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CIRCULAR ELASTIC PLATES WITH PRESCRIBED AXISYMMETRIC EDGE DISPLACEMENTS

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ABSTRACT. Three canonical (singular) solutions are obtained for the axisymmetric elastostatics of an orthotropic circular plate with its only edge constrained from any displacement. Each three-dimensional elastostatic problem is first reduced to a scalar integral equation which is then solved numerically by orthogonal collocation. Moments of stresses at the edge of the plate for the canonical problems are calculated to be used in the formulation of the proper displacement boundary conditions for two-dimensional thin or thick plate theory solutions. These moments are also used here to determine the interior solution of a circular plate subject to a point load at the center of the plate's upper face.

1. Introduction. The exact solution of the equations of three-dimensional elastostatics for an isotropic homogeneous flat plate is known to consist of an *interior* component and an *edge zone* component. The interior solution is significant throughout the plate, while the edge zone solution is significant only in a narrow region adjacent to the edges of the plate [**3, 5, 6, 7, 9, 22, 23**]. The exact interior solution may be taken in the form of the Lévy solution [**14**]; various thin and thick plate theories are different truncated parametric series expansions of this exact solution in powers of a small thickness parameter [**3, 5, 9, 22–24**]. Except for cases with simple geometries and load symmetries, the exact edge zone solution is rather intractable.

By itself, the interior solution component generally cannot be made to fit the prescribed boundary conditions at an edge of the plate. When the edge data are given in terms of stresses, Saint Venant's principle is usually invoked to provide a set of boundary conditions for the Lévy solution or the various approximating plate theories (see [**21**] and [**25**] for example). Strictly speaking, the principle is not applicable to plate problems as the perimeter of the plate is not small compared to the

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characteristic length of the problem such as the plate span. Recent developments in [**10–12**] justify the application of Saint Vernant's principle for plates under suitable geometric and load conditions but also delineate its limitation.

Much less is known about plate theory solutions when edge data are prescribed in terms of displacements. Saint Venant's principle is not useful here. Until the developments in [**9–12**], the only general approach available for this class of problems had been to perform a matched asymptotic solution for a given three-dimensional problem. Unfortunately, the inner (expansion) solution needed to fit the prescribed data in this approach is difficult to obtain and too complex to be useful in practice [**3, 22, 23**]. For this reason, there has been for some time an effort to find an analogue of Saint Venant's principle for stress data to assign an appropriate portion of the prescribed displacement or mixed data to the interior solution so that it can be determined independent of the boundary layer (or the inner) solution. A correct method for doing this appeared for the first time in 1984 [**9**] and was subsequently implemented for some nontrivial isotropic plate problems in [**10–12, 19, 20**]. The theoretical basis for the method is described in [**12, 17**].

We see from the developments of [**9–12**] that, for the correct formulation of boundary conditions at a cylindrical edge of the plate, we need the solution of a certain number of canonical problems for the particular plate. The form of the canonical problems is fixed by the type of edge data (such as stresses, displacements, or mixed data) but is independent of any change in the prescribed distributions of the data. The canonical problems for displacement data generally do not admit a simple exact solution. Numerical solutions for the canonical problems have been obtained for isotropic plates in plane strain deformations (i.e., the strip problem) in [**8**] (see also references given in [**9**]). Similar results have also been obtained for orthotropic strips [**19, 20**]. Applications of these results to specific physical problems can be found in [**9–12, 19, 20**].

In this paper circular plates in axisymmetric deformations with prescribed displacement edge data will be treated. The central issue will again be a set of correct displacement boundary conditions for the interior solution or its various plate theory approximations. For added generality, we allow the circular plate to be orthotropic in the formulation and reduction of the boundary value problem. A brief summary of the procedure for deducing the correct edge conditions introduced in [**10–12**] will be given in Section 3 to show why we need the solution of a number of canonical boundary value problems.

Because of axisymmetry, these canonical problems are boundary value problems in two independent variables. In Section 4 we recast these problems by finite Hankel transforms to prepare for their reduction to one-dimensional problems. A scalar integral equation for the unknown radial edge stress is deduced from the Hankel transform solution in Section 5. This integral equation is then solved numerically in Section 6 by an orthogonal collocation method similar to that used for the plane strain problem in [**19, 20**]. The nature of the stress singularity at the rim of the cylinder needed for an efficient accurate numerical solution has already been analyzed previously in [**18**].

Applications of the solution of these canonical problems are illustrated in Sections 7 and 8. In Section 7 we see how the moments of stresses of the canonical problems appear naturally in the proper displacement boundary conditions for plate theories. In Section 8 we obtain the interior solution for the problem of a circular plate clamped at the outer edge and subject to a point force at the center of the plate's upper surface, again with the help of the solution of a number of canonical problems.

The boundary value problems for our circular plate can also be solved by the method of eigenfunction expansions. Because of solution singularities at the rims, the method will have convergence difficulty there. We have not experienced any numerical instability or convergence difficulty in our integral equation approach.

2. Axisymmetric deformation of short orthotropic circular cylinders. In cylindrical coordinates the behavior of an orthotropic circular cylinder in axisymmetric deformation is governed by the following sets of conditions:

(i) Two equilibrium equations for the four relevant stress components $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$ and $\sigma_{rz} = \sigma_{zr}$

$$(r\sigma_{rr})_{,r} - \sigma_{\theta\theta} + r\sigma_{rz,z} = 0, \qquad (r\sigma_{rz})_{,r} + r\sigma_{zz,z} = 0 \tag{2.1}$$

where a comma in an equation indicates partial differentiation so that

$(\ )_{,t} \equiv \partial(\ )/\partial t$.

(ii) Four stress-strain relations relating the four strain components ε_{rr}, $\varepsilon_{\theta\theta}$, ε_{zz} and $\varepsilon_{rz} = \varepsilon_{zr}$ to the stress components [**15**]:

(2.2)
$$\varepsilon_{rr} = a_{11}\sigma_{rr} + a_{12}\sigma_{\theta\theta} + a_{13}\sigma_{zz}, \qquad \varepsilon_{\theta\theta} = a_{21}\sigma_{rr} + a_{22}\sigma_{\theta\theta} + a_{23}\sigma_{zz}$$
$$\varepsilon_{zz} = a_{31}\sigma_{rr} + a_{32}\sigma_{\theta\theta} + a_{33}\sigma_{zz}, \qquad \varepsilon_{rz} = a_{44}\sigma_{rz}.$$

(iii) Four strain-displacement relations defining the strain components in terms of the radial and axial displacement components, u_r and u_z

$$\varepsilon_{rr} = u_{r,r}, \qquad \varepsilon_{\theta\theta} = \frac{1}{r}u_r, \qquad \varepsilon_{zz} = u_{z,z}, \tag{2.3}$$
$$\varepsilon_{rz} = \varepsilon_{zr} = u_{r,z} + u_{z,r}.$$

(iv) Appropriate boundary conditions on the surface of the elastic body. Note that we have taken the equilibrium equations (2.1) to be homogeneous as any distributed load in the interior of the body may be removed by particular integrals. For simplicity, we also limit ourselves for the most part of this paper to transversely isotropic materials with $a_{22} = a_{11}$, $a_{21} = a_{12}$ and $a_{32} = a_{23} = a_{31} = a_{13}$ where

$$a_{11} = \frac{1}{E}, \qquad a_{12} = -\frac{\nu}{E}, \qquad a_{33} = \frac{1}{E_3}, \tag{2.4}$$
$$a_{13} = -\frac{\nu_3}{E_3}, \qquad a_{44} = \frac{1}{G_3}.$$

The elastic moduli E, E_3, G_3, ν and ν_3 in (2.4) are assumed to be constant; E_3, E and G_3 are necessarily positive.

We consider a short cylinder (or flat circular plate) which extends from $z = -h$ to $z = h$ in the transverse direction and from $r = r_i$ to $r = r_0$ in the radial direction with $r_0 > r_i \geq 0$. The two faces $z = \pm h$ are free of surface tractions so that

$$z = \pm h: \quad \sigma_{rz} = \sigma_{zz} = 0. \tag{2.5}$$

In a later section we will discuss specific admissible combinations of stress and displacement conditions along the cylindrical (constant r) edge(s) to complete the problem formulation.

It is not difficult to verify that the strain measures given by (2.3) satisfy the following two compatibility equations

$$(r\varepsilon_{\theta\theta})_{,r} = \varepsilon_{rr}, \qquad \varepsilon_{rr,zz} + \varepsilon_{zz,rr} = \varepsilon_{rz,rz}. \tag{2.6}$$

As in the isotropic case, the compatibility equations (2.6) and equilibrium equations (2.1) are satisfied identically by the stress function representation

$$\begin{aligned}
\sigma_{rr} &= -\frac{\partial}{\partial z}\left[\frac{\partial^2}{\partial r^2} + \frac{\alpha_2}{r}\frac{\partial}{\partial r} + \alpha_1\frac{\partial^2}{\partial z^2}\right]\phi, \\
\sigma_{zz} &= \frac{\partial}{\partial z}\left[\alpha_3\Delta_r + \alpha_4\frac{\partial^2}{\partial z^2}\right]\phi, \\
\sigma_{\theta\theta} &= -\frac{\partial}{\partial z}\left[\alpha_2\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \alpha_1\frac{\partial^2}{\partial z^2}\right]\phi, \\
\sigma_{rz} = \sigma_{zr} &= \frac{\partial}{\partial r}\left[\Delta_r + \alpha_1\frac{\partial^2}{\partial z^2}\right]\phi,
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
&\Delta_r(\;) \equiv (\;)_{,rr} + r^{-1}(\;)_{,r}, && \beta\alpha_1 = a_{13}(a_{11} - a_{12}), \\
&\beta\alpha_2 = a_{13}(a_{13} + a_{44}) - a_{12}a_{33}, && \beta\alpha_4 = a_{11}^2 - a_{12}^2 \\
&\beta\alpha_3 = a_{13}(a_{11} - a_{12}) + a_{11}a_{44}, && \beta = a_{11}a_{33} - a_{13}^2
\end{aligned} \tag{2.8}$$

with ϕ being a solution of the fourth order partial differential equation

$$\begin{aligned}
&\left[\Delta_r^2 + (\alpha_1 + \alpha_3)\Delta_r\frac{\partial^2}{\partial z^2} + \alpha_4\frac{\partial^4}{\partial z^4}\right]\phi \\
&\qquad \equiv \left[\Delta_r + \frac{1}{s_1^2}\frac{\partial^2}{\partial z^2}\right]\left[\Delta_r + \frac{1}{s_2^2}\frac{\partial^2}{\partial z^2}\right]\phi = 0
\end{aligned} \tag{2.9}$$

where

$$\{s_1^2, s_2^2\} = \frac{1}{2\alpha_4}[(\alpha_1 + \alpha_3) \pm \sqrt{(\alpha_1 + \alpha_3)^2 - 4\alpha_4}]. \tag{2.10}$$

For isotropic materials, we have $2\alpha_4 = \alpha_1 + \alpha_3 = 2$ and $s_1 = s_2 = 1$ so that equation (2.9) reduces correctly to a biharmonic equation with axisymmetry. With $\alpha_1 = \alpha_2 = -\nu/(1-\nu)$ and $\alpha_3 = (2-\nu)/(1-\nu)$, the representation (2.7) also reduces correctly to the corresponding representation for the isotropic case (cf. (5.2) in [**10**]). For our more general orthotropic plate, the positive definiteness of the strain energy of the plate requires $\beta > 0$ and $\alpha_4 > 0$.

Five different exact solutions of (2.9) have been obtained in [**17**]; their linear combination is also an exact solution. The corresponding stress and displacement fields are

(2.11a,b)
$$\sigma^I_{rz} = \frac{c}{r}(\alpha_4 - \alpha_1\alpha_3)(h^2 - z^2), \qquad \sigma^I_{zz} \equiv 0$$

(2.11c)
$$\sigma^I_{rr} = a\left\{\frac{z}{r^2}\right\} + b\left\{\frac{1}{r^2}\right\} + c\left\{\frac{1}{3}(1-\alpha_2)\alpha_3\frac{z^3}{r^2} + \beta_1 z\ln(r)\right\} + d\{z\} + e$$

(2.11d)
$$\sigma^I_{\theta\theta} = -a\left\{\frac{z}{r^2}\right\} - b\left\{\frac{1}{r^2}\right\} + c\left\{\frac{1}{3}(\alpha_2 - 1)\alpha_3\frac{z^3}{r^2} + \beta_1 z\ln(r) + (\alpha_2 - 1)\alpha_4 z\right\} + d\{z\} + e$$

(2.12a)
$$u^I_r = a\left\{(a_{12} - a_{11})\frac{z}{r}\right\} + b\left\{(a_{12} - a_{11})\frac{1}{r}\right\} + d\{(a_{12} + a_{11})zr\} + e\{(a_{12} + a_{11})r\} + c\left\{\frac{1}{3}(a_{12} - a_{11})(1-\alpha_2)\alpha_3\frac{z^3}{r} + (a_{12} + a_{11})\beta_1 zr\ln(r) + a_{11}(\alpha_2 - 1)\alpha_4 rz\right\}$$

(2.12b)
$$u^I_z = a\left\{(a_{11} - a_{12})\ln\left(\frac{r}{r_0}\right)\right\} + d\left\{\frac{1}{2}(a_{12} + a_{11})(r_0^2 - r^2) + a_{13}z^2\right\} + e\{2a_{13}z\} + w_0$$

$$
\begin{aligned}
&+ c\left\{ a_{44}(\alpha_4 - \alpha_1\alpha_3)h^2 \ln\left(\frac{r}{r_0}\right) \right. \\
&\quad - \frac{1}{2}(a_{12} + a_{11})\beta_1[r^2 \ln(r) - r_0^2 \ln(r_0)] \\
&\quad + \frac{1}{4}(r^2 - r_0^2)[\beta_1(a_{11} + a_{12}) - 2a_{11}\alpha_4(\alpha_2 - 1)] \\
&\quad \left. + a_{13}z^2 \left[\frac{1}{2}(\alpha_2 - 1)\alpha_4 + \beta_1 \ln(r)\right] \right\}
\end{aligned}
$$

where $\beta_1 = (\alpha_2 + 1)\alpha_4 - 2\alpha_1\alpha_3$ and a, b, c, d, e, and w_0 are six arbitrary constants with w_0 being axial displacement of the ring $\{r = r_0, z = 0\}$. It is easy to see from (2.11a,b) that they also satisfy the stress-free conditions (2.5) at the top and bottom faces of the plate. As such, it is the axisymmetric solution of (2.9) and (2.5) for a circular plate of radius r_0 which is significant throughout the plate; it is called the *interior* (or outer) solution of the boundary value problem.

Evidently, it is not possible to specify the six arbitrary constants in the exact (interior) solution (2.11) and (2.12) to fit an admissible set of edge data at edge $r = r_0$, involving higher (than fifth) powers of z or nonpolynomial dependence on z. Additional *residual* (edge zone) solution components have been obtained in [**17**] to allow for an eigenfunction expansion solution which is expected to fit any admissible data at $r = r_0$. However, the determination of these residual solution (components) is known to be difficult and costly even for the isotropic case. At the same time, the effects of the residual solution are confined to a narrow layer adjacent to the edge(s) of the plate if the plate is sufficiently thin (see [**17**] for the precise nature of the exponential decay). For many physical problems, the actual interior solution or its thin or thick plate approximation suffices for all practical purposes. Considerable effort has therefore been made over the years to determine how to assign an appropriate portion of the edge data to the interior solution without any reference to the (residual) edge zone solution. For axisymmetric deformations of orthotropic circular plates, the proper assignment has been obtained in [**17**] for two of the four possible combinations of admissible edge data with the help of certain first integrals of the equations of linear elastostatics (see also [**10**]). For the proper assignment in the case of axisymmetric displacement edge data (by way of the reciprocal theorem in linear elasticity), the numerical solutions of a number of three-dimensional elastostatic problems involving axisymmetric deformations of circular cylinders are needed. The determination of accurate solutions for these canonical problems is the principal concern of this paper.

3. Necessary conditions for an edge zone state and canonical problems. In the absence of interior loading, the reciprocal theorem of linear elasticity, applied to the two axisymmetric torsionless elastostatic states $\{\sigma_{ij}^{(1)}, u_k^{(1)}\}$ and $\{\sigma_{ij}^{(2)}, u_k^{(2)}\}$ over the annular region $R_m = \{(r_i <)r_m \le r \le r_0, |z| \le h\}$ of our circular plate, takes the form

$$\sum_{j=1}^{3} \int_{S_m}\!\!\int \{\sigma_{nj}^{(1)} u_j^{(2)} - \sigma_{nj}^{(2)} u_j^{(1)}\}\, dS = 0 \tag{3.1}$$

where S_m is the surface of the R_m and n denotes the outward normal direction. Our plate is traction-free at $z = \pm h$; hence, the integrals over the two faces do not contribute to the surface integral of (3.1), leaving us with

$$2\pi \int_{-h}^{h} [r\{\sigma_{rr}^{(1)} u_r^{(2)} + \sigma_{rz}^{(1)} u_z^{(2)} - \sigma_{rr}^{(2)} u_r^{(1)} - \sigma_{rz}^{(2)} u_z^{(1)}\}]_{r_m}^{r_0}\, dz = 0. \tag{3.2}$$

Suppose state (2) has at most an algebraic growth $h/r_0 \to 0$ and state (1) is an edge zone state which becomes exponentially small as $h/r_0 \to 0$. In that case, the contributions from the integral over the cylindrical surface $r = r_m$ is negligible so that, except for exponentially small terms, (3.2) may be written as

$$2\pi r_0 \int_{-h}^{h} [\sigma_{rr}^{(1)} u_r^{(2)} + \sigma_{rz}^{(1)} u_z^{(2)} - \sigma_{rr}^{(2)} u_r^{(1)} - \sigma_{rz}^{(2)} u_z^{(1)}]_{r=r_0}\, dz = 0. \tag{3.3}$$

(A more careful analysis taking into account stress singularities which may exist at the rims of the cylinder [**16, 18**] is parallel to a similar analysis for the plane strain case in [**9**] and will not be carried out herein.)

Consider the case of prescribed axisymmetric edge displacements at $r = r_0$

The residual state $\{\sigma_{ij}^R \equiv \sigma_{ij} - \sigma_{ij}^I, u_j^R \equiv u_j - u_j^I\}$, i.e., the difference between the actual elastostatic state $\{\sigma_{ij}, u_j\}$ induced by the data and its interior solution component $\{\sigma_{ij}^I, u_j^I\}$ of Section 2, is expected to be an edge zone phenomenon [**3, 5, 7, 22, 23**]. Let state (1) in (3.3) be this residual state and state (2) satisfy the condition of zero edge displacements at $r = r_0$

$$u_r^{(2)}(r_0, z) = u_z^{(2)}(r_0, z) = 0, \qquad |z| < h. \tag{3.5}$$

In that case, the reciprocal relation requires

$$\int_{-h}^{h} [\sigma_{rr}^{(2)} \bar{u}_r^R + \sigma_{rz}^{(2)} \bar{u}_z^R]_{r=r_0}\, dz = 0 \tag{3.6}$$

(except for exponentially small terms) with $\bar{u}_t^R \equiv \bar{u}_t(z) - u_t^I(r_0, z)$. Thus, for $\{\sigma_{ij}^R, u_j^R\}$ to be an edge zone phenomenon, the corresponding edge data $\{\bar{u}_r^R(z), \bar{u}_z^R(z)\}$ must satisfy (3.6) for any state (2) which meets the requirements stipulated above.

Evidently, there are infinitely many candidates for state (2) satisfying (i) the governing equations of axisymmetric, torsionless linear orthotropic elasto-statics (2.1), (2.2) and (2.3), (ii) the traction-free conditions (2.5) on the two faces, (iii) algebraic growth constraint as $h/r_0 \to 0$, and (iv) homogeneous displacement edge conditions (3.5). For any one of these candidates, (3.6) may be rewritten as

$$\int_{-h}^{h} [\sigma_{rr}^{(2)} u_r^I + \sigma_{rz}^{(2)} u_z^I]_{r=r_0}\, dz = \int_{-h}^{h} [\sigma_{rr}^{(2)} \bar{u}_r(z) + \sigma_{rz}^{(2)} \bar{u}_z(z)]_{r=r_0}\, dz. \tag{3.7}$$

In this form, the reciprocal relation effectively assigns a portion of the displacement edge data to the interior solution. It is analogous to the assignments of stress edge data (in resultant force and moment) by Saint Venant's principle in three-dimensional elasticity. Saint Venant's principle is now known to be inappropriate for axisymmetric deformation of circular plates [**10, 11**] except asymptotically in the limiting case as $h \to 0$, i.e., for sufficiently thin plates).

From the form of the interior solution, we see that three such assignments are needed for the case of a circular plate with no holes

$(0 \leq r \leq r_0)$; they determine the three constants w_0, d and e while a, b and c must be set to zero for boundedness. (The exceptional case of a point load is treated in Section 8). For an annular plate $0 \leq r_i \leq r \leq r_0)$ for which a, b and c are not required to vanish, three similar assignments are needed (and available) for the edge data at $r = r_i$ for the simultaneous determination of all six constants.

We define three special interior states for our circular plate by setting appropriate values for the parameters $\{a, b, \dots\}$ in (2.11) and (2.12) as follows:

$$\text{State (a): } \{a = 1, \text{ all other five constants are set to zero}\}$$

(3.8) $$\text{State (b): } \{b = 1, \text{ all other five constants are set to zero}\}$$

$$\text{State (c): } \{c = 1, \text{ all other five constants are set to zero}\}.$$

Then we choose the three (2)-states to be the solution of three canonical problems which satisfy the clamped edge conditions at $r = r_0$ and have the same singular behavior as the special states (a), (b) and (c), respectively, as $r \to 0$. Our main task is to obtain accurate solutions for these three (2)-states.

By superposition, we may work with the *residua problems* corresponding to the difference between the desired (2) state and the corresponding special state with the same singular behavior at $r = 0$. The residual solutions are nonsingular everywhere. They allow us to focus our attention on boundary value problems of an orthotropic circular plate free of tractions at the faces with finite stresses and displacements at the center of the plate and with the following prescribed inhomogeneous displacement edge data:

(3.9a) $$(a) \quad u_r(r_0, z) \equiv [u_r^{(2)} - u_a]_{r=r_0} = (a_{11} - a_{12}) \frac{z}{r_0}$$

(3.9b) $$u_z(r_0, z) \equiv [u_z^{(2)} - w_a]_{r=r_0} = 0$$

(3.10a) $$(b) \quad u_r(r_0, z) \equiv [u_r^{(2)} - u_b]_{r=r_0} = (a_{11} - a_{12}) \frac{1}{r_0}$$

(3.10b) $$u_z(r_0, z) \equiv [u_z^{(2)} - w_b]_{r=r_0} = 0$$

(3.11a)

$$
\begin{aligned}
(c)\quad u_r(r_0,z) &\equiv [u_r^{(2)} - u_c]_{r=r_0} \\
&= \frac{1}{3}\alpha_3(1-\alpha_2)(a_{11}-a_{12})\frac{z^3}{r_0} \\
&\quad - (a_{11}+a_{12})\beta_1 r_0 z \ln(r_0) \\
&\quad - \alpha_4(\alpha_2-1)a_{11}r_0 z
\end{aligned}
$$

(3.11b)

$$
\begin{aligned}
u_z(r_0,z) &\equiv [u_z^{(2)} - w_c]_{r=r_0} \\
&= \left[\frac{1}{2}(1-\alpha_2)\alpha_4 - \beta_1\ln(r_0)\right]a_{13}z^2
\end{aligned}
$$

where u_t and w_t are defined to be u_r^I and u_z^I, respectively, for the special state t. Any state (2) defined above does not depend on the prescribed edge data at $r = r_0$ of the original problem. Hence, it is determined once and for all. For the same plate geometry (h/r_0) and material (a_{ij}), it may be used for the determination of the interior solution of the same type of boundary value problem with any distributions of displacement edge data.

To obtain the solution for any one of the three canonical problems described above, we note that σ_{zz} satisfies the same fourth order partial differential equation (2.9) as ϕ so that

$$
\left[\Delta_r + \frac{1}{s_1^2}\frac{\partial^2}{\partial z^2}\right]\left[\Delta_r + \frac{1}{s_2^2}\frac{\partial^2}{\partial z^2}\right]\sigma_{zz} = 0 \tag{3.12}
$$

because the operator $\partial(\ )/\partial z$ commutes with $\Delta_r(\ )$. Also, the second equilibrium equation (2.1b) may be written as $r\sigma_{zz,z} = -\sigma_{rz} - r\sigma_{rz,r}$. Therefore, the condition $\sigma_{rz}(r,\pm h) \equiv 0$ implies $\sigma_{zz,z}(r,\pm h) \equiv 0$ and the traction-free conditions in (2.5) may be written as

$$
\sigma_{zz}(r,\pm h) = \sigma_{zz,z}(r,\pm h) = 0, \qquad 0 \le r \le r_0. \tag{3.13}
$$

Evidently, it is less cumbersome to formulate the canonical problems in terms of $s(r,z) \equiv \sigma_{zz}(r,z)$ instead of $\phi(r,z)$. We will use this alternate formulation to facilitate a reduction of the canonical problems to the solution of a scalar Fredholm integral equation of the second kind for the edge value of the radial stress component where displacement data are prescribed. Given $\sigma_{zz}^I \equiv 0$ (see (2.11b)), $s(r,z)$ is also the transverse normal stress of any one of the residual problems.

4. Finite Hankel transforms. Any one of the three canonical problems for the pure displacement edge data case may be specified by (i) the partial differential equation (3.12) for the axial stress component $s(r,z)$ (of state (2) and of the corresponding residual problem), (ii) the traction-free conditions (3.13), (iii) the displacement edge conditions

$$u_r(r_0,z)=u(z), \qquad u_z(r_0,z)=w(z), \tag{4.1}$$

for suitable $u(z)\equiv -u_\alpha$ and $w(z)\equiv -w_\alpha$, $\alpha=a,b$ or c (see equations (3.9)–(3.11)) and (iv) regularity conditions on stresses and displacements in the plate interior. For an efficient solution of these problems, we introduce the finite Hankel transform $S_i(z)$, $i=1,2,3,\ldots$, of $s(r,z)$:

$$S_i(z)=\int_0^{r_0} rJ_0(\rho_i r)s(r,z)\,dr\equiv H[s] \tag{4.2}$$

where $J_0(x)$ is the zero$^{\text{th}}$ order Bessel function of the first kind and $x_i\equiv\rho_i r_0$, $i=1,2,3,\ldots$, are the zeros of $J_0(x)$. By the inversion formula, we may express s in terms of $\{S_i(z)\}$ by

$$s(r,z)=\frac{2}{r_0^2}\sum_{i=1}^{\infty}S_i(z)\frac{J_0(\rho_i r)}{[J_1(\rho_i r_0)]^2} \tag{4.3}$$

where $J_1(x)$ is the first order Bessel function of the first kind.

It is not difficult to show by repeated integration by parts that

$$H[\Delta_r s]=r_0\rho_i s(r_0,z)J_1(\rho_i r_0)-\rho_i^2 S_i(z) \tag{4.4}$$

$$\begin{aligned} H[\Delta_r\Delta_r s]&=r_0\rho_i[\Delta_r s]_{r=r_0}J_1(\rho_i r_0)\\ &\quad -r_0\rho_i^3 s(r_0,z)J_1(\rho_i r_0)+\rho_i^4 S_i(z)\end{aligned} \tag{4.5}$$

where we have made use of the two relations between J_0 and J_1,

$$\frac{d}{dx}[J_0(x)]=-J_1(x), \qquad \frac{d}{dx}[xJ_1(x)]=xJ_0(x), \tag{4.6}$$

as well as the boundedness of stresses and their derivatives at $r=0$. With (4.4) and (4.5), we may now transform the partial differential equation (3.12) into

$$\alpha_4 S_i''''-\rho_i^2(\alpha_1+\alpha_3)S_i''+\rho_i^4 S_i=f_i(z) \tag{4.7}$$

where $(\ )' \equiv d(\ )/dz$ and

(4.8) $$f_i(z) = r_0\rho_i J_1(\rho_i r_0)\{\rho_i^2 s(r_0,z) - (\alpha_1+\alpha_3)s''(r_0,z) - [\Delta_r s]_{r=r_0}\}.$$

We may also write the traction-free conditions (3.13) as

(4.9) $$S_i(\pm h) = 0, \qquad S_i'(\pm h) = 0.$$

The exact solution of (4.7) and (4.9) may be taken in the form of the Green's function representation

(4.10) $$\begin{aligned} S_i(z) &= \int_{-h}^{h} G_i(z,y)f_i(y)\,dy \\ &= \rho_i r_0 J_1(\rho_i r_0)\int_{-h}^{h}\{[\rho_i^2 G_i - (\alpha_1+\alpha_3)G_{i,yy}]s(r_0,y) \\ &\qquad - G_i\Delta_r s(r_0,y)\}\,dy. \end{aligned}$$

The explicit expression for the Green's function $G_i(z,y)$ is given in an appendix of this paper.

It is possible to express the edge values $s(r_0,z)$ and $\Delta_r s(r_0,z)$ in terms of the edge value of u_r, u_z and σ_{rr}, denoted by $u(z), w(z)$ and $\sigma(z)$, respectively. The two stress-strain relations (2.2b) and (2.2c) may be solved for the hoop and axial stress components, $\sigma_{\theta\theta}$ and s, to get

(4.11) $$\begin{aligned} s &= \frac{a_{11}}{\beta}u_{z,z} - \frac{a_{13}}{\beta}\frac{u_r}{r} - \alpha_1\sigma_{rr}, \\ \sigma_{\theta\theta} &= -\frac{a_{13}}{\beta}u_{z,z} + \frac{a_{33}}{\beta}\frac{u_r}{r} - \alpha_5\sigma_{rr} \end{aligned}$$

with $\beta\alpha_5 = a_{12}a_{33} - a_{13}^2$. The stress-stress function relations

(4.12) $$\begin{aligned} \alpha_3\Delta_r\phi_{,z} + \alpha_4\phi_{,zzz} &= s, \\ (\alpha_2+1)\Delta_r\phi_{,z} + 2\alpha_1\phi_{,zzz} &= -(\sigma_{rr}+\sigma_{\theta\theta}) \end{aligned}$$

may also be solved for $\Delta_r\phi_{,z}$ and $\phi_{,zzz}$ in terms of u_r, u_z and σ_{rr}:

(4.13) $$\begin{aligned} \Delta_r\phi_{,z} &= A_{11}u_{z,z} + A_{12}\frac{u_r}{r} + A_{13}\sigma_{rr}, \\ \phi_{,zzz} &= A_{21}u_{z,z} + A_{22}\frac{u_r}{r} + A_{23}\sigma_{rr} \end{aligned}$$

where $\beta_2 A_{11} = \alpha_4 a_{13} - 2\alpha_1 a_{11}$, $\beta_2 A_{12} = -\alpha_4 a_{33} + 2\alpha_1 a_{13}$, $A_{13} = (a_{11} - a_{12})A_{12}$, $\beta_2 A_{21} = -\alpha_3 a_{13} + (\alpha_2 + 1)a_{11}$, $\beta_2 A_{22} = \alpha_3 a_{33} - (\alpha_2 + 1)a_{13}$ and $A_{23} = (a_{11} - a_{12})A_{22}$ with $\beta_2 = (a_{11} - a_{12})^2(1 - \alpha_2)$. Note that the dimension of A_{13} and A_{23} are different from that of the other A_{ij}. By (4.13), the expression

$$\begin{aligned} \Delta_r s &= [\alpha_3 \Delta_r(\Delta_r \phi) + \alpha_4(\Delta_r \phi)_{,zz}]_{,z} \\ &= [(\alpha_4 - \alpha_1\alpha_3 - \alpha_3^2)\Delta_r \phi_{,z} - \alpha_3\alpha_4 \phi_{,zzz}]_{,zz} \end{aligned} \tag{4.14}$$

may also be expressed in terms of $u_{z,z}$, u_r/r and σ_{rr}. We may write this expression as

$$\Delta_r s = [\gamma_1 u_{z,z} + \gamma_2 \frac{u_r}{r} + \gamma_3 \sigma_{rr}]_{,zz} \tag{4.15}$$

with $\gamma_i = (\alpha_4 - \alpha_1\alpha_3 - \alpha_3^2)A_{1i} - \alpha_3\alpha_4 A_{2i}$, $i = 1, 2, 3$. In view of (4.11a) and (4.15), we now rewrite (4.10) in terms of u, w and σ:

$$\begin{aligned} S_i(z) = \rho_i r_0 J_1(\rho_i r_0) \int_{-h}^{h} \Big[& G_{iw}(z,y) w'(y) \\ & + G_{iu}(z,y) \frac{u(y)}{r_0} + G_{i\sigma}(z,y)\sigma(y) \Big] \, dy \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} G_{iw} &= -\gamma_1 G_{i,yy} + \frac{\alpha_{11}}{\beta}[\rho_i^2 G_i - (\alpha_1 + \alpha_3) G_{i,yy}] \\ G_{iu} &= -\gamma_2 G_{i,yy} - \frac{\alpha_{13}}{\beta}[\rho_i^2 G_i - (\alpha_1 + \alpha_3) G_{i,yy}] \\ G_{i\sigma} &= -\gamma_3 G_{i,yy} - \alpha_1[\rho_i^2 G_i - (\alpha_1 + \alpha_3) G_{i,yy}]. \end{aligned} \tag{4.17}$$

Upon substituting (4.16) into the inversion formula (4.3), we get, after some rearrangements,

$$\begin{aligned} s(r,z) = \int_{-h}^{h} \Big[& K_w(r;z,y) w'(y) + K_u(r;z,y) \frac{u(y)}{r_0} \\ & + K_\sigma(r;z,y)\sigma(y) \Big] \, dy \end{aligned} \tag{4.18}$$

with

$$K_\xi(r;z,y) = \frac{2}{r_0}\sum_{i=1}^{\infty} \frac{\rho_i J_0(\rho_i r)}{J_1(\rho_i r_0)} G_{i\xi}(z,y) \tag{4.19}$$

for $\xi = w$, u or σ.

5. Reduction to a scalar integral equation. Equation (4.18) is not yet the solution for s as the radial stress at the edge r_0, $\sigma_{rr}(r_0, z) \equiv \sigma(z)$ is unknown for the canonical displacement boundary value problems of interest here. Our ultimate goal is to deduce an equation for this unknown and thus solve the boundary value problem. We begin this last phase of our reduction by integrating (2.1b) to get

$$\begin{aligned} \sigma_{rz}(r,z) &= -\frac{1}{r}\int_0^r t[s(t,z)]_{,z}\,dt \\ &= \int_{-h}^{h}\left[\hat{K}_w w'(y) + \hat{K}_u \frac{u(y)}{r_0} + \hat{K}_\sigma \sigma(y)\right] dy \end{aligned} \tag{5.1}$$

where the expressions for $\hat{K}_\xi(r;z,y)$, with $\xi = w, u, \sigma$, are obtained with the help of the known identity [**1**]

$$\int_0^r tJ_0(\rho_i t)\,dt = \frac{r}{\rho_i} J_1(\rho_i r), \tag{5.2}$$

to be

$$\begin{aligned} \hat{K}_\xi(r;z,y) &= -\frac{1}{r}\int_0^r t[K_\xi(t;z,y)]_{,z}\,dt \\ &= -\frac{2}{r_0}\sum_{i=1}^{\infty} \frac{J_1(\rho_i r)}{J_1(\rho_i r_0)} G_{i\xi,z}(z,y). \end{aligned} \tag{5.3}$$

Next, the equilibrium equation (2.1a) gives $\sigma_{\theta\theta}$ in terms of σ_{rr} and σ_{rz}. The resulting expression may be used to eliminate $\sigma_{\theta\theta}$ from the following stress-displacement relations

$$\begin{aligned} u_{r,r} &= a_{11}\sigma_{rr} + a_{12}(r\sigma_{rr})_{,r} + a_{12} r\sigma_{rz,z} + a_{13}s = \varepsilon_{rr} \\ \frac{1}{r}u_r &= a_{12}\sigma_{rr} + a_{11}(r\sigma_{rr})_{,r} + a_{11} r\sigma_{rz,z} + a_{13}s = \varepsilon_{\theta\theta}. \end{aligned} \tag{5.4}$$

The combination $ru_{r,r} + u_r = (ru_r)_{,r}$ of the above two equations can be integrated from 0 to r to get

$$(5.5)\qquad \begin{aligned} ru_r = (a_{11}+a_{12})r^2\sigma_{rr} + (a_{11}+a_{12})\int_0^r t^2\sigma_{rz,z}(t,z)\,dt \\ + 2a_{13}\int_0^r s(t,z)t\,dt \end{aligned}$$

where the boundedness conditions on stresses and displacements at $r=0$ have been used. With (4.18) and (5.1), and with the known identities (5.2) and

$$(5.6)\qquad \int_0^r t^2 J_1(\rho_i t)\,dt = \frac{r^2}{\rho_i}J_2(\rho_i r)$$

(which can be found in [**1**]), we now rewrite (5.5) as

$$(5.7)\quad ru_r = (a_{11}+a_{12})r^2\sigma_{rr} + \int_{-h}^{h}\left[k_w w'(y) + k_u\frac{u(y)}{r_0} + k_\sigma\sigma(y)\right]dy$$

where

$$(5.8)\qquad \begin{aligned} k_\xi(r;z,y) = \sum_{i=1}^{\infty}\Bigg\{4a_{13}\frac{rJ_1(\rho_i r)}{r_0 J_1(\rho_i r_0)}G_{i\xi}(z,y) \\ - 2(a_{11}+a_{12})\frac{r^2 J_2(\rho_i r)}{\rho_i r_0 J_1(\rho_i r_0)}G_{i\xi,zz}(z,y)\Bigg\}. \end{aligned}$$

At the edge $r=r_0$, equation (5.7) becomes
(5.9)

$$\begin{aligned} r_0 u(z) = (a_{11}+a_{12})r_0^2\sigma(z) \\ + \int_{-h}^{h}\left[\bar{k}_w(z,y)w'(y) + \bar{k}_u(z,y)\frac{u(y)}{r_0} + \bar{k}_\sigma(z,y)\sigma(y)\right]dy \end{aligned}$$

with

$$(5.10)\qquad \begin{aligned} \bar{k}_\xi(z,y) = 4\sum_{i=1}^{\infty}\left[a_{13}G_{1\xi}(z,y) - \frac{1}{\rho_i^2}(a_{11}+a_{12})G_{i\xi,zz}(z,y)\right] \\ = k_\xi(r_0;z,y) \end{aligned}$$

where $\xi = w, u$ or σ and the identity $J_2(\rho_i r_0)/J_1(\rho_i r_0) = 2/(\rho_i r_0)$ has been used. Equation (5.9) is a Fredholm integral equation of the second kind for $\sigma(z) \equiv \sigma_{rr}(r_0, z)$ when displacements edge data (4.1) are prescribed (or for $u(z) \equiv u_r(r_0, z)$ when $\sigma_{rr}(r_0, z)$ and $u_z(r_0, z)$ are prescribed).

With the solution of (5.9), we will have all three edge quantities $w(z), u(z)$ and $\sigma(z)$ to be used in (4.18) and (5.1) for the solution by quadratures of the transverse normal and shear stresses $s(r, z)$ and $\sigma_{rz}(r, z)$ of the residual problem. Upon eliminating u_r from (5.4b) and (5.7), we obtain

(5.11)

$$a_{11}\sigma_{rr,r} = \frac{1}{r^3}\int_{-h}^{h}\left[k_w w'(y) + k_u\frac{u(y)}{r_0} + k_\sigma \sigma(y)\right]dy - a_{11}\sigma_{rz,z} - \frac{a_{13}}{r}s$$

so that σ_{rr} is also determined by quadratures. The equilibrium equation (2.1a) then gives $\sigma_{\theta\theta}$ as a linear combination of σ_{rr} and σ_{rz} and their partial derivatives. The radial displacement u_r is then given by (5.7). Finally, the transverse displacement u_z is obtained by integrating $a_{44}\sigma_{rz} = u_{r,z} + u_{z,r}$ with respect to r and using $u_z(r_0, z) = w(z)$. All field quantities of interest are therefore completely determined once we have the solution of (5.9).

6. Numerical solutions for canonical problems.

For the numerical solution of the canonical problems formulated in Section 3, symmetry and antisymmetry about the midplane $z = 0$ inherent in the stress and displacement fields allow us to work more economically with one integral equation on the half interval $[0, h]$ instead. From the expression for $u(z)$ (the radial displacement at $r = r_0$) given by (5.9), we get

(6.1)

$$r_0 u(z) = (a_{11} + a_{12})r_0^2\sigma(z) + \int_0^h \left[\tilde{k}_w(z, y)w'(y) + \tilde{k}_u(z, y)\frac{u(y)}{r_0} + \tilde{k}_\sigma(z, y)\sigma(y)\right]dy$$

with

$$\tilde{k}_\xi(z, y) = \bar{k}_\xi(z, y) \pm \bar{k}_\xi(z, -y), \qquad \xi = w, u, \sigma \tag{6.2}$$

where "+" and "−" correspond to the symmetric and antisymmetric case. For displacement boundary value problems, the integral equation

(6.1) is solved for $\sigma(z)$ numerically by the method of orthogonal collocation.

With the singularity exponent β determined in [**18**], the unknown $\sigma(z)$ can be written as $(h-z)^{-\beta}\bar{\sigma}(z)$, where $\bar{\sigma}(z)$ is a bounded function and represented by a linear combination of Jacobi polynomials $J_n^{(0,-\beta)}(z)$. The solution is then evaluated at the zeros of the Jacobi polynomials. The number of terms in Fourier-Bessel series for the kernels $\tilde{k}_w, \tilde{k}_u$ and $\tilde{k}_\sigma$ to be retained for convergence is no more than 80 for all cases computed. Various integrals are evaluated numerically by Gaussian quadratures. By taking advantage of the symmetry property of the solution, eight collocation points suffice for an accurate solution of the integral equation for all problems investigated. The details for the numerical solution are similar to those for the corresponding plane strain problem [**19, 20**] and will not be repeated here.

The first 20 zeros of the Bessel function $J_0(x)$, $x_i \equiv r_0\rho_i$, $i = 1, 2, 3, \ldots$, are taken from the table in [**1**], and the others are calculated from the asymptotic expansion

$$r_0\rho_i \sim T_i + \frac{1}{8T_i} - \frac{31}{384T_i^3} + \frac{3779}{15360T_i^5} \tag{6.3}$$

with $T_i = (4i-1)/4\pi$.

The computer code developed to implement the numerical scheme has been used to solve the three residual problems of Section 3 for the six plate materials given in Table 1. The inplane shear modulus in all cases is calculated from $G = E/2(1+\nu)$. The data for ortho 1 are those of Douglas fir with an in-plane isotropy approximation. The data of ortho 2 are those of ortho 1 with transverse and in-plane Young's modulus interchanged. Ortho 3 and 4 correspond to ortho 1 and 2 with the smaller Young's modulus adjusted so that E and E_3 are of the same order of magnitude. The two isotropic materials correspond to ortho 1 and ortho 2 with $E_3 \to E$, $G_3 \to G$ and $\nu_3 \to \nu$. (In actual computation, we set $G_3 = G$, $\nu_3 = \nu$ and then let $E_3 \to E$ to avoid any complication associated with the two double roots in (2.10).) Rapid and stable convergence was observed for all computed solutions as we increase the number of collocation points. The results of these computations and their application in plate theories will be discussed in the next section.

The complete solution of a canonical problem is obtained by the superposition of the solution of the residual problem and the corresponding singular interior solution. For graphical presentation, we denote by $\sigma_{ij}^{(\xi)}(r,z)$, $\xi = a, b, c$, the stress component σ_{ij} of the canonical problem (ξ) defined in Section 3 with the multiplicative factor ξ restored. This solution generally depends on geometrical and material parameters h, r_0 and a_{ij}. It is not difficult to see however that, with $R = r/r_0$ and $Z = z/h$, the dimensionless stress components $\tilde{\sigma}_{ij}^{(\xi)}(R,Z)$ defined by

$$\sigma_{ij}^{(a)}(r,z) = \frac{a}{h}\tilde{\sigma}_{ij}^{(a)}(R,Z), \qquad \sigma_{ij}^{(b)}(r,z) = \frac{b}{h^2}\tilde{\sigma}_{ij}^{(b)}(R,Z) \tag{6.4}$$
$$\sigma_{ij}^{(c)}(r,z) = ch\tilde{c}_{ij}^{(c)}(R,Z)$$

depend only on the dimensionless parameter $\varepsilon = h/r_0$ and the four dimensionless material parameters α_i, $i = 1,2,3,4$, introduced in (2.8).

TABLE 1. Elastic moduli for six plate materials.

Material	E lb/in^2	E_3 lb/in^2	G_3 lb/in^2	ν	ν_3
(1) Ortho 1	0.05 E_3	1.56×10^6	0.078 E_3	0.287	0.449
(2) Ortho 2	1.56×10^6	0.05 E	0.078 E	0.287	0.02245
(3) Ortho 3	0.5 E_3	1.56×10^6	0.078 E_3	0.287	0.449
(4) Ortho 4	1.56×10^6	0.5 E	0.078 E	0.287	0.2245
(5) Iso 1	7.8×10^4	7.8×10^4	3.03×10^4	0.287	0.287
(6) Iso 2	1.56×10^6	1.56×10^6	6.06×10^5	0.287	0.287

We show in Figures 1–3 the distribution of the dimensionless radial stress components at the edge $r = r_0$ for each of the three canonical problems and the six materials of Table 1 and $h/r_0 = \varepsilon = 0.1$. Except for Iso 2, the solutions for all three problems seem insensitive to the material properties. In all cases, the variation of $\tilde{\sigma}^{(\xi)}(Z) \equiv \tilde{\sigma}_{rr}^{(\xi)}(1,Z)$, $\xi = a, b, c$, with $Z = z/h$, is nearly straight, except at the corner $Z = 1$ where there is a stress singularity. (The nature of this singularity has been analyzed in [**18**] and incorporated in the solution process similar to the procedure used in [**19, 20**]. The results suggest that the interior solution dominates except near the corners. In the case of problem

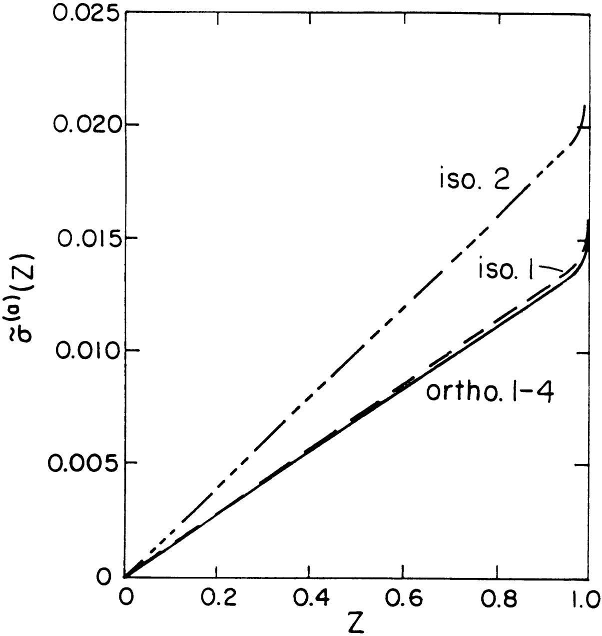

FIGURE 1. Dimensionless edge radial stress distributions of six different circular plates for canonical problem (a).

(c) where the exact interior solution is cubic in z/h, the linear term appears to dominate the bending action of the plate. The distribution of the dimensionless transverse shear stress $\tilde{\tau}^{(\xi)}(Z) \equiv \tilde{\sigma}_{rz}^{(\xi)}(1, Z)$ is more sensitive to material properties as shown in Figures 4–6. As we might expect, the isotropic cases exhibit larger transverse shear for all three problems because the material is more stiff in the transverse direction. Other stress and displacement components have also been computed.

7. Displacement boundary conditions for plate theories. We indicated in previous sections that the principal motivation for our research is to obtain the solution of relevant canonical problems for the formulation of the correct boundary conditions for thin and thick plate theories. For boundary conditions given in terms of displacements,

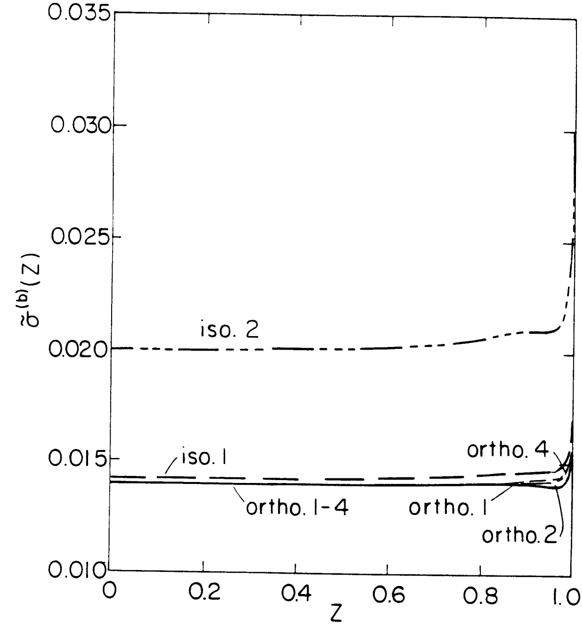

FIGURE 2. Dimensionless edge radial stress distributions of six different circular plates for canonical problem (b).

$u_r(r_0, z) = \bar{u}_r(z)$, $u_z(r_0, z) = \bar{u}_z(z)$, we have from the condition (3.6)

$$\text{(7.1)}\quad \int_{-h}^{h} \{\sigma_{rr}^{(2)}(r_0, z) u_r^I(r_0, z) + \sigma_{rz}^{(2)}(r_0, z) u_z^I(r_0, z)\}\, dz = \int_{-h}^{h} \{\sigma_{rr}^{(2)}(r_0, z)\bar{u}_r(z) + \sigma_{rz}^{(2)}(r_0, z)\bar{u}_z(z)\}\, dz$$

for any suitable state (2) as described in Section 3. The canonical problems (a), (b) and (c) are three such (2) states. For these (2) states, and with the expressions (2.12) for u_r^I and u_z^I, we may write (7.1) as

$$\text{(7.2)}\quad b\{(a_{12} - a_{11}) r_0^{-1} N_{b0}\} + e\{r_0(a_{12} + a_{11}) N_{b0} + 2a_{13} T_{b1}\} = \mu_b$$

$$\text{(7.3)}\quad a\{(a_{12} - a_{11}) r_0^{-1} N_{a1}\} + c\{c_{N1} N_{a1} + c_{N3} N_{a3} + c_{T2} T_{a2}\} + d\{r_0(a_{12} + a_{11}) N_{a1} + a_{13} T_{a2}\} = \mu_a$$

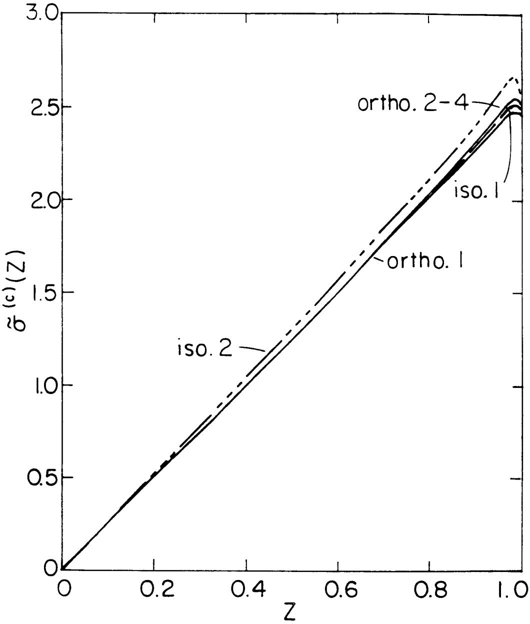

FIGURE 3. Dimensionless edge radial stress distributions of six different circular plates for canonical problem (c).

$$a\{(a_{12}-a_{11})r_0^{-1}N_{c1}\}+c\{c_{N1}N_{c1}+c_{N3}N_{c3}+c_{T2}T_{c2}\} + d\{r_0(a_{12}+a_{11})N_{c1}+a_{13}T_{c2}\}+w_0\left\{\frac{4h^3(\alpha_4-\alpha_1\alpha_3)}{3r_0}\right\}=\mu_c \tag{7.4}$$

where

$$c_{N1}=(a_{12}+a_{11})[(\alpha_2+1)\alpha_4-2\alpha_1\alpha_3]r_0\ln(r_0) + a_{11}(\alpha_2-1)\alpha_4 r_0 \tag{7.5}$$

$$c_{T2}=a_{13}\left\{[(\alpha_2+1)\alpha_4-2\alpha_1\alpha_3]\ln(r_0)+\frac{\alpha_4}{2}(\alpha_2-1)\right\} \tag{7.6}$$

$$c_{N3}=\frac{\alpha_3}{3r_0}(a_{12}-a_{11})(1-\alpha_2) \tag{7.7}$$

$$N_{\xi k}=\int_{-h}^{h}\sigma_{rr}^{(\xi)}(r_0,z)z^k\,dz\equiv\zeta_{\xi k}\tilde{N}_{\xi k}, \tag{7.8}$$

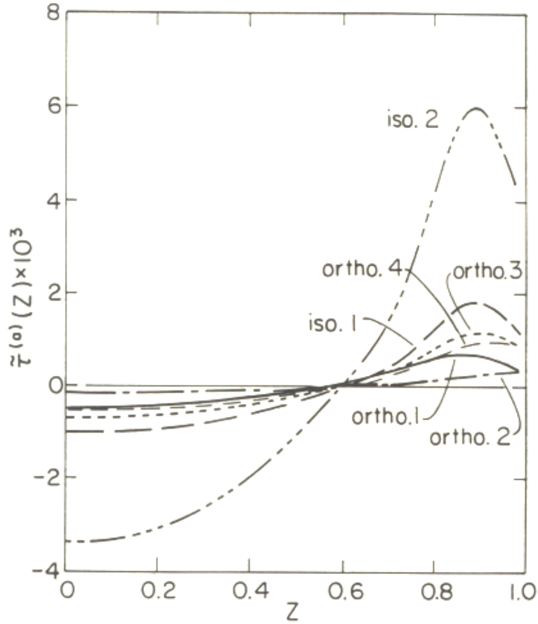

FIGURE 4. Dimensionless edge transverse shear stress distributions of six different circular plates for canonical problem (a).

$$T_{\xi k} = \int_{-h}^{h} \sigma_{rz}^{(\xi)}(r_0, z) z^k \, dz \equiv \zeta_{\xi k} \tilde{T}_{\xi k}, \tag{7.9}$$

$$\mu_\xi = \int_{-h}^{h} \{\sigma_{rr}^{(\xi)}(r_0, z)\bar{u}_r(z) + \sigma_{rz}^{(\xi)}(r_0, z)\bar{u}_z(z)\} \, dz \tag{7.10}$$

with $\xi = a, b$ or c, $k = 0, 1, 2$ or 3, and, by (6.4), $\zeta_{ak} = ah^k$, $\zeta_{bk} = bh^{-1+k}$, $\zeta_{ck} = ch^{2+k}$. The moments of stresses $\tilde{N}_{\xi k}$ and $\tilde{T}_{\xi k}$ defined in (7.8) and (7.9) are given in Table 2 for the six materials in Table 1 for $\varepsilon = h/r_0 = 0.1$ (with $\tilde{N}_{ck}^* = \tilde{N}_{ck}/(\alpha_4 - \alpha_1\alpha_3)$ and $\tilde{T}_{c2}^* = \tilde{T}_{c2}/(\alpha_4 - \alpha_1\alpha_3)$. Note that $N_{\xi 0}$ and $N_{\xi 1}$ are the values of the conventional N_{rr} and M_{rr} at the edge r_0 for the ξ case while $T_{\xi 0}$ is $Q_r(r_0)$.

Given the prescribed edge displacements $\bar{u}_r(z)$ and $\bar{u}_z(z)$ (and the edge stresses of the canonical problems $\tilde{\sigma}_{rr}^{(\xi)}(1, Z)$ and $\tilde{\sigma}_{rz}^{(\xi)}(1, Z)$ for $\xi = a, b$ and c), the right-hand members of (7.2)–(7.4) are known

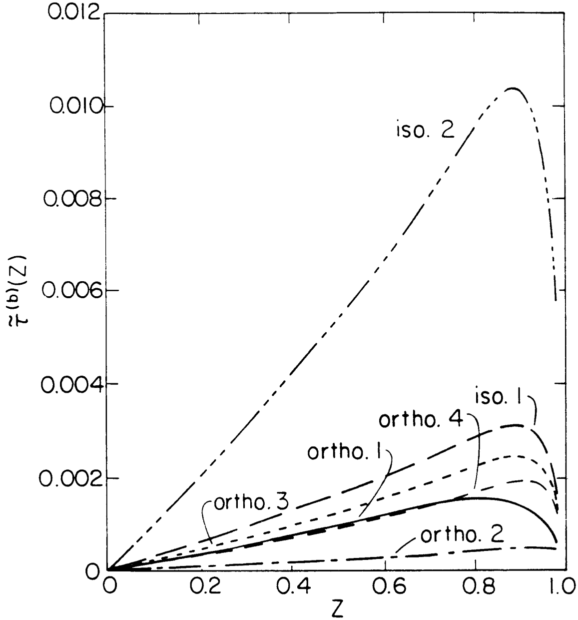

FIGURE 5. Dimensionless edge transverse shear stress distributions of six different circular plates for canonical problem (b).

constants; so are the expressions in braces on the left. Hence, we have effectively *three* linear equations for the six unknown constants a, b, c, d, e and w_0. For the special case where $\bar{u}_r(z) = U_0 + z\Phi_0$ and $\bar{u}_z(z) = W_0$, the expression (7.10) becomes

$$\mu_{\mathrm{a}} = \Phi_0 M_{rr}^{(a)}(r_0) + W_0 Q_r^{(\mathrm{a})}(r_0), \qquad \mu_{\mathrm{b}} = U_0 N_{rr}^{(\mathrm{b})}(r_0),$$

(7.11)

$$\mu_{\mathrm{c}} = \Phi_0 M_{rr}^{(\mathrm{c})}(r_0) + W_0 Q_r^{(\mathrm{c})}(r_0).$$

Since $\sigma_{rr}^{(\mathrm{b})}, \sigma_{rz}^{(\mathrm{a})}, \sigma_{rz}^{(\mathrm{c})}$ are even in z and $\sigma_{rz}^{(\mathrm{b})}, \sigma_{rr}^{(\mathrm{a})}, \sigma_{rr}^{(\mathrm{c})}$ are odd in z, we have $Q_r^{(\mathrm{b})}(r_0) = M_{rr}^{(\mathrm{b})}(r_0) = N_{rr}^{(\mathrm{a})}(r_0) = N_{rr}^{(\mathrm{c})}(r_0) = 0$. It follows that W_0 and Φ_0 induce only plate bending (as we would expect).

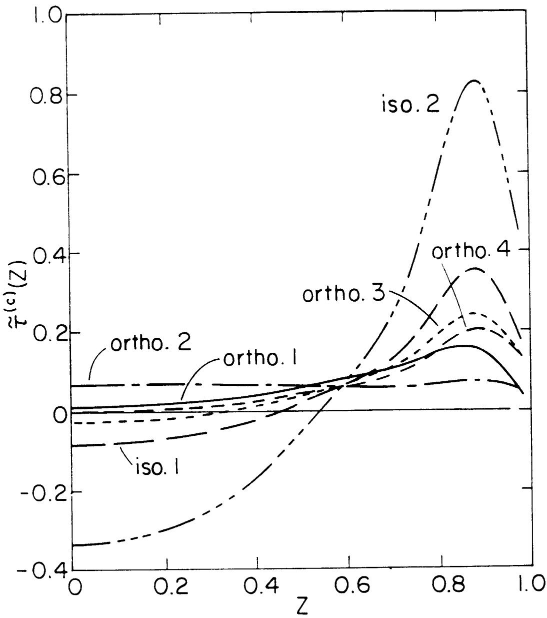

FIGURE 6. Dimensionless edge transverse shear stress distributions of six different circular plates for canonical problem (c).

TABLE 2. The moments of stresses as weight coefficients.

Plate Material	Ortho 1	Iso 1	Iso 2	Ortho 2	Ortho 3	Ortho 4
$\tilde{N}_{a1}$	.9368-2	.9544-2	.1340-1	.9370-2	.9365-2	.9369-2
$\tilde{N}_{a3}$	.5623-2	.5741-2	.8069-2	.5631-2	.5627-2	.5632-2
$\tilde{N}_{b0}$	.2812-1	.2872-1	.4074-1	.2814-1	.2816-1	.2819-1
$\tilde{N}^*_{c1}$	.1683+1	.1693+1	.1774+1	.1700+1	.1692+1	.1702+1
$\tilde{N}^*_{c3}$	.1010+1	.1019+1	.1070+1	.1025+1	.1019+1	.1027+1
$\tilde{T}_{a2}$	.2452-3	.5680-3	.1850-2	.9613-4	.3876-3	.3206-3
$\tilde{T}_{b1}$	.1105-2	.2151-2	.7037-2	.3291-3	.1677-2	.1280-2
$\tilde{T}^*_{c2}$	.6750-1	.1234	.2601	.4338-1	.9357-1	.8250-1

For annular plates, three more equations for the six constants can be obtained from a condition similar to (7.1) (or a corresponding condition for other types of edge data already discussed in [**10**, **17**]) for the other edge of the plate. (Thus, we need another set of weight factors corresponding to $\tilde{N}_{\xi k}$ and $\tilde{T}_{\xi k}$ as well as μ_ξ for whatever prescribed edge conditions at the other edge $r = r_i$.) Together, the six equations determine the six unknown constants and hence the interior solution (2.11)–(2.12). In the next section we obtain the interior solution of a problem for which this second set of three equations is relatively simple.

8. A clamped circular plate under a point load. Consider a circular plate of radius r_0 and thickness $2h$. The plate is constrained from any displacement along its only edge $r = r_0$ (generally called a clamped edge) and is subject to a vertical point force of magnitude P downward at the center of the upper surface of the plate. We are interested in the correct interior solution for this problem and therefore in the determination of the six unknown constants in (2.11)–(2.12) by the assigned edge data $\bar{u}_r(z) \equiv 0$, $\bar{u}_z(z) \equiv 0$. The solution for c is an immediate consequence of the overall equilibrium requirement. For any r_i in the interval $(0, r_0)$, with $h \ll r_i < r_0$, we must have

$$\text{(8.1)} \qquad \int_{-h}^{h} \sigma_{rz}^{I}(r_i, z)\, dz \cong \int_{-h}^{h} \sigma_{rz}(r_i, z)\, dz = \frac{P}{2\pi r_i}.$$

With σ_{rz}^{I} given by (2.11a), we have, except for exponentially small terms,

$$\text{(8.2)} \qquad c = \frac{3P}{8\pi h^3(\alpha_4 - \alpha_1\alpha_3)}.$$

We still need two more conditions to supplement (7.2)–(7.4). We get them by the technique used in [**11**] for the corresponding isotropic problem. By linearity, the present problem may also be decomposed into two uncoupled problems for this purpose: one with an inward point load of magnitude $P/2$ at both $\{r = 0, z = h\}$ and $\{r = 0, z = -h\}$, and the other with a point load of magnitude $P/2$ inward at $\{r = 0, z = h\}$ and outward at $\{r = 0, z = -h\}$. We apply the reciprocal theorem to a circular portion of the plate $0 \le r \le r_i (< r_0)$ taking state (1) to be the solution of the first problem (with symmetric point loads). For this case, we take state (2) to be the interior solution (2.11)–(2.12) with $e = 1$ and $a = b = c = d = w_0 = 0$. For r_i not near the edge of r_0, we have

$$(8.3)\quad 2\pi r_i \int_{-h}^{h} [\sigma_{rz}^I u_z^{(2)} + \sigma_{rr}^I u_r^{(2)} - \sigma_{rz}^{(2)} u_z^I - \sigma_{rr}^{(2)} u_r^I]_{r=r_i}\, dz \\ + 2\pi \int_0^{r_i} [\bar{\sigma}_{zz} u_z^{(2)} + \bar{\sigma}_{zr} u_r^{(2)} - \sigma_{zz}^{(2)} u_z^{(1)} - \sigma_{zr}^{(2)} u_r^{(1)}]_{-h}^{h} r\, dr = 0$$

except for exponentially small terms introduced when the actual solution of the problem (state (1)) is replaced by the corresponding interior solution at $r = r_i$. On the two faces $z = \pm h$, all the stresses involved in the second integral except $\bar{\sigma}_{zz}(r, \pm h)$ vanish. The prescribed $\bar{\sigma}_{zz}$ on the two faces are the two point loads at the center of the faces and, for the purpose of our analysis, may be taken in the form of $(\pm P/2)/\pi\delta^2$ where δ is a small radius which will be shrunk to zero for the final results. Upon substituting into (8.3) the expressions from (2.11)–(2.12) for the nonvanishing quantities involved, we get

$$8\pi h a_{11} b - 2P a_{13} h = 0,$$

(each term corresponding to the value of one of the integrals) and therewith

$$(8.4)\qquad b = \frac{P a_{13}}{4\pi a_{11}}.$$

Note that the expression for b is independent of r_i and reduces correctly to the result for an isotropic plate (see equation (6.3) in [**11**]).

Similarly, we get an expression for the constant a by applying the reciprocal theorem to the second problem with antisymmetric point loads over a circular disc portion of the plate of radius r_i. We take as state (1) the solution for the given point load problem. At $r = r_i$, it may be approximated by the interior solution up to exponentially small terms for r_i away from the actual plate edge. As state (2), we take the interior solution (2.11)–(2.12) with $d = -1$, $w_0 = a_{13}h^2 + r_0^2(a_{11} + a_{12})/2$ and $a = b = c = e = 0$. For this particular choice of state (2), the limiting value of the second integral in (8.3) as $\delta \to 0$ is zero. It is a straightforward calculation to get from the first integral an expression for a in terms of c independent of r_i:

$$a = -\frac{h^2 a_{13}}{5 a_{11}} \left(4 - \frac{\alpha_3}{\alpha_1}\right)(\alpha_1 \alpha_3 - \alpha_4) c.$$

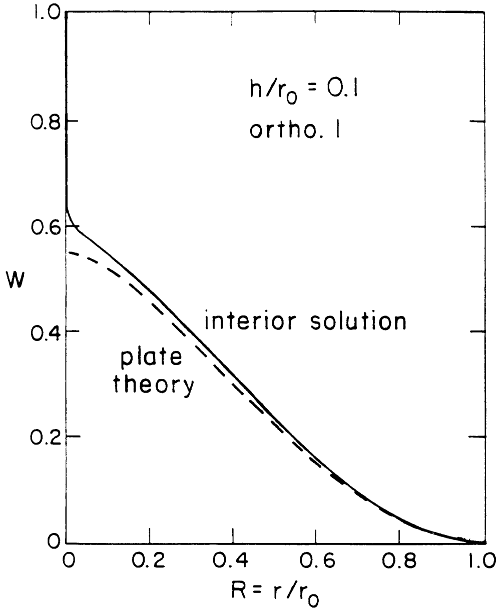

FIGURE 7. Normalized interior midplane displacement distribution of a clamped plate subject to a concentrated force at the center.

Upon substituting the expression for c from (8.2), we get

$$a = \frac{3Pa_{13}}{40\pi h a_{11}}\left(4 - \frac{\alpha_3}{\alpha_1}\right). \tag{8.5}$$

The expression (8.5) reduces to

$$a = -\frac{3P(2+3\nu)}{40\pi h} \tag{8.6}$$

for an isotropic plate. This result and the expressions for b and c for an isotropic material,

$$c = \frac{3P(1-\nu)^2}{8\pi h^3}, \qquad b = -\frac{\nu P}{4\pi}, \tag{8.7}$$

obtained from (8.2) and (8.4), are consistent with the expressions in (5.2,3) of [**11**] for A and B and the expression (6.3) for B', keeping in mind that the bending stiffness factor D is $2Eh^3/3(1-\nu^2)$ for a plate of thickness $2h$ as in our case. With (8.2), (8.4) and (8.7) giving a, b and c in terms of P and the plate parameters, the three equations (7.2)–(7.4) now uniquely determine d, e, and w_0.

A graph of the midplane transverse displacement distribution

$$W(R) \equiv \frac{E_3 h^3}{P r_0^2} u_z(R, 0) \tag{8.8}$$

is given in Figure 7 for the material ortho 1 listed in Table 1. The corresponding graph by a thin orthotropic plate theory similar to the classical Kirchhoff theory for isotropic plates [**4, 13**] is also shown there for comparison. The two results are in reasonable agreement except near the origin where the interior solution has a singularity there.

Appendix

The Green's Functions $G_i(z, y)$**.** For a fixed ρ_i, we take the Green's function for the boundary value problem (4.7) and (4.9) to be the function $G_i(z, y)$ which satisfies (4.7) with $f_i(z)$ being the Dirac delta function $\delta(z - y)$ and the homogeneous boundary conditions

$$G_i(\pm h, y) = 0, \qquad G_{i,z}(\pm h, y) = 0. \tag{A1}$$

The solution of this new BVP is

$$G_i(z, y) = \begin{cases} G_n(z, y; \rho_i), & -h \leq z \leq y \\ G_p(z, y; \rho_i), & y \leq z \leq h \end{cases} \tag{A2}$$

where

$$\begin{aligned} G_t(z, y; \rho_i) &= A_t \sinh(s_1 \rho_i z) + B_t \sinh(s_2 \rho_i z) \\ &\quad + C_t \cosh(s_1 \rho_i z) + D_t \cosh(s_2 \rho_i z), \qquad t = n, p. \end{aligned} \tag{A3}$$

The two boundary conditions on G_i at $z = -h$ determine A_n and B_n in terms of C_n and D_n. Similarly, the two boundary conditions at $z = h$ determine A_p and B_p in terms of C_p and D_p. The remaining four unknown constants are to be chosen to ensure that the continuity of G_i, G_i' and G_i'' (with $(\ )' \equiv (\ )_{,z}$) across the junction $z = y$ and the jump condition

$$[G_i''']_{z=y-}^{y+} \equiv [G_i'''(y+, y) - G_i'''(y-, y)] = \frac{1}{\alpha_4} \tag{A4}$$

are met. Evidently, all the constants A_t, B_t, C_t and D_t, $t = n, p$, depend on the location of the point source y so that we may write the final expression for $G_i(z, y)$ in the form (A.2) with

$$G_p(z, y; \rho_i) = \mu_G\{s_2 f_1(-z; \rho_i; s_1, s_2) f_2(y; \rho_i; s_1, s_2) + s_1 f_1(-z; \rho_i; s_2, s_1) f_2(y; \rho_i; s_2, s_1)\} \tag{A5}$$

$$G_n(z, y; \rho_i) = G_p(-z, -y; \rho_i) \tag{A6}$$

with

$$\mu_G = [2\alpha_4 s_1 s_2 \rho_i^3 (s_2^2 - s_1^2) f_3 f_4]^{-1} \tag{A7}$$

$$f_1 = f_3 \cosh(s_1 \rho_i z) + f_6 \sinh(s_1 \rho_i z) - s_1 \sinh(s_2 \rho_i z) \tag{A8}$$

$$f_2 = s_1 \cosh(s_2 \rho_i y) + f_5 \cosh(s_1 \rho_i y) + f_4 \sinh(s_1 \rho_i y) \tag{A9}$$

$$f_3 = s_2 \sinh(s_1 \rho_i h) \cosh(s_2 \rho_i h) - s_1 \sinh(s_2 \rho_i h) \cosh(s_1 \rho_i h) \tag{A10}$$

$$f_4 = s_2 \sinh(s_2 \rho_i h) \cosh(s_1 \rho_i h) - s_1 \sinh(s_1 \rho_i h) \cosh(s_2 \rho_i h) \tag{A11}$$

$$f_5 = s_2 \sinh(s_2 \rho_i h) \sinh(s_1 \rho_i h) - s_1 \cosh(s_2 \rho_i h) \cosh(s_1 \rho_i h) \tag{A12}$$

$$f_6 = s_2 \cosh(s_2 \rho_i h) \cosh(s_1 \rho_i h) - s_1 \sinh(s_2 \rho_i h) \sinh(s_1 \rho_i h). \tag{A13}$$

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