 0, \quad B\nabla^4 K + \frac{1}{R}\nabla^2 w = 0. \tag{6}$$

The symmetric appearance of K and w in these equations is one of the consequences of the s–g duality of shell theory [see Wan and Weinitschke (1988) for a detailed description]. With K and B as the s–g duals of w and $-D$, respectively, each of the two equations in (6) is the s–g dual of the other equation. That is, we can obtain one equation from the other by replacing w, K, D and B by their dual quantities K, w, $-B$ and $-D$, respectively. With the additional dual relations listed in Table 1, the three stress–strain relations [eqn (2a)] with the resultants expressed in terms of K by eqn (4), become the s–g duals of the three stress–strain relations in eqn (3a) expressed in terms of w through use of eqn (3b).

To take advantage of the s–g duality, we now express the general solution of the system (6) in terms of two harmonic functions ϕ and ψ together with a third function χ in the form

$$w = \phi + A_0\chi + A_1\nabla^2\chi, \quad K = \psi + B_0\chi + B_1\nabla^2\chi, \tag{7}$$

with χ as the solution of

$$R^2 DB\nabla^4\chi + \chi = 0 \tag{8}$$

and with the coefficients A_0, A_1, B_0, B_1 related by

$$A_1 = RBB_0, \quad B_1 = -RDA_0. \tag{9}$$

In Reissner (1980a,b,c, 1981, 1986) and Reissner and Reissner (1982), it was stipulated that $A_0 = 1$ and $B_0 = 0$, and therewith $A_1 = 0$ and $B_1 = -RD$ for the problem of transverse twisting as well as for the problem of membrane shearing. In what follows, the χ-terms in eqn (7) will instead be taken in the form

$$A_0\chi + A_1\nabla^2\chi = \chi + R\sqrt{DB}\nabla^2\chi, \tag{10a}$$

$$B_0\chi + B_1\nabla^2\chi = \sqrt{\frac{D}{B}}(\chi - R\sqrt{DB}\nabla^2\chi) \tag{10b}$$

for the problem of transverse twisting, and in the form

$$A_0\chi + A_1\nabla^2\chi = \sqrt{\frac{B}{D}}(\chi - R\sqrt{DB}\nabla^2\chi), \tag{11a}$$

$$B_0\chi + B_1\nabla^2\chi = (\chi + R\sqrt{DB}\nabla^2\chi) \tag{11b}$$

Table 1. Static–geometric dual quantities

M_{rr}	$M_{\theta\theta}$	$M_{r\theta}$	$M_{\theta r}$	N_{rr}	$N_{\theta\theta}$	$N_{r\theta}$	$N_{\theta r}$	K	D	ν_M
$\varepsilon_{\theta\theta}$	ε_{rr}	$-\varepsilon_{\theta r}$	$-\varepsilon_{r\theta}$	$-\kappa_{\theta\theta}$	$-\kappa_{rr}$	$\kappa_{\theta r}$	$\kappa_{r\theta}$	w	$-B$	$-\nu_N$

for the problem of membrane shearing, to bring out the s–g duality inherent in these problems.

3. UNIFORM TRANSVERSE TWISTING AND MEMBRANE SHEARING OF A SHALLOW SPHERICAL SHELL

Given the well-known classical formulas for transverse twisting and in-plane shearing of a flat plate,

$$w_p = \frac{T}{2D(1-\nu_M)}xy, \quad K_p = -Sxy \tag{12}$$

where $x = r\cos\theta$ and $y = r\sin\theta$, we have, as a consequence of the relations $\nabla^2 w_p = 0$ and $\nabla^2 K_p = 0$, that w_p and K_p are also solutions of the corresponding problems for a shallow spherical shell.

With eqn (7) as the general solution of the shell eqns (6), it is then possible to obtain closed-form solutions for problems of stress concentrations due to a circular hole or a rigid insert, similar to the classical solutions for the corresponding flat plate problems. Apart from the effect of shell curvature on the numerical values of stress concentration factors, the results for the shell problems in Reissner (1980a, b, c, 1981, 1986) and Reissner and Reissner (1982) were found to be of particular interest because of the following aspects of the analysis. While the solutions of shell problems for "sufficiently thin" shells were known previously to consist of edge zone solution contributions and interior solution contributions, with the latter being either of the "membrane" type or of the "inextensional bending" type, some of the present stress concentration problems were of a more intriguing nature. The interior solution contribution for these problems was such that the shell interior was subdivided into three domains. One of these was of the inextensional bending type, a second was of the membrane type, with a third (transition) domain in which inextensional bending and membrane actions were of equal significance.

The stress concentration problems with the aforementioned unusual three-domain property turned out to be the ones associated with surprisingly large stress concentration factors, compared to the problems without the three-domain property. For the two transverse twisting problems, the one with a rigid insert is of the three-domain type and the one for the circular hole is of the usual one-domain type. The opposite is true for the two membrane-shear problems. In what follows we formulate and solve a more general problem in a way to make possible a study of the transitions from problems of the unusual kind to problems of the usual kind.

Consider a shell with outer boundary $r = \infty$ and inner boundary $r = a$. For sufficiently large values of r, the solution should be independent of what is prescribed for $r = a$. We may therefore stipulate, on the basis of eqn (12), as conditions for the outer boundary

$$r \to \infty: \quad w = \tfrac{1}{2}\tau r^2 \sin 2\theta, \quad K = 0 \tag{13}$$

for the problem of transverse twisting, and

$$r \to \infty: \quad w = 0, \quad K = -\tfrac{1}{2}Sr^2 \sin 2\theta \tag{14}$$

for the problem of membrane shearing. The curvature parameter $\tau = T/2D(1-\nu_M)$ is introduced here as the s–g dual of $-S$ so that the two conditions in eqn (14) are now the s–g duals of the two conditions in eqn (13).

The corresponding conditions for the edge of a circular hole or rigid insert were previously stated in the form

$$r = a: \quad N_{rr} = N_{r\theta} = P_r = M_{rr} = 0 \tag{15}$$

where $P_r = Q_r + M_{r\theta,\theta}/r$, and

$$r = a: \quad u_r = u_\theta = w = w_{,r} = 0, \tag{16}$$

respectively. For our subsequent analysis, we will re-state these conditions in terms of w and K.

With eqns (3), (4) and (5), the traction-free conditions in eqn (15) are satisfied by requiring

$$r = a: \begin{cases} K = 0, \quad K_{,r} = 0 & (17a, b) \\ -D[(\nabla^2 w)_{,r} + (1-\nu_M)(r^{-2}w_{,r} - r^{-3}w)_{,\theta\theta}] = 0 & (17c) \\ -D[\nabla^2 w - (1-\nu_M)(r^{-1}w_{,r} + r^{-2}w_{,\theta\theta})] = 0. & (17d) \end{cases}$$

Equations (17a, b) have natural physical interpretations. The left-hand side of eqn (17b) is the resultant force in the direction tangent to the edge $r = a$ along the edge curve. In the absence of this resultant tangential force, the left-hand side of eqn (17a) is the resultant torque along the edge $r = a$ turning about the normal of the midsurface of the shell.

Of the vanishing displacement conditions in eqn (16), the last two are already in terms of w and $w_{,r}$ and in the form of the s–g duals of eqns (17a, b). In order to obtain the s–g duals of eqns (17c, d), we make use of the fact that the conditions of vanishing u_r and u_θ, in conjunction with the conditions of vanishing w and $w_{,r}$, turn out to be equivalent to two conditions in terms of membrane strains, in the form (Wan, 1968)

$$\varepsilon_{\theta\theta} = 0, \quad re_\theta \equiv (r\varepsilon_{\theta\theta})_{,r} - \varepsilon_{rr} - 2\varepsilon_{r\theta,\theta} = 0. \tag{18a, b}$$

That eqn (18b) is the dual of the third relation in eqn (15) follows from the fact that the moment equilibrium equation $(rM_{rr})_{,r} - M_{\theta\theta} + M_{r\theta,\theta} = rQ_r$ implies the relation, $rP_r = (rM_{rr})_{,r} - M_{\theta\theta} + 2M_{r\theta,\theta}$. Accordingly, the auxiliary variable e_θ has the role of the s–g dual of P_r (Wan, 1968; Wan and Weinitschke, 1988).

Upon expressing the strain conditions (18a, b) in terms of the stress resultants in eqn (4) with the help of eqn (2), we have the displacement conditions (16) as s–g duals of the traction conditions in eqn (17) in the form

$$r = a: \begin{cases} w = 0, \quad w_{,r} = 0 & (19a, b) \\ B[(\nabla^2 K)_{,r} + (1+\nu_N)(r^{-2}K_{,r} - r^{-3}K)_{,\theta\theta}] = 0 & (19c) \\ B[\nabla^2 K - (1+\nu_N)(r^{-1}K_{,r} + r^{-2}K_{,\theta\theta})] = 0. & (19d) \end{cases}$$

Given that the problem of transverse twisting of a shell with a circular hole in Reissner (1980a, 1981), henceforth designated as the TH problem, is governed by the two partial differential equations (PDE) in eqn (6), the far field conditions (13) and the edge conditions (17), we observe that the s–g dual of this boundary value problem (BVP) consists of the same two PDEs together with the far field conditions (14) and the edge condition (19). This dual BVP is just the BVP for the problem of membrane shearing of a shell with a rigid circular insert in Reissner (1980c, 1986), henceforth designated as the SI problem. Its solution can therefore be obtained from that for the TH problem in Reissner (1980a, 1981) without a separate analysis or calculation by an application of the s–g duality rules, i.e. by replacing all quantities in the solution in Reissner (1980a, 1981) by their s–g dual quantities. Similarly, it can be seen that the problem of transverse twisting of a shell with a rigid insert (TI), as analyzed in Reissner (1980c, 1986), is the s–g dual of the problem of membrane shearing of a shell with a circular hole (SH) in Reissner (1980b, 1981) and its solution could therefore have been obtained directly from the results in Reissner (1980c, 1986).

4. A CLASS OF ELASTIC SUPPORT EDGE CONDITIONS

The form of eqns (17) and (19) suggests that we consider the following system of conditions at $r = a$, which contains both eqns (17) and (19) as special cases;

$$r = a: \begin{cases} -\alpha_{s1}a^3[(\nabla^2 w)_{,r} + (1-\nu_M)(r^{-2}w_{,r} - r^{-3}w)_{,\theta\theta}] + \alpha_{d1}w = 0 & (20a) \\ -\alpha_{s2}a[\nabla^2 w - (1-\nu_M)(r^{-1}w_{,r} + r^{-2}w_{,\theta\theta})] + \alpha_{d2}w_{,r} = 0 & (20b) \\ \beta_{d1}a^3[(\nabla^2 K)_{,r} + (1+\nu_N)(r^{-2}K_{,r} - r^{-3}K)_{,\theta\theta}] + \beta_{s1}K = 0 & (20c) \\ \beta_{d2}a[\nabla^2 K - (1+\nu_N)(r^{-1}K_{,r} + r^{-2}K_{,\theta\theta})] + \beta_{s2}K_{,r} = 0. & (20d) \end{cases}$$

In this system, α and β are dimensionless weighting factors which remain to be chosen. Without loss of generality, it may be assumed that $\alpha_{di}^2 + \alpha_{si}^2 = 1$ and $2\beta_{di}^2 + \beta_{si}^2 = 1$. Furthermore, by considering β_{di} and β_{sj} as the s–g duals of $-\alpha_{si}$ and $-\alpha_{dj}$, respectively, the edge conditions (20c, d) become the s–g duals of the edge conditions (20a, b).

The two conditions (20a, b) are the conventional elastic-support conditions for transverse deformations,

$$\bar{\alpha}_{s2}M_{rr} + \bar{\alpha}_{d2}w_{,r} = 0 \quad \text{and} \quad \bar{\alpha}_{s1}P_r + \bar{\alpha}_{d1}w = 0. \tag{21}$$

Equations (20c, d) describe a certain type of in-plane elastic-support. The meaning of eqn (20d) is that the hoop strain at a point along the $r = a$ boundary is proportional to the resultant tangential force at that point. The meaning of eqn (20c) is that the effective curvature change e_θ [see eqn (18b)] along $r = a$ is proportional to the resultant torque turning about the normal to the midsurface, when there is no tangential resultant edge force contributing to that torque.

Instead of a duality between two sets of four conditions (each set associated with a different physical problem) as in eqns (17) and (19), we now have an internal duality between two sets of two conditions, so that the set of four conditions in eqn (20) for the elastically supported shell is its own dual.

It is convenient from here on to introduce a dimensionless coordinate $\rho = r/a$ and to consider the Laplace operator as involving ρ rather than r. The function χ in eqn (7) is now subject to the differential equation

$$\nabla^4\chi + \mu^4\chi = 0 \tag{22}$$

with μ as in eqn (1). For $\mu \gg 1$, the singular perturbation structure of eqn (22) implies that χ is significant only near an edge of the shell.

We take account of eqns (10), (11) and (12), now for $\rho \to \infty$, by writing with suitably redefined potential functions ϕ and ψ

$$w_T = \tfrac{1}{2}\tau a^2(\phi_T + \chi^+), \quad K_T = \tfrac{1}{2}\tau a^2\sqrt{\frac{D}{B}}(\mu^t\psi_T + \chi^-) \tag{23a, b}$$

and

$$K_S = -\tfrac{1}{2}Sa^2(\psi_S + \chi^-), \quad w_S = -\tfrac{1}{2}Sa^2\sqrt{\frac{B}{D}}(\mu^s\phi_S + \chi^+) \tag{24a, b}$$

$$\nabla^2\chi^{\pm} = -\eta_{\pm}\mu^2\chi^{\mp} = \mp\mu^2\chi^{\mp}. \tag{26}$$

Introduction of the coefficients μ^t and μ^s in eqns (23b) and (24b), with exponents t and s which remain to be chosen, is not a necessity but turns out to be helpful in connection with the asymptotic consideration of the problems of twisting and shearing in the range $1 \ll \mu$. Note that with $\{\eta_+, \mu^t, \chi^+, \phi_T, \psi_T\}$ as the s–g duals of $\{\eta_-, \mu^s, \chi^-, \psi_S, \phi_S\}$, w_T and K_T are now the s–g duals of K_s and w_s, respectively.

On the basis of eqns (23) and (24), and with an anticipation of the properties of the solution contribution χ, the conditions (13) and (14) can be written in the form

$$\rho \to \infty: \quad \begin{cases} \phi_T \sim \rho^2 \sin 2\theta, & \psi_T \sim 0, \quad \chi \sim 0, \quad (27a) \\ \phi_S \sim 0, & \psi_S \sim \rho^2 \sin 2\theta, \quad \chi \sim 0. \quad (27b) \end{cases}$$

With this, and in view of the homogeneity of the conditions in eqn (20), we will have as expressions for ϕ_T, ψ_T, ϕ_S and ψ_S

$$\phi_T = (\rho^2 + c_1\rho^{-2})\sin 2\theta \equiv \Phi_T(\rho)\sin 2\theta, \quad \psi_T = c_2\rho^{-2}\sin 2\theta \equiv \Psi_T(\rho)\sin 2\theta \tag{28a, b}$$

$$\phi_S = c_1\rho^{-2}\sin 2\theta \equiv \Phi_S(\rho)\sin 2\theta, \quad \psi_S = (\rho^2 + c_2\rho^{-2})\sin 2\theta \equiv \Psi_S(\rho)\sin 2\theta \tag{28c, d}$$

and, on the basis of eqn (22), as an expression for χ

$$\chi = [c_5K_2(\lambda\rho) + \bar{c}_5K_2(\bar{\lambda}\rho)]\sin 2\theta \equiv X(\rho)\sin 2\theta \tag{29}$$

where $\lambda = \sqrt{i\mu} = \mu(1+i)/\sqrt{2}$, and $c_5 = c_3 + ic_4$ with complex conjugates $\bar{\lambda}$ *and* $\bar{c}_5$.

With r replaced by ρ and with ϕ, ψ and χ proportional to $\sin 2\theta$, the boundary conditions at $\rho = 1$ in eqn (19) may now be written as

$$\alpha_{s1}[\eta_+\mu^2\chi^-_{,\rho} + 4(1-\nu_M)(\phi_{T,\rho} - \phi_T + \chi^+_{,\rho} - \chi^+)] + \alpha_{d1}[\phi_T + \chi^+] = 0, \tag{30a}$$

$$\alpha_{s2}[\eta_+\mu^2\chi^- + (1-\nu_M)(\phi_{T,\rho} - 4\phi_T + \chi^+_{,\rho} - 4\chi^+)] + \alpha_{d2}[\phi_{T,\rho} + \chi^+_{,\rho}] = 0, \tag{30b}$$

$$-\beta_{d1}[\eta_-\mu^2\chi^+_{,\rho} + 4(1+\nu_N)\{\mu^t(\psi_{T,\rho} - \psi_T) + (\chi^-_{,\rho} - \chi^-)\}] + \beta_{s1}[\mu^t\psi_T + \chi^-] = 0, \tag{30c}$$

$$-\beta_{d2}[\eta_-\mu^2\chi^+ + (1+\nu_N)\{\mu^t(\psi_{T,\rho} - 4\psi_T) + (\chi^-_{,\rho} - 4\chi^-)\}] + \beta_{s2}[\mu^t\psi_{T,\rho} + \chi^-_{,\rho}] = 0, \tag{30d}$$

for the problem of transverse twisting, and

$$\alpha_{s1}[\eta_+\mu^2\chi^-_{,\rho} + 4(1-\nu_M)\{\mu^s(\phi_{S,\rho} - \phi_S) + (\chi^+_{,\rho} - \chi^+)\}] + \alpha_{d1}[\mu^s\phi_S + \chi^+] = 0, \tag{31a}$$

$$\alpha_{s2}[\eta_+\mu^2\chi^- + (1-\nu_M)\{\mu^s(\phi_{S,\rho} - 4\phi_S) + (\chi^+_{,\rho} - 4\chi^+)\}] + \alpha_{d2}[\mu^2\phi_{S,\rho} + \chi^+_{,\rho}] = 0, \tag{31b}$$

$$-\beta_{d1}[\eta_-\mu^2\chi^+_{,\rho} + 4(1+\nu_N)(\psi_{S,\rho} - \psi_S + \chi^-_{,\rho} - \chi^-)] + \beta_{s1}[\psi_S + \chi^-] = 0, \tag{31c}$$

$$-\beta_{d2}[\eta_-\mu^2\chi^+ + (1+\nu_N)(\psi_{S,\rho} - 4\psi_S + \chi^-_{,\rho} - 4\chi^-)] + \beta_{s2}[\psi_{S,\rho} + \chi^-_{,\rho}] = 0, \tag{31d}$$

for the problem of membrane shearing.

The introduction of eqns (28) and (29) into eqn (30) or eqn (31) leads, in both cases, to a system of four simultaneous equations for the four constants of integration c_1, c_2, c_3 and c_4.

5. STRESS RESULTANT AND STRESS COUPLE CONCENTRATION FORMULAS

Our principal interest, insofar as applications of the foregoing are concerned, has to do with results pertaining to stress concentrations along the edge $r = a$. The direct stress concentration factors for the problem of membrane shear have previously been defined in the form

$$k_m^{\mathrm{SH}} = \frac{N_{\theta\theta}^{\mathrm{SH}}(1,\pi/4)}{N_{\theta\theta}(\infty,\pi/4)}, \quad k_m^{\mathrm{SI}} = \frac{N_{rr}^{\mathrm{SI}}(1,\pi/4)}{N_{rr}^{\mathrm{SI}}(\infty,\pi/4)} \tag{32a, b}$$

and the bending stress concentration factors for the problem of transverse twisting in the form

$$k_b^{\mathrm{TH}} = \frac{M_{\theta\theta}^{\mathrm{TH}}(1,\pi/4)}{M_{\theta\theta}^{\mathrm{TH}}(\infty,\pi/4)}, \quad k_b^{\mathrm{TI}} = \frac{M_{rr}^{\mathrm{TI}}(1,\pi/4)}{M_{rr}^{\mathrm{TI}}(\infty,\pi/4)}. \tag{32c, d}$$

For the definition of bending stress concentration factors k_b^{SH} and k_b^{SI} for the membrane shear problem and the direct stress concentration factors, k_m^{TH} and k_m^{TI} for the transverse twisting problem, we limit ourselves here, as in Reissner (1980a, b), to the establishment of results based on the assumption of a linear distribution of midsurface parallel stresses across the thickness of the shell, so as to have

$$k_b^{\mathrm{SH}} = \frac{6M_{\theta\theta}^{\mathrm{SH}}(1,\pi/4)}{hN_{\theta\theta}^{\mathrm{SH}}(\infty,\pi/4)}, \quad k_m^{\mathrm{TH}} = \frac{hN_{\theta\theta}^{\mathrm{TH}}(1,\pi/4)}{6M_{\theta\theta}^{\mathrm{TH}}(\infty,\pi/4)} \tag{32e, f}$$

and

$$k_b^{\mathrm{SI}} = \frac{6M_{rr}^{\mathrm{SI}}(1,\pi/4)}{hN_{rr}^{\mathrm{SI}}(\infty,\pi/4)}, \quad k_m^{\mathrm{TI}} = \frac{hN_{rr}^{\mathrm{TI}}(1,\pi/4)}{6M_{rr}^{TI}(\infty,\pi/4)} \tag{32g, h}$$

where we note that the s–g dual of k_m^{SH} is not k_b^{TI} since

$$\text{s–g dual}\left[\frac{N_{\theta\theta}^{\mathrm{SH}}(1,\pi/4)}{N_{\theta\theta}^{\mathrm{SH}}(\infty,\pi/4)}\right] = \frac{\kappa_{rr}^{\mathrm{TI}}(1,\pi/4)}{\kappa_{rr}^{\mathrm{TI}}(\infty,\pi/4)} = \frac{[M_{rr}^{\mathrm{TI}}(1,\pi/4)-\nu_M M_{\theta\theta}^{\mathrm{TI}}(1,\pi/4)]}{[M_{rr}^{\mathrm{TI}}(\infty,\pi/4)-\nu_M M_{\theta\theta}^{\mathrm{TI}}(\infty,\pi/4)]}. \tag{33}$$

Given that $M_{\theta\theta}^{\mathrm{TI}}(\infty,\pi/4) = -M_{rr}^{\mathrm{TI}}(\infty,\pi/4)$ and $\kappa_{\theta\theta}(a,\theta) = [aw_{,r}(a,\theta)+w_{,\theta\theta}(a,\theta)]/a^2 = 0$ so that $M_{\theta\theta}^{\mathrm{TI}}(1,\pi/4) = \nu_M M_{rr}^{\mathrm{TI}}(1,\pi/4)$, eqn (33) becomes

$$\text{s–g dual}\,[k_m^{\mathrm{SH}}] = (1-\nu_M)\frac{M_{rr}^{\mathrm{TI}}(1,\pi/4)}{M_{rr}^{\mathrm{TI}}(\infty,\pi/4)} = (1-\nu_M)k_b^{\mathrm{TI}}. \tag{34a}$$

Similarly, we obtain from straightforward calculations the following s–g dualities;

$$\text{s–g dual}\ [k_b^{\mathrm{SH}}] = 3(1-\nu_N)k_m^{\mathrm{TI}}, \tag{34b}$$

$$\text{a–g dual}\ [k_m^{\mathrm{TH}}] = -\frac{1-\nu_N}{3}k_b^{\mathrm{SI}}, \tag{34c}$$

$$\text{s–g dual}\ [k_b^{\mathrm{TH}}] = (1-\nu_N)k_m^{\mathrm{SI}}, \tag{34d}$$

where we have made use of $\varepsilon_{\theta\theta}(a,\theta) = 0$ for a rigid insert.

6. ASYMPTOTIC ANALYSIS FOR LARGE VALUES OF μ

For sufficiently large values of μ, we may again approximate eqn (22) by

$$\chi_{,\rho\rho\rho\rho} + \mu^4\chi = 0, \tag{35a}$$

since we expect a sharp gradient in the radial direction near the edge. The solution of eqn (35a) with appropriate behavior for $\rho \to \infty$ is

$$\chi = e^{-\mu\zeta}(c_3 \cos \mu\zeta + c_4 \sin \mu\zeta) \sin 2\theta \equiv X(\rho) \sin 2\theta. \tag{35b}$$

From eqn (35b) and eqn (28), we obtain

$$\chi^{\pm} \sim e^{-\mu\zeta}[c_3(c+\eta_{\pm}s)+c_4(s-\eta_{\pm}c)] \sin 2\theta \equiv X^{\pm}(\rho) \sin 2\theta, \tag{36}$$

where $c = \cos \mu\zeta$ and $s = \sin \mu\zeta$, with

$$X^{\pm}(1) = c_3 - c_4\eta_{\pm}, \tag{37}$$

$$X^{\pm}_{,\rho}(1) = \frac{\mu}{\sqrt{2}}[c_3(-1+\eta_{\pm})+c_4(1+\eta_{\pm})]. \tag{38}$$

Upon substituting eqns (37), (38), (28a) and (28b) into eqns (30) and (31), both sets of (asymptotic) boundary conditions become four simultaneous equations for the determination of the four constants of integration c_1, c_2, c_3, c_4.

In this section, we consider two problems which have been investigated previously. Our concern with them here is to show the s–g, duality between the two problems which had not previously been indicated. Two other problems for which the factors of stress concentration are of order μ, rather than of order μ^0 and μ^2 as in the earlier problems (Reissner, 1980a, b, c, 1981, 1986; Reissner and Reissner, 1982), will be analyzed in the next section.

6.1. *Effect of a rigid inclusion for the problem of transverse twisting*

Setting $\beta_{si} = \alpha_{si} = 0$ in eqn (30) and considering the order of magnitude relation $\chi_{,\rho} = O(\mu\chi)$, we obtain from eqn (30), except for terms of relative order $1/\mu^2$, and the expressions for ϕ and ψ in eqn (28a)

$$\eta_{-}\mu^2\chi^{+}_{,\rho}+4(1+v_N)\mu^t(\psi_{T,\rho}-\psi_T) = 0, \qquad \phi_T+\chi^{+} = 0 \tag{39a, b}$$

$$\eta_{-}\mu^2\chi^{+}+(1+v_N)\mu^t(\psi_{T,\rho}-4\psi_T) = 0, \quad \phi_{T,\rho}+\chi^{+}_{,\rho} = 0 \tag{39c, d}$$

for $\rho = 1$. We set $t = 2$ in the above to balance the two types of terms in eqns (39a) and (39c) and use two of the relations in eqn (39) to eliminate $\chi^{+}(1,\theta)$ and $\chi^{+}_{,\rho}(1, \theta)$ from the other two to obtain

$$\eta_{-}\phi_T-(1+v_N)(\psi_{T,\rho}-4\psi_T) = 0, \quad \eta_{-}\phi_{T,\rho}-(1+v_N)(\psi_{T,\rho}-\psi_T) = 0 \tag{40}$$

at $\rho = 1$ for the determination of the two constants of integration c_1 and c_2 in the interior solution contributions. With $\phi_T(1,\theta)$ and $\psi_T(1,\theta)$ satisfying eqn (40), we subsequently obtain the two constants c_3 and c_4 in χ, except for terms of relative order $1/\mu$, with the help of the two conditions

$$\rho = 1: \quad \chi^{+} \sim -\phi_T, \quad \chi^{+}_{,\rho} \sim 0. \tag{41}$$

We now have as expressions for the resultant and for the couple which enter into the stress concentration factors,

$$M_{rr}(1,\theta) = -D\nabla^2 w_T(1,\theta) = \tfrac{1}{2}D\tau\mu^2\chi^{-}(1,\theta) = \frac{\mu^2 T}{4(1-v_M)}\chi^{-}(1,\theta) \tag{42a}$$

$$N_{rr}(1,\theta) = \frac{\nabla^2 K_T(1,\theta)}{1+\nu_N} = \frac{\tau\mu^2}{2(1+\nu_N)}\sqrt{\frac{D}{B}}\chi^+(1,\theta) = \frac{\mu^2 T}{4(1-\nu^2)}\frac{\chi^+(1,\theta)}{\sqrt{DB}}. \tag{42b}$$

Here we have used the relation $\nabla^2 K = N_{rr}+N_{\theta\theta}$ with $N_{\theta\theta}(1,\theta) = \nu_N N_{rr}(1,\theta)$ since $\varepsilon_{\theta\theta} = 0$ at $\rho = 1$. The expressions (42a, b) imply bending and membrane stress concentration factors of the order of magnitude μ^2 for isotropic shells with the expressions for k_b^{TI} and k_m^{TI} obtained from eqns (32d, h) in the form;

$$k_b^{\mathrm{TI}} = \frac{M_{rr}^{\mathrm{TI}}(1,\pi/4)}{M_{rr}^{\mathrm{TI}}(\infty,\pi/4)} \sim \frac{\mu^2}{2(1-\nu_M)}\chi^-\left(1,\frac{\pi}{4}\right), \tag{43a}$$

$$k_m^{\mathrm{TI}} = \frac{h}{6}\frac{N_{rr}^{\mathrm{TI}}(1,\pi/4)}{M_{rr}^{\mathrm{TI}}(\infty,\pi/4)} \sim \frac{\mu^2}{\sqrt{12(1-\nu^2)}}\chi^+\left(1,\frac{\pi}{4}\right). \tag{43b}$$

6.2. *Effect of a hole for the problem of membrane shearing*

The traction-free conditions at $r = a$ correspond to $\alpha_{di} = \beta_{di} = 0$ in eqn (31). With ϕ and ψ now as in eqn (28b), we have as asymptotic boundary conditions at $\rho = 1$ for this case

$$\eta_+\mu^2\chi^-_{,\rho} + 4(1-\nu_M)\mu^2(\phi_{S,\rho}-\phi_S) = 0, \quad \psi_S+\chi^- = 0, \tag{44a, b}$$

$$\eta_+\mu^2\chi^- + (1-\nu_M)\mu^s(\phi_{S,\rho}-4\phi_S) = 0, \quad \psi_{S,\rho}+\chi^-_{,\rho} = 0. \tag{44c, d}$$

Upon setting now $s = 2$, again to balance terms in eqns (44a, c), we obtain as two boundary conditions for ϕ_S and ψ_S

$$\eta_+\psi_{S,\rho} - 4(1-\nu_M)(\phi_{S,\rho}-\phi_S) = 0, \quad \eta_+\psi_S - (1-\nu_M)(\phi_{S,\rho}-4\phi_S) = 0, \tag{45}$$

and then as boundary conditions for χ^-, except for terms of relative order $1/\mu$,

$$\rho = 1: \quad \chi^- \sim -\psi_S, \quad \chi^-_{,\rho} \sim 0. \tag{46}$$

With $M_{rr} = 0$ and $N_{rr} = 0$ for $\rho = 1$, we now have the relevant stress concentration quantities

$$M_{\theta\theta}(1,\theta) = -(1+\nu_M)D\nabla^2 w_S = -\tfrac{1}{2}S\mu^2(1+\nu_M)\sqrt{DB}\chi^-(1,\theta) \tag{47a}$$

$$N_{\theta\theta}(1,\theta) = \nabla^2 K_S = -\tfrac{1}{2}S\mu^2\chi^+(1,\theta). \tag{47b}$$

These expressions again imply bending and membrane stress concentration factors of order of magnitude μ^2, with the following asymptotic expressions for k_m^{SH} and k_b^{SH} obtained by the s–g duality with the help of eqns (33) and (43);

$$k_m^{\mathrm{SH}} = \text{s–g dual }[(1-\nu_M)k_b^{\mathrm{TI}}] \sim \frac{\mu^2}{2}\chi^+\left(1,\frac{\pi}{4}\right), \tag{48a}$$

$$k_b^{\mathrm{SH}} = \text{s-g dual }[3(1-\nu_N)k_m^{\mathrm{TI}}] \sim \frac{\mu^2}{2}\sqrt{\frac{3+3\nu_M}{1-\nu_N}}\chi^-\left(1,\frac{\pi}{4}\right) \tag{48b}$$

consistent with the results of Reissner (1980b, c).

We refer to Reissner (1980c, 1986) for a discussion of the peculiar near and far field behavior of the interior solution of the foregoing two problems, involving the transition to and from, and the coexistence of inextensional bending and membrane state domains.

7. SHELLS WITH AN INNER EDGE TRANSVERSELY FIXED AND TANGENTIALLY FREE

Because of its relative simplicity and because the asympototic results for this class of problems are intermediate relative to the previously found $O(1)$ and $O(\mu^2)$ behavior, we consider in this section the special case $\alpha_{s1} = \alpha_{s2} = \beta_{d1} = \beta_{d2} = 0$. We may evidently stipulate $t = s = 0$ for the values of the exponents in eqns (30) and (31), which then reduce to

$$\phi_{,\rho} + \chi^{+}_{,\rho} = 0, \quad \psi_{,\rho} + \chi^{-}_{,\rho} = 0 \tag{49a, b}$$

$$\phi + \chi^{+} = 0, \quad \psi + \chi^{-} = 0 \tag{49c, d}$$

for $(\phi, \psi) = (\phi_S, \psi_S)$ as in eqns (28c, d), as well as for $(\phi, \psi) = (\phi_T, \psi_T)$ as in eqns (28a, b).

Note that the conditions (49b, d) [corresponding to $K(1, \theta) = K_{,\rho}(1, \theta) = 0$] imply that the edge $\rho = 1$ of the shell is tangentially traction-free. At the same time, the conditions (49a, c) stipulate the vanishing of transverse edge deflections and of the edge slope of the shell [corresponding to $w(1, \theta) = w_{,\rho}(1, \theta) = 0$]. Hence, we have effectively a (tangential) roller-support at the edge $r = a$.

We may use eqn (38) to deduce from eqns (49a, b)

$$\sqrt{2}\mu c_4 \sin 2\theta = -\phi_{,\rho}(1, \theta), \quad \sqrt{2}\mu c_3 \sin 2\theta = -\psi_{,\rho}(1, \theta), \tag{50a, b}$$

for a leading term approximation, and use eqn (37) to deduce from eqns (49c, d)

$$(c_4 - c_3) \sin 2\theta = \phi(1, \theta), \quad -(c_3 + c_4) \sin 2\theta = \psi(1, \theta). \tag{50c, d}$$

From the four relations in eqns (50a–d), we obtain as boundary conditions for the interior solution contributions ϕ and ψ;

$$\rho = 1: \quad \psi + \frac{1}{\sqrt{2}\mu}(\psi_{,\rho} - \phi_{,\rho}) = 0, \qquad \phi + \frac{1}{\sqrt{2}\mu}(\psi_{,\rho} - \phi_{,\rho}) = 0. \tag{51a, b}$$

For the problem of transverse twisting with $\phi = \phi_T$ and $\psi = \psi_T$ designated as the TR problem, it follows from eqns (51a, b) that, except for higher order terms in $1/\mu$,

$$\rho = 1: \quad \phi_T = 0, \quad \psi_T = \frac{1}{\sqrt{2}\mu}\phi_{T,\rho} \tag{52a, b}$$

and, therewith,

$$c_1 = -1, \quad c_2 = 2\sqrt{2}/\mu \tag{53a, b}$$

and, in accordance with eqns (50a, b),

$$c_3 = 4/\mu^2 \sim 0, \quad c_4 = -2\sqrt{2}/\mu. \tag{53c, d}$$

Expressions for the relevant stress concentration quantities for this TR problem are now

$$M_{rr}^{\mathrm{TR}}(1, \theta) = -D\nabla^2 w_T(1, \theta) = \tfrac{1}{2}D\tau\mu^2\chi^{-}(1, \theta)$$

$$\sim \tfrac{1}{2}D\tau\mu^2(c_3 + c_4) = -\sqrt{2}D\tau\mu = \frac{-T\mu}{\sqrt{2(1 - \nu_M)}} \tag{54a}$$

in view of eqns (23a), (26) and (53c, d), and

$$N_{\theta\theta}^{\mathrm{TR}}(1,\theta) = \nabla^2 K_T(1,\theta) = \tfrac{1}{2}\tau\mu^2\sqrt{\frac{D}{B}}\chi^+(1,\theta)$$

$$\sim \tfrac{1}{2}\tau\mu^2\sqrt{\frac{D}{B}}(c_3-c_4) = \tau\sqrt{2}\mu\sqrt{\frac{D}{B}} = \frac{T\mu}{\sqrt{2DB}(1-\nu_M)} \tag{54b}$$

in view of eqns (23b), (26) and (53c, d). It is apparent that, for this case, we encounter stress concentration factors of order of magniude μ for isotropic shells with sufficiently large values of μ, rather than of order μ^0 or μ^2,

$$k_b^{\mathrm{TR}} = \frac{M_{rr}^{\mathrm{TR}}(1,\pi/4)}{M_{rr}^{\mathrm{TR}}(\infty,\pi/4)} \sim -\frac{\sqrt{2}\mu}{1-\nu_M},\quad k_m^{\mathrm{TR}} = \frac{hN_{\theta\theta}^{\mathrm{TR}}(1,\pi/4)}{6M_{rr}^{\mathrm{TR}}(\infty,\pi/4)} \sim \sqrt{\frac{2+2\nu_N}{3-3\nu_M}}\mu. \tag{54c, d}$$

For the problem of membrane shearing with roller support (designated as the SR problem) with $\phi = \phi_s$ and $\psi = \psi_s$ in accordance with eqns (28c, d), we can proceed as in the TR problem above and obtain asymptotic results analogous to eqns (53) and (54). However, the same results can be written down immediately by appealing to the s–g duality between the SR problem and the TR problem. In particular, we obtain from the s–g duals of eqn (52a, b) a leading term asymptotic form of eqns (49c, d),

$$\rho = 1:\quad \psi_S = 0,\quad \phi_S = \frac{1}{\sqrt{2}\mu}\psi_{S,\rho}. \tag{55a, b}$$

Correspondingly, we have from the s–g duals of eqns (53a, b) [see also eqn (28)]

$$c_1 = 2\sqrt{2}/\mu,\quad c_2 = -1 \tag{56a, b}$$

and, from eqns (53c, d) [see also eqn (36)],

$$c_3 = 2\sqrt{2}\mu,\quad c_4 = 4/\mu^2 \sim 0 \tag{56c, d}$$

The s–g duals of $M_{rr}^{\mathrm{TR}}(1,\theta)$ and $N_{\theta\theta}^{\mathrm{TR}}(1,\theta)$ are

$$\varepsilon_{\theta\theta}^{\mathrm{SR}}(1,\theta) \sim -SB\sqrt{2}\mu,\quad \kappa_{rr}^{\mathrm{SR}}(1,\pi/4) \sim S\sqrt{\frac{B}{D}}\sqrt{2}\mu. \tag{57a, b}$$

With $\varepsilon_{\theta\theta}^{\mathrm{SR}}(1,\theta) = BN_{\theta\theta}^{\mathrm{SR}}(1,\theta)$ and $\kappa_{rr}^{\mathrm{SR}}(1,\theta) = M_{rr}^{\mathrm{SR}}(1,\theta)/D$, we then have as expressions for the relevant stress concentration quantities

$$M_{rr}^{\mathrm{SR}}(1,\pi/4) = S\mu\sqrt{2DB},\quad N_{\theta\theta}^{\mathrm{SR}}(1,\pi/4) = -\sqrt{2}S\mu, \tag{58a, b}$$

with

$$k_m^{\mathrm{SR}} = \frac{N_{\theta\theta}^{\mathrm{SR}}(1,\pi/4)}{N_{\theta\theta}^{SR}(\infty,\pi/4)} \sim \sqrt{2}\mu,\quad k_b^{\mathrm{SR}} = \frac{6M_{rr}^{\mathrm{SR}}(1,\pi/4)}{hN_{\theta\theta}^{\mathrm{SR}}(\infty,\pi/4)} \sim \frac{6\mu\sqrt{2DB}}{h} = \sqrt{\frac{6}{1-\nu^2}}\mu. \tag{59a, b}$$

In contrast to the results for the two problems in Section 6, we now have, from eqns (53) and (56), inextensional bending and membrane states throughout the exterior of the edge zone of roller supported shells for the problem of twisting as well as for the problem of shearing, respectively.

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