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CORRECT ASYMPTOTIC THEORIES FOR THE AXISYMMETRIC DEFORMATION OF THIN AND MODERATELY THICK CYLINDRICAL SHELLS

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Abstract—A refined shell theory is formulated for the elastostatics of long, moderately thick cylindrical shells in axisymmetric deformation. This theory corresponds to a two-term outer asymptotic expansion of the exact solution for small values of the dimensionless shell thickness parameter. The complexity of the known exact solution for the three-dimensional elasticity problem has stimulated an interest in thin and thick shell theories to provide accurate approximate solutions in the shell interior without any reference (or matching) to the inner asymptotic solution. The principal difficulty in developing a shell theory lies in the determination of an appropriate set of two-dimensional boundary conditions for the shell solution from the prescribed edge data of the three-dimensional theory. The derivations of boundary conditions for the thin shell theory and the refined shell theory constitute a main contribution of this paper. Correct boundary conditions obtained for the first time include (i) displacement edge conditions for thin shell theory, and (ii) stress and two types of mixed edge conditions for the refined shell theory. Several applications of the refined theory are given to show that corrections to the thin shell solution can be important.

1. INTRODUCTION

The elastostatic theory of thin shells is a two-dimensional system of differential equations and boundary conditions which determines a first approximation for the three-dimensional elastostatic behavior of thin shell structures. Thin shell theory was originally derived from elasticity theory with the help of a set of (self-contradictory) assumptions known as the *Kirchhoff–Love hypotheses* [see Love (1944) and Wan and Weinitschke (1988) for examples]. Saint-Venant's principle is often invoked to justify the adequacy of the thin shell approximate solution away from the edges of the shell (Love, 1944; Timoshenko and Goodier, 1969). It is therefore of fundamental interest in theoretical and applied mechanics to deduce thin shell theory as a logical first approximation of elasticity theory for the shell interior, free of *ad hoc* assumptions and inconsistencies and with the possibility of establishing shell theories of higher order accuracy. A method for accomplishing this important goal was developed for the first time by Johnson and Reissner (1958). This method has since become a standard approach to an asymptotic solution of the three-dimensional elasticity problem for the *interior* of the shell away from its edges [see Gol'denveizer (1968), Green (1962a, b), Reiss (1960, 1962), and Johnson (1963)].

In their pioneering work, Johnson and Reissner describe their parametric series expansion method of solution for the three-dimensional boundary value problem by way of the axisymmetric deformation problem for circular cylindrical shells. The method introduces suitable scale factors to nondimensionalize stress and displacement variables and thereby transforms the governing equations of elasticity theory into a dimensionless form which contains a dimensionless parameter $\varepsilon = h/2R$, the ratio of the half-shell-thickness to the cylindrical midsurface radius. For thin cylindrical shells, ε is a small parameter and parametric expansions of the dimensionless stress and displacement variables in powers of ε are appropriate. The relevant differential equations for the leading terms of these expansions

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are identical to the thin shell equations for the problem, now obtained without the Kirchhoff–Love hypotheses. For this reason, the conventional thin shell theory is also known as the *first approximation shell theory*.

Higher order correction terms in the Johnson–Reissner parametric series solution are found to be determined by the same system of lower dimensional (shell) equations but now with the previously determined lower order solutions as forcing terms. These higher order terms provide the thick shell corrections to the thin shell solution. When combined with the thin shell solution, we have then a more accurate approximate solution in the shell interior for the three-dimensional elasticity problem needed for thicker shells.

The thin shell solution or the higher order parametric series solution of Johnson and Reissner generally cannot be made to fit the edge conditions of the three-dimensional elasticity problem, prescribed at an end section of a cylindrical shell for example. Johnson and Reissner acknowledged in the introduction of their paper the need for a supplementary solution to complete the actual solution of the elasticity problem. However, they chose not to treat this problem but limit themselves to the restricted class of stress boundary value problems for which Saint-Venant's principle applies to individual terms in the parametric series solution. For these stress boundary value problems, the thin shell solution and its higher order corrections are completely determined by the stress resultants and couples of the prescribed edge stresses. For small values of ε, the corresponding supplementary solution is expected to be confined to a narrow layer of the order of the shell thickness adjacent to the edges of the shell. In that case, the parametric series solution would provide an accurate approximate solution for the elasticity problem in the interior of the shell; hence, it is also called the *interior solution* of the elasticity problem. The same approach was also adopted by Johnson (1963) in a later paper on unsymmetric deformation of cylindrical shells. [Note that the interior solution corresponds to the *outer* (asymptotic expansion of the exact) *solution* in the method of matched asymptotic expansions.]

A method for obtaining the supplementary (Saint-Venant boundary layer) solution for the stress boundary value problem was developed by Reiss (1960) also by way of axisymmetric deformations of circular cylindrical shells. In general, Reiss' results show that the boundary conditions for the nth order correction term in the Johnson–Reissner type parametric series representation for the interior solution require the supplementary boundary layer solution of the $(n-1)$th order. The situation is much more complex for problems with prescribed edge displacements (or mixed stress and displacement edge data). Saint-Venant's principle, even if it should apply, is not useful as we do not know the stress resultants and couples at the edge. Moreover, the Reiss approach requires the *simultaneous* determination of terms of the same order in both the interior expansions and boundary layer expansions for problems with prescribed edge displacements. For such problems, even the thin shell solution, i.e. the leading term interior solution, cannot be obtained by Reiss' method without knowing the supplementary boundary layer solution for the same problem.

For many problems in applications, we do not need to know the boundary layer solution explicitly. For these problems, it is important to have a method for obtaining the interior component of the exact solution, or its various truncated parametric series approximations, without an explicit determination of any part of the supplementary solution component. The principal aim of the present paper is to develop such a method by applying the reciprocal theorem of elasticity in a way similar to our recent approach for the corresponding problem for flat plates (Gregory and Wan, 1984, 1985a, b, 1988; Lin and Wan, 1988). The extension of the *method of decaying residual solution* for flat plates to shell problems requires some finesse since the interior solution itself contains decaying components which would be lost by a straightforward application of the technique used in the flat plate case.

To bring out the essential features of the method of decaying residual solution for shells and to obtain explicit results for applications, we also consider in this paper axisymmetric deformation problems of long (and circumferentially complete) circular cylindrical shells. Our final result will be a two-term outer asymptotic solution which should be useful and adequate for moderately thick shells. This two-term asymptotic theory in ε will be called the *refined shell theory* henceforth. To establish this refined shell theory, we will take advantage of the availability of explicit exact (eigen-) solutions of the governing equations of three-dimensional elasticity theory which satisfy the stress free conditions on the cylindrical surfaces of the shell. Two-term asymptotic approximations accurate to order ε will be obtained from the asymptotic expansions of these exact solutions [and not from the differential equations of elasticity as in Johnson and Reissner (1958) or Reiss (1960)]. This will be done for both the interior components (the shell-eigenfunctions) and the boundary layer components. The latter are the analogue of the Papkovich–Fadle eigenfunctions for a semi-infinite strip in plane strain elasticity theory and will be called the PF-eigenfunctions.

The two-term asymptotic expressions for the eigenfunctions have a vastly simpler form than their exact counterparts and can be used to construct an approximate solution for a prescribed set of end conditions correct to relative order $O(\varepsilon)$, even in the Saint-Venant boundary layer. However, unless the $O(h)$ layer part of the solution is of special significance, the more interesting and important problem is to determine the refined shell theory solution directly from the prescribed end data without any consideration of the supplementary boundary layer solution. Needed for this purpose are the appropriate two-dimensional boundary conditions for the refined shell theory along an edge of the shell for given stress, displacement or mixed edge data. The derivation of these boundary conditions constitutes the main contribution of this paper.

Explicit boundary conditions for the refined shell theory are obtained by the method of decaying residual solution for prescribed stress and mixed conditions at an end of the circular cylindrical shell. These conditions take a particularly simple and elegant form when pure stress data are prescribed at an end section of the shell. Johnson and Reissner limited themselves to stress boundary value problems for which the stress boundary conditions effectively require the interior solution to have the same stress resultants and couples as those of the data. Our results justify the appropriateness of these conditions for thin shell theory. For the refined shell theory, it is appropriate to equate the stress resultants and couples only when Poisson's ratio is zero. Thus, Saint-Venant's principle is of limited applicability for stress boundary value problems beyond thin shell theory.

For pure displacement end data, we obtain the appropriate boundary conditions explicitly only for thin shell theory; these conditions are new to the literature [see Nair and Reissner (1977, 1978) for related results for the limiting case of transversely inextensible circular cylindrical shells]. For the refined (or any other thick) shell theory, the appropriate displacement boundary conditions are determined in terms of the solution of a few canonical boundary value problems for the same shell. The two-term asymptotic expressions for the PF-eigenfunctions are needed for the solution of these canonical problems.

Applications of the preceding results to four examples are given in Section 5. They show that there can be a substantial difference between the thin shell solution and the refined shell theory solution for a given problem. For displacement end data, the conventional displacement boundary conditions for thin shell theory are generally inadequate except for data which vary linearly across the thickness.

In Section 6, we estimate the maximum influence of the refined stress boundary conditions when only the stress resultants and couples are known at an end section. Strictly speaking, knowledge of these resultants and couples alone is insufficient and the $O(\varepsilon)$ correction terms could have arbitrarily large coefficients. However, by stipulating physically reasonable restrictions on the pointwise distribution of the stress data, we are able to obtain simple and useful bounds on the influence of the refined boundary conditions.

2. THE EXACT THREE-DIMENSIONAL THEORY

2.1. *Statement of the problem*

We consider in this paper a thick semi-infinite cylindrical shell occupying the region $a \leqslant r \leqslant b$, $0 \leqslant \theta \leqslant 2\pi$, $z \geqslant 0$, where r, θ, z are cylindrical polar co-ordinates. The shell is composed of homogeneous, isotropic, linearly elastic material and is subjected to an axially symmetric loading which gives rise to a small axially symmetric deformation; the linear theory of small deformations is assumed with $u_\theta = \tau_{r\theta} = \tau_{z\theta} = 0$ throughout. We shall suppose that any tractions on the surfaces $r = a, b, z \geqslant 0$ (and also any body force) can be removed from the problem by subtracting off appropriate solutions for the *infinite* cylindrical shell. This reduces the problem for the semi-infinite shell to one in which the homogeneous conditions

$$\tau_{rr} = \tau_{rz} = 0 \tag{1}$$

hold on $r = a, b, z \geqslant 0$, while one of the following sets of conditions is prescribed on the end surface $z = 0$, $a \leqslant r \leqslant b$:

$$\text{Case A}: \quad \tau_{zz}(r,\theta,0) = \bar{\tau}_{zz}(r), \quad \tau_{rz}(r,\theta,0) = \bar{\tau}_{rz}(r). \tag{2a}$$

$$\text{Case B}: \quad \tau_{zz}(r,\theta,0) = \bar{\tau}_{zz}(r), \quad u_r(r,\theta,0) = \bar{u}_r(r). \tag{2b}$$

$$\text{Case C}: \quad u_z(r,\theta,0) = \bar{u}_z(r), \quad \tau_{rz}(r,\theta,0) = \bar{\tau}_{rz}(r). \tag{2c}$$

$$\text{Case D}: \quad u_r(r,\theta,0) = \bar{u}_r(r), \quad u_z(r,\theta,0) = \bar{u}_z(r). \tag{2d}$$

Furthermore, by subtracting off a suitable simple tension solution, we can restrict the prescribed $\bar{\tau}_{zz}(r)$ in Cases A, B to satisfy

$$\int_a^b \bar{\tau}_{zz}(r) r \, \mathrm{d}r = 0, \tag{3}$$

so that the *resultant applied force in the z-direction is zero*. In Cases C and D we shall impose this condition as an additional constraint upon the solution. With this restriction, it follows from the general theory of Saint-Venant's principle for cylinders (Horgan and Knowles, 1983) that as $z \to \infty$

$$\tau_{ij}(r,\theta,z) = O(\mathrm{e}^{-kz}) \tag{4}$$

for some positive constant k, uniformly in r, θ. For each of the Cases A to D, the boundary value problem specified by (1)–(4) has a unique solution for the induced stresses.

2.2. *The exact eigenvalues and eigenfunctions*

The eigenfunctions appropriate to the above boundary value problems are axially symmetric equilibrium fields $\{\boldsymbol{\tau}, \mathbf{u}\}$ defined in the semi-infinite shell which

(i) satisfy the traction free conditions

$$\tau_{rr} = \tau_{rz} = 0 \tag{5}$$

on $r = a, b$ (while the condition $\tau_{r\theta} = 0$ is satisfied identically),

(ii) have the z-dependence $\mathrm{e}^{-\lambda z}$, where the eigenvalue λ satisfies $\mathrm{Re}\{\lambda\} > 0$.

These eigenfunctions were first presented by Prokopov (1949) and we shall complete and extend Prokopov's development by also using the Love stress function $\phi(r, z)$ [see Timoshenko and Goodier (1951)].

As stipulated above, $\phi(r, z)$ has the form

$$\psi = AJ_0(\lambda r) + BrJ_1(\lambda r) + CY_0(\lambda r) + DrY_1(\lambda r), \tag{8}$$

where J_n, Y_n are Bessel functions, and A, B, C, D are constants. The conditions (5) imply that ϕ must satisfy

$$\frac{\partial}{\partial z}\left\{\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right\} = 0, \qquad \frac{\partial}{\partial r}\left\{(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right\} = 0 \tag{9, 10}$$

on $r = a$ and $r = b$, where ν is Poisson's ratio. Substituting (6), (8) into (9), (10) for $r = a$ and $r = b$ yields a set of four linear homogeneous equations for the constants A, B, C, D. For a nontrivial solution, the determinant $\Delta(\lambda, a, b, \nu)$ of this system must vanish. After some rearrangements with the help of the Wronskian formula $J_0(u)Y_1(u) - J_1(u)Y_0(u) = -2/\pi u$, we obtain

$$\Delta = \lambda^4 a^2 b^2 D_{00}^2 + \{\lambda^2 a^2 - 2(1-\nu)\}\{\lambda^2 b^2 - 2(1-\nu)\}D_{11}^2$$
$$+ \lambda^2 a^2\{\lambda^2 b^2 - 2(1-\nu)\}\left\{D_{01}^2 - \frac{4}{\pi^2\lambda^2 a^2}\right\} + \lambda^2 b^2\{\lambda^2 a^2 - 2(1-\nu)\}\left\{D_{10}^2 - \frac{4}{\pi^2\lambda^2 b^2}\right\}, \tag{11a}$$

where

$$D_{mn} = J_m(\lambda a)Y_n(\lambda b) - J_n(\lambda b)Y_m(\lambda a). \tag{11b}$$

Note that the logarithmic terms in D_{mn} cancel and we have $\Delta \to -4(1-\nu^2)(b^2-a^2)^2/\pi^2 a^2 b^2$ as $\lambda \to 0$. Hence Δ is actually an entire function of λ^2 whose power series expansion has real coefficients. It follows that, for each choice of a, b, ν, the roots of $\Delta(\lambda, a, b, \nu) = 0$ are the eigenvalues which position themselves symmetrically about both the real and imaginary axes of the complex λ-plane. The general theory of entire functions of exponential growth can be used to show that $\Delta = 0$ must have infinitely many roots. It may be further shown that the roots of large modulus are close to the large modulus roots of $\sin^2\lambda(b-a) - \lambda^2(b-a)^2 = 0$, which have the asymptotic form

$$\lambda \sim \frac{1}{b-a}\left[\pm(n+\tfrac{1}{2})\pi \pm \mathrm{i}\log\{(2n+1)\pi\}\right] + O\left(\frac{\log n}{n}\right) \tag{12}$$

as $n \to \infty$.

For each eigenvalue λ, we can now construct the Love stress function ϕ of the corresponding eigenfunction to obtain

$$\psi(r, \lambda, a, b, \nu) = \lambda r\left[\{\lambda^2 a^2 - 2(1-\nu)\}D_{11}D_{11}^{ar} + \lambda^2 a^2 D_{01}D_{01}^{ar} - \frac{2}{\pi}\lambda b D_{10}^{rb}\right]$$
$$+ \left[\lambda^3 a^2 b D_{00}D_{00}^{ar} + \lambda b\{\lambda^2 a^2 - 2(1-\nu)\}D_{10}D_{10}^{ar} + \frac{2}{\pi}\{\lambda^2 a^2 - 2(1-\nu)\}D_{01}^{rb}\right]$$
$$+ 2(1-\nu)\left[\{\lambda^2 a^2 - 2(1-\nu)\}D_{11}D_{10}^{ar} + \lambda^2 a^2 D_{01}D_{00}^{ar} - \frac{2}{\pi}\lambda b D_{00}^{rb}\right], \tag{13}$$

where

$$D_{mn}^{pq}(\lambda) = J_m(\lambda p)Y_n(\lambda q) - J_n(\lambda q)Y_m(\lambda p), \tag{14}$$

with $D_{mn}^{ab} = D_{mn}$. The Love stress function ϕ is then given by (6). The corresponding stress and displacement fields can now be found by substituting this ϕ into the formulae given in Timoshenko and Goodier (1951).

2.3. *Solution of boundary value problems*

Let us denote the stress and displacement fields corresponding to the eigenvalue λ by $\{\boldsymbol{\tau}^\lambda(r)\,e^{-\lambda z}, \mathbf{u}^\lambda(r)\,e^{-\lambda z}\}$, where the dependence on a, b, v is now understood. For each of the boundary value problems A, B, C, D specified by (1)–(4), expand the required solution in the form

$$\boldsymbol{\tau}(r,z) = \sum_{\lambda \in L} a_\lambda \boldsymbol{\tau}^\lambda(r)\,e^{-\lambda z}, \quad \mathbf{u}(r,z) = \sum_{\lambda \in L} a_\lambda \mathbf{u}^\lambda(r)\,e^{-\lambda z}, \tag{15}$$

where L is the set of roots of $\Delta = 0$ lying in the half-plane $\mathrm{Re}\,(\lambda) > 0$. The expressions (15) automatically satisfy (1), (3), (4) and it remains to choose the coefficients $\{a_\lambda\}$ so as to satisfy the end conditions (2) in the appropriate case.† The procedure for determining the $\{a_\lambda\}$ depends upon the case being considered:

(i) For Cases B, C (which have mixed stress/displacement data on $z = 0$) one may exploit the bi-orthogonality of the eigenfunctions [see Gregory (1983) and Gregory and Gladwell (1984)] to give a simple formula for the coefficients $\{a_\lambda\}$.

(ii) For Case A (pure stress data), no closed formula for $\{a_\lambda\}$ is available, but the end-data (2a) can be fitted approximately by truncating the series (15) and fitting the data (2a) by collocation [see Gregory and Gladwell (1989)].

(iii) In Case D (pure displacement data) truncation and collocation can again be carried out, but special methods are needed [see Gregory and Gladwell (1982)] if the stresses near the end $z = 0$ are required.

This completes our treatment of the exact theory. The main purpose of this brief treatment is to establish the eigenvalues and eigenfunctions needed in the subsequent development of a refined shell theory.

3. A REFINED SHELL THEORY

The main disadvantages of the exact three-dimensional theory are (i) the complexity of the eigenfunctions, and (ii) the numerical difficulty in calculating them. Although the eigenfunctions themselves were first presented by Prokopov (1949), it was not until the 1970s that sufficient computing power became available to calculate them directly. For example Steven (1973) treated a specimen problem for Case A, while Redekop (1977) solved a more complicated problem in which the cylindrical shell was sealed by a hemispherical cap. Such investigations, though interesting in themselves, do not make it much easier for a subsequent researcher to solve even a closely related problem.

At present the only practical alternative to the exact theory is the *thin shell theory* (and then, as we shall see, only for Case A) based on asymptotic considerations. This is an approximate theory which is valid and accurate at a distance of order h *away from the edge*(*s*) *of the shell* in the limit as the dimensionless shell thickness ε tends to zero. It offers a massive simplification, but is appropriate only for $\varepsilon \ll 1$ (less than 1/10 in engineering applications). Recent results in plate theory (Gregory and Wan, 1985a, b) suggest that the thin shell theory is, in principle, adequate only for Case A. The refined shell theory developed in this paper will confirm this observation and provide a properly formulated thin shell theory for other types of end data as byproducts.

A refined shell theory will be developed in Sections 3 and 4 to bridge the gap between thin shell theory and the exact theory. This refined theory is much simpler to apply than the exact theory, and yet is more accurate than thin shell theory because it retains terms of order ε relative to the corresponding terms of thin shell theory. A pleasing feature of this refined shell theory is that (unlike the exact theory) it is simple enough to allow closed form analytic solutions to be found (see Section 5). An essential aspect in the formulation of our refined shell theory is the derivation in Section 4 of the appropriate boundary conditions

† We are presuming that this set of eigenfunctions is *complete*. This has never been proved, but the situation is analogous to that for the semi-infinite elastic strip, for which rigorous completeness results have been established [see Gregory (1980a, b)].

for the two-dimensional theory from the given three-dimensional data. Even for the first approximation thin shell theory, the development in Section 4 provides new and fundamentally important results.

3.1. *The PF-eigenvalues and eigenfunctions*

Let the mid-radius R and a dimensionless thickness ε of the cylindrical shell be defined by

$$a = R(1-\varepsilon), \quad b = R(1+\varepsilon), \tag{16}$$

so that $R = (a+b)/2$ and the actual shell thickness $h = 2R\varepsilon$. The refined shell theory is developed for small ε in the formulae of Section 2. In the present section we shall investigate the asymptotic behavior of the "PF-eigenvalues and eigenfunctions"; these are eigenfunctions whose eigenvalues λ behave like ε^{-1} as $\varepsilon \to 0$. They are related to the plane strain Papkovich–Fadle eigenfunctions for the semi-infinite strip of width h.

Results of previous investigations suggest that we introduce the dimensionless eigenvalue w by

$$\lambda = \frac{w}{R\varepsilon}, \tag{17}$$

where (complex) w is *bounded away from zero*; thus $|\lambda| \to \infty$ as $\varepsilon \to 0$. With the alternative representation

$$D_{mn} = \frac{i}{2}[H_m^{(1)}(\lambda a)H_n^{(2)}(\lambda b) - H_n^{(1)}(\lambda b)H_m^{(2)}(\lambda a)], \tag{18}$$

where $H_k^{(1)}, H_k^{(2)}$ are Hankel functions of the first and second kinds (Watson, 1966), we may use the asymptotic expansions for these functions to obtain

$$\begin{aligned}\frac{\pi}{2}\frac{w(1-\varepsilon^2)^{1/2}}{\varepsilon}D_{00} = \cos 2w\left[\frac{1}{4w}\varepsilon^2 + \left(\frac{1}{4w} - \frac{27}{64w^3}\right)\varepsilon^4 + O(\varepsilon^6)\right]\\ + \sin 2w\left[1 - \frac{1}{8w^2}\varepsilon^2 + \left(\frac{27}{128w^4} - \frac{13}{32w^2}\right)\varepsilon^4 + O(\varepsilon^6)\right],\end{aligned} \tag{19}$$

with similar expressions for the other D_{mn}. Throughout this paper, all such evaluations were performed with the assistance of computer algebra. The system used was MAPLE (1988) installed on an Amdahl 5890 mainframe computer at the University of Manchester, England.

Define the modified determinant $\Delta^{\mathrm{PF}}(w, \varepsilon, \nu)$ by

$$\Delta^{\mathrm{PF}} = \frac{\pi^2}{4}\Delta\left(\frac{w}{R\varepsilon}, R(1-\varepsilon), R(1+\varepsilon), \nu\right), \tag{20}$$

where $\Delta(\lambda, a, b, \nu)$ is defined by (11). On substituting the expressions like (19) for D_{mn} into (11), and then simplifying, we obtain

$$\begin{aligned}\Delta^{\mathrm{PF}} = (\sin^2 2w - 4w^2) + \frac{1}{2w^2}[(8\nu-7)(2w^2 + w\sin 2w\cos 2w)\\ + (8\nu^2 - 8\nu - 1)\sin^2 2w]\varepsilon^2 + O(\varepsilon^4).\end{aligned} \tag{21}$$

The asymptotic formula (21) is uniformly valid (as $\varepsilon \to 0$) for w on any bounded set, bounded away from $w = 0$, and lying in the first quadrant; this last restriction can be

Table 1. Exact and approximate roots of $\Delta^{PF} = 0$ to five decimal places for the case $\varepsilon = 0.1$, $\nu = 0.3$

n	w_n	Expression (24)	Exact root of $\Delta^{PF} = 0$
1	2.10620+i1.12536	2.10958+i1.12546	2.10962+i1.12546
2	3.74884+i1.38434	3.75052+i1.38434	3.75054+i1.38434
3	5.35627+i1.55157	5.35740+i1.55155	5.35741+i1.55155

dropped however, since both sides of (21) have the same reflective symmetries in the co-ordinate axes of w.

It follows from (21) that, as $\varepsilon \to 0$, the roots of Δ^{PF} in any fixed *bounded* region of the w-plane are close† to roots of

$$\Delta_0^{PF} \equiv \sin^2 2w - 4w^2 = 0. \tag{22}$$

The roots of (22) are well known from the theory of the semi-infinite strip: There is a quadruple root at $w = 0$, but no other real or pure imaginary roots; the complex roots are simple and occur in symmetrical sets of four; if the roots lying in the first quadrant, and ordered by increasing real part, are denoted by $\{w_n\}$, $n = 1, 2, \ldots$, then the root w_n satisfies the equation

$$\sin 2w_n = (-1)^n 2w_n, \tag{23}$$

with the asymptotic behavior of w_n for large n given by (12) (for $b-a = 2$). The relation (21) can be used to find an approximation to the root of Δ^{PF} which is close to w_n as $\varepsilon \to 0$:

$$w = w_n + \left[\frac{(16\nu^2 - 8\nu - 9) + (-1)^n(8\nu - 7)\cos 2w_n}{8w_n(1 - (-1)^n \cos 2w_n)}\right]\varepsilon^2 + O(\varepsilon^4). \tag{24}$$

We note that, since the expansion (21) for Δ^{PF} is in even powers of ε, the "correction" to w_n is of order $O(\varepsilon^2)$ as $\varepsilon \to 0$.

Table 1 compares numerical values of w_n, the corrected expression (24), and the exact roots of $\Delta^{PF} = 0$ when $\varepsilon = 0.1$, $\nu = 0.3$ and $n = 1, 2, 3$. The w_n were found by performing a (complex) Newton iteration with eqn (22), using as starting values the (large n) asymptotic expression (12); the exact roots were then found by performing a Newton iteration with the equation $\Delta^{PF} = 0$, using as starting values the (small ε) asymptotic expression (24). For each root w of $\Delta^{PF} = 0$, the corresponding PF-eigenvalue is given by $\lambda = w/(R\varepsilon)$.

For each PF-eigenvalue λ, there corresponds a PF-eigenfunction whose Love stress function ϕ is given by (6), (13). As $\varepsilon \to 0$, the formula (13) for ψ can be evaluated asymptotically by the same sort of procedure as was used above for Δ; the details are more complicated, since (13) also involves the radial co-ordinate r. The results of this calculation for the eigenfunction stresses and displacements are given in Appendix A; a significant feature is that the "correction terms" are of order $O(\varepsilon)$ as $\varepsilon \to 0$ (unlike the eigenvalues where the correction term has order $O(\varepsilon^2)$).

3.2. *The "shell" eigenvalue and eigenfunction*

We must still account for the four roots of $\Delta = 0$ which are of order $o(\varepsilon^{-1})$ as $\varepsilon \to 0$. (The existence of these roots is established by (21), since (22) has a quadruple root at the origin.) We shall show that these roots are a symmetrical set of four ($\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}$ with $0 < \arg \lambda < \pi/2$) and that, as $\varepsilon \to 0, \lambda$ is close to the eigenvalue predicted by thin shell theory.

Previous results suggest that, for the shell eigenvalues, we should re-scale and set

† More precisely: If C is any simple closed contour, not passing through any root of (22), then, for all sufficiently small ε, the number of zeros of Δ^{PF} lying in C is equal to the number of zeros of Δ_0^{PF} lying in C.

$$\lambda = \frac{W}{R\varepsilon^{1/2}}, \tag{25}$$

where (complex) W is bounded away from zero; then $|\lambda| \to \infty$ as $\varepsilon \to 0$. We can now proceed through a process similar to that in Section 3, but using (25) instead of (17). As $\varepsilon \to 0$, this case is more ill-conditioned than that of the PF-eigenvalues. Because of cancellations, it is necessary to retain nine terms in each of the expansions of Bessel functions in order to obtain a two term approximation to Δ. In addition, the exponential term is now ε-dependent. However MAPLE (1988) can handle such difficulties and we find, for instance, that

$$D_{00} = \frac{2}{\pi}\frac{1}{W}\left(\frac{\varepsilon}{1-\varepsilon^2}\right)^{1/2}[2W\varepsilon^{1/2} - \tfrac{4}{3}W^3\varepsilon^{3/2} + (\tfrac{4}{15}W^5 - \tfrac{1}{3}W)\varepsilon^{5/2} + (\tfrac{2}{15}W^3 - \tfrac{8}{315}W^7)\varepsilon^{7/2} + O(\varepsilon^{9/2})] \tag{26}$$

with similar expressions for the other D_{mn}.

Define the modified determinant $\Delta^{\mathrm{S}}(W, \varepsilon, \nu)$ by

$$\Delta^{\mathrm{S}} = -\frac{3\pi^2}{64}\varepsilon^{-2}\Delta\left(\frac{W}{R\varepsilon^{1/2}}, R(1-\varepsilon), R(1+\varepsilon), \nu\right), \tag{27}$$

where $\Delta(\lambda, a, b, \nu)$ is defined by (11). On substituting the expressions such as (26) for the D_{mn} into (11) and then simplifying, we obtain

$$\Delta^{\mathrm{S}} = (W^4 + 4\omega^4) - W^2(\tfrac{16}{3}\omega^4 + \tfrac{8}{15}W^4)\varepsilon + O(\varepsilon^2), \tag{28}$$

where the constant $\omega(\nu)$ is given by $\omega = \{3(1-\nu^2)/4\}^{1/4}$. The asymptotic formula (28) is uniformly valid (as $\varepsilon \to 0$) for W on any bounded complex set, bounded away from zero. It follows that, as $\varepsilon \to 0$, the roots of $\Delta^{\mathrm{S}} = 0$ lying in any fixed *bounded* region of the W-plane are close to roots of

$$\Delta_0^{\mathrm{S}} \equiv W^4 + 4\omega^4 = 0. \tag{29}$$

Equation (29) has only the symmetrical set of four roots $W = (\pm 1 \pm \mathrm{i})\omega$ and we shall focus attention on the zero of Δ^{S} which (as $\varepsilon \to 0$) is close to W_0, where

$$W_0 = (1+\mathrm{i})\omega. \tag{30}$$

It follows from (28) that this zero of Δ^{S} is given by

$$W = \omega(1+\mathrm{i})\{1 - \tfrac{2}{5}\mathrm{i}\omega^2\varepsilon\} + O(\varepsilon^2) \tag{31}$$

as $\varepsilon \to 0$. We note that the "correction" to W_0 is of order $O(\varepsilon)$, unlike the PF-eigenvalues.

The limiting value W_0 gives the eigenvalue

$$\lambda_0 = \frac{\omega(1+\mathrm{i})}{R\varepsilon^{1/2}} \tag{32}$$

which is the value predicted by thin shell theory; (31) then gives the corrected value

$$\lambda = \frac{\omega(1+\mathrm{i})}{R\varepsilon^{1/2}}\{1-\tfrac{2}{5}\mathrm{i}\omega^2\varepsilon+O(\varepsilon^2)\} \tag{33}$$

as $\varepsilon \to 0$. The formula (33) differs from the corrected value of λ derived by Prokopov (1949). However, Prokopov assumed that only powers of W up to W^4 should appear in his expansion of Δ; this assumption is now seen to err in the $O(\varepsilon)$ term [see (28)] which is significant for the refined theory.

Table 2 compares the thin shell eigenvalue (32), the corrected shell eigenvalue (33), and the exact shell eigenvalue for $R = 1$, $\nu = 0.3$ and for a range of values of ε; the exact shell eigenvalue was calculated by performing a Newton iteration with the equation $\Delta = 0$, using as the starting value the expression (28). These numerical results clearly confirm the adequacy of this asymptotic expression for the range of ε values there.

The Love stress function of the shell eigenfunction is given by (6), (13), where λ is now taken from (33). The formula (13) for ψ can now be evaluated asymptotically as $\varepsilon \to 0$ by substituting (25) into (13) and proceeding as for Δ. The details are complicated as the radial co-ordinate r now appears; also it is found that 11 terms have to be taken in the Bessel function expansions. The results of this calculation for the eigenfunction stresses and displacements are given in Appendix B. The *leading* terms are found to be the same as the predictions of thin shell theory. The correction terms [which are of order $O(\varepsilon)$] differ from those derived by Prokopov (1949). This is in part because Prokopov's $O(\varepsilon)$ correction of the thin shell approximation for λ requires a significant modification. In addition, Prokopov's expression for τ_{rz}, for example, contains no term in ξ^4 [compare with (B13), (B14)].

4. BOUNDARY CONDITIONS FOR THE REFINED SHELL THEORY

By using the formulae in Appendices A, B, one may now replace the terms in the eigenfunction expansions (15) by their refined approximations, and then proceed to solve the boundary value problems A–D as in the exact theory. This avoids the computational complexity of the exact theory and (as the first example in Section 5 shows) also produces a substantial improvement in accuracy over thin shell theory throughout the whole shell for moderately small values of ε.

However our main purpose here is not to proceed with this essentially numerical process, but to create an *analytical refined shell theory* valid and more accurate than thin shell theory in the shell interior. Such a theory is possible (as is thin shell theory) because of the differing rates of decay of the shell and PF-eigenfunctions as z increases. The decay of the PF-eigenfunctions is governed by the real parts of the PF-eigenvalues $\{\lambda_n\}$. It follows (see Section 3) that the "decay length" of a general linear combination of PF-eigenfunctions is $R\varepsilon/\Re(w_1) \approx 0.24h$, where h is the shell thickness; thus this combination will be insignificant (compared to its value at $z = 0$) for (say) $z > h$. We shall refer to such an elastic state in the shell as being *rapidly decaying*. On the other hand, the shell eigenfunction (see Section 3) has the decay length $R\varepsilon^{1/2}/\omega \approx 0.8(Rh)^{1/2}$; this is long, compared to the PF-decay length, when h/R is small.

In the present context, the object of a shell theory is to determine, in the expansions (15), the coefficient of the shell eigenfunction (and its conjugate). These two terms dominate the solution for $z > h$ when h/R is small. For the stress boundary value problem, thin shell

Table 2. Exact and approximate values for the shell eigenvalue (to four decimal places) for the case $R = 1$, $\nu = 0.3$

ε	Thin shell theory (32)	Refined shell theory (33)	Exact root of $\Delta = 0$
0.08	3.2135+i3.2135	3.2985+i3.1286	3.3029+i3.1324
0.04	4.5446+i4.5446	4.6047+i4.4845	4.6062+i4.4859
0.02	6.4270+i6.4270	6.4695+i6.3846	6.4700+i6.3851
0.01	9.0892+i9.0892	9.1192+i9.0592	9.1194+i9.0593

theory is supposed to determine these coefficients (and eigenfunctions) correct to order $O(1)$ as $\varepsilon \to 0$, while the refined shell theory of the present paper determines them correct to order $O(\varepsilon)$. As the examples in Section 5 show, the differences between the thin and refined shell theories can be substantial, even when thin shell theory is known to be asymptotically valid. More importantly, the method for determining the coefficients of the shell eigenfunction for the refined theory and its conjugate is not a simple extension of that for the thin shell theory. This is so even for the pure stress end data case for which the conventional thin shell theory will be shown to provide the leading asymptotic approximation.

For the displacement boundary value problem, however, the correct determination of the coefficients of the shell eigenfunction and its conjugate in (15) is actually not known up to now, even in thin shell theory [see Nair and Reissner (1978) for related developments]. In the more conventional formulation of shell theories in the form of a system of two-dimensional differential equations and boundary conditions, there is no two-dimensional edge condition known to be the generally appropriate replacement for Case D end data except when the data are linearly distributed across the shell thickness. We will show that the correct edge conditions generally do not correspond to the presently accepted displacement boundary conditions for thin shell theory even for very thin shells.

The formulation of the proper two-dimensional edge conditions for the different types of end data constitutes the heart of the present paper. In developing these edge conditions, it is possible to allow the prescribed data to depend generally upon ε, as well as on the radial co-ordinate ξ. The methods described below do apply to such general data but, in order to simplify the form of the final conditions, we will usually assume that the data is of order $O(1)$ and also does not depend on ε. The modifications required for the more general case are clear enough.

4.1. *Case A—Pure stress data*

Suppose that the data on $z = 0$, $a < r < b$ is that prescribed by (2a). It is tempting to try to determine the refined shell solution by requiring that its *stress resultants and couples at* $z = 0$ should equal those of the precribed data $\bar{\tau}_{zz}(r)$, $\bar{\tau}_{rz}(r)$. The analysis below proves that such an approach generally does not lead to the correct asymptotic solution except for the leading terms (as $\varepsilon \to 0$), thus justifying its use only in *thin* shell theory.

The derivation of the correct conditions to be applied at the end $z = 0$ is similar to that in thick *plate* theory [see Gregory and Wan (1985a, b)]. The problem is equivalent to asking what conditions the end-data functions $\bar{\tau}_{zz}(r)$, $\bar{\tau}_{rz}(r)$ must satisfy in order that the resulting state of stress in the shell should be *rapidly decaying* as $z \to \infty$. By using the elastic reciprocal theorem as in Gregory and Wan (1985a), it follows that end-data which generates a rapidly decaying state must satisfy necessary conditions of the form

$$\int_a^b [U_r \bar{\tau}_{rz} + U_z \bar{\tau}_{zz}] r \, \mathrm{d}r = 0. \tag{34}$$

In (34), U_r, U_z are the end displacements (on $z = 0$) of any axially symmetric elastic field $\{\mathbf{T}, \mathbf{U}\}$ in the shell, which satisfies the boundary conditions

$$T_{rr} = T_{rz} = 0 \quad \text{on} \quad \{r = a, b, z > 0\} \tag{35}$$

$$T_{rz} = T_{zz} = 0 \quad \text{on} \quad \{z = 0, a < r < b\} \tag{36}$$

and that $\mathbf{T}$ is *not* rapidly increasing† as $z \to \infty$. As this latter requirement is significantly different from the *algebraic* growth restriction for the corresponding plate problem, the construction of $\{\mathbf{T}, \mathbf{U}\}$ requires more finesse. We will now construct *two* suitable states $\{\mathbf{T}, \mathbf{U}\}$ correct to order $O(\varepsilon)$.

† That is, not increasing as rapidly as any of the *increasing* PF-eigenfunctions. Note that $\{\mathbf{T}, \mathbf{U}\}$ may (and in fact must) grow exponentially as $z \to \infty$.

Let $\boldsymbol{\tau}^{\lambda}(r)\,\mathrm{e}^{-\lambda z}$ denote the shell eigenfunction (with eigenvalue λ lying in the first quadrant) and define

$$\boldsymbol{\tau}^{R}(r,z) = \Re\{\boldsymbol{\tau}^{\lambda}\,\mathrm{e}^{-\lambda z}\}, \qquad \boldsymbol{\tau}^{I}(r,z) = \Im\{\boldsymbol{\tau}^{\lambda}\,\mathrm{e}^{-\lambda z}\}. \tag{37, 38}$$

Consider the stress field

$$[\boldsymbol{\tau}^{R}]_{\lambda\to-\lambda}+\boldsymbol{\tau}^{R}, \tag{39}$$

where $[\boldsymbol{\tau}]_{\lambda\to-\lambda}$ denotes $\boldsymbol{\tau}$ with λ replaced by $-\lambda$. The first term *grows* exponentially as $z\to\infty$, but is not "rapidly increasing" (see footnote on p. 1967). On $r=a,b$, the stress field (39) satisfies (35), while on $z=0$ its rz-component is zero by symmetry (see Appendix B). The zz-component of (39) on $z=0$ has the value $2\tau^{R}_{zz}(r,0)$, where [see eqn (B11)]

$$\tau^{R}_{zz}(\xi,0) = -\tfrac{2}{5}E\xi(11+5\xi^{2})\varepsilon+O(\varepsilon^{2}) \tag{40}$$

as $\varepsilon\to 0$; here E is Young's modulus, and the dimensionless radial co-ordinate ξ $(-1\leqslant\xi\leqslant 1)$ is defined by $r=R(1+\xi\varepsilon)$. The stress field (39) is therefore an approximation [correct to order $O(1)$] to a possible $\mathbf{T}$ for (34).

To obtain a better approximation correct to order ε, consider

$$[\boldsymbol{\tau}^{R}]_{\lambda\to-\lambda}+\boldsymbol{\tau}^{R}-\tfrac{56}{15}\,\omega^{2}[\boldsymbol{\tau}^{R}+\boldsymbol{\tau}^{I}]\varepsilon. \tag{41}$$

The zz-component of the stress field (41) on $z=0$ is given by

$$\tfrac{4}{5}E\xi(3-5\xi^{2})\varepsilon+O(\varepsilon^{2}), \tag{42}$$

while the rz-component is now $O(\varepsilon^{5/2})$.

Now imagine that

$$\tau_{\zeta\zeta} = \tfrac{4}{5}E\xi(3-5\xi^{2}), \quad \tau_{\xi\zeta}=0 \tag{43}$$

is the prescribed end-data for the plane semi-infinite elastostatic strip $-1\leqslant\xi\leqslant 1$, $\zeta\geqslant 0$, whose long sides are traction free. *This set of end-data is self-equilibrating in* (ξ,ζ)-*space* and so [see Gregory (1980b)] can be expanded as a series of the Papkovich–Fadle eigenfunctions *for this strip*. Take this set of expansion coefficients, but now replace the strip PF-eigenfunctions with the *shell* PF-eigenfunctions of Appendix A. The leading term of these shell PF-eigenfunctions (as $\varepsilon\to 0$) coincides with the strip eigenfunctions, and so we will generate a rapidly decaying stress state $\boldsymbol{\tau}^{\mathrm{PF}}$ in the shell whose zz- and rz-components on $z=0$ are given by $\tfrac{4}{5}E\xi(3-5\xi^{2})+O(\varepsilon)$ and $O(\varepsilon)$, respectively.

An approximate $\mathbf{T}$, which is appropriate for our refined shell theory when used with (34), can now be constructed in the form

$$\mathbf{T} = [\boldsymbol{\tau}^{R}]_{\lambda\to-\lambda}+\boldsymbol{\tau}^{R}-\tfrac{56}{15}\,\omega^{2}[\boldsymbol{\tau}^{R}+\boldsymbol{\tau}^{I}]-\boldsymbol{\tau}^{\mathrm{PF}}\varepsilon+\boldsymbol{\tau}^{\mathrm{ERR}}, \tag{44}$$

where the zz- and rz-components of $\boldsymbol{\tau}^{\mathrm{ERR}}$ on $z=0$ are of order $O(\varepsilon^{2})$ as $\varepsilon\to 0$, and all terms except the first are exponentially decaying as $z\to\infty$. The displacement field at $z=0$ corresponding to the stress field (44) has the form

$$U_{r} = 4(1-\nu\xi\varepsilon)-\tfrac{112}{15}\,\omega^{2}\varepsilon+O(\varepsilon^{3/2}), \quad U_{z} = \text{constant}+O(\varepsilon^{3/2}), \tag{45}$$

as $\varepsilon\to 0$. A pleasing feature of (45) is that the *displacement* field corresponding to $\boldsymbol{\tau}^{\mathrm{PF}}\varepsilon$ is of order $O(\varepsilon^{2})$ [see Appendix A, eqns (A3), (A4)], and so makes no explicit appearance; *this means that we do not need to solve the plane semi-infinite strip problem for the end-data* (43). On substituting (45) into (34) and changing the integration variable to ξ, we obtain

$$\int_{-1}^{1} (1+\xi\varepsilon)[(1-\nu\xi\varepsilon-\tfrac{28}{15}\omega^2\varepsilon+O(\varepsilon^{3/2}))\bar{\tau}_{rz}(\xi)+(\text{constant}+O(\varepsilon^{3/2}))\,\bar{\tau}_{zz}(\xi)]\,\mathrm{d}\xi = 0. \quad (46)$$

In view of the restriction (11), the constant $O(1)$ term in U_z makes no contribution to (46). Also the term $-28\omega^2\varepsilon/15$ can be removed from (46) by multiplying the whole equation by $(1+\frac{28}{15}\omega^2\varepsilon)$. Finally, by replacing $\boldsymbol{\tau}^{\mathrm{ERR}}$ in (44) by an appropriate combination of $\boldsymbol{\tau}^R$, $\boldsymbol{\tau}^I$, together with a new error term, it may be shown that the two $O(\varepsilon^{3/2})$ terms in (46) may in fact be improved to $O(\varepsilon^2)$. The condition (46) then becomes

$$\int_{-1}^{1} (1+\xi\varepsilon)(1-\nu\xi\varepsilon)\bar{\tau}_{rz}(\xi)\,\mathrm{d}\xi = O(\varepsilon^2) \quad (47)$$

as $\varepsilon \to 0$. [In order to simplify the writing, we have assumed that the data $\bar{\tau}_{rz}(\xi)$, $\bar{\tau}_{zz}(\xi)$ is of order $O(1)$ as $\varepsilon \to 0$; if this is not so, then the error term $O(\varepsilon^2)$ may be changed. Also, if $\bar{\tau}_{rz}$ and $\bar{\tau}_{zz}$ depend on ε analytically, we should truncate their Taylor series in ε after the first two terms. Similar remarks apply to the data in Cases B, C, D treated later.] The requirement (47) is our first necessary condition that the data (2a) should generate a rapidly decaying state; when working with our refined shell theory, correct to order $O(\varepsilon)$, the right-hand side of (47) is negligible.

A second necessary condition for a rapidly decaying state can be obtained in a similar way by beginning with the stress field

$$[\boldsymbol{\tau}^R+\boldsymbol{\tau}^I]_{\lambda\to-\lambda}-[\boldsymbol{\tau}^R+\boldsymbol{\tau}^I] \quad (48)$$

instead of (39). The final form of this condition is

$$\int_{-1}^{1} (1+\xi\varepsilon)\xi(1-\tfrac{1}{2}\nu\xi\varepsilon)\bar{\tau}_{zz}(\xi)\,\mathrm{d}\xi = O(\varepsilon^2) \quad (49)$$

as $\varepsilon \to 0$, where, as before, the dimensionless radial co-ordinate ξ is defined by $r = R(1+\xi\varepsilon)$.

The limiting forms of the conditions (47) and (49) as $\varepsilon \to 0$ can be written as (on restoring the radial variable r)

$$Q_z \equiv \int_a^b \bar{\tau}_{rz}(r)\frac{r}{R}\,\mathrm{d}r = 0, \qquad M_z \equiv \int_a^b (r-R)\bar{\tau}_{zz}(r)\frac{r}{R}\,\mathrm{d}r = 0, \quad (50, 51)$$

where Q_z, M_z are the stress resultant and couple (across the shell thickness) as defined, for example, by Johnson and Reissner (1958). These limiting necessary conditions for a rapidly decaying state are thus equivalent to the usual appeal to Saint-Venant's principle [see Timoshenko and Goodier (1951) for example]. In the refined theory however, we see that Saint-Venant's principle does not yield the appropriate correction terms (except when $\nu = 0$). The situation is summarized by the following theorem:

Theorem. Suppose that $\bar{\tau}_{rz}(r)$, $\bar{\tau}_{zz}(r)$ have the same order of magnitude† *as $\varepsilon \to 0$. If the data $\bar{\tau}_{rz}(r)$, $\bar{\tau}_{zz}(r)$ induce a rapidly decaying state, then*

(a) *The corresponding stress resultant Q_z and stress couple M_z must vanish (so that the conventional use of Saint-Venant's principle is justified) for the* **thin** *shell theory.*

(b) *The modified stress resultant $\hat{Q}_z$ and modified couple $\hat{M}_z$ given below must vanish for the* **refined** *theory:*

† This condition can be relaxed to allow $\bar{\tau}_{rz}$, $\bar{\tau}_{zz}$ to have orders of magnitude which differ by ε^α as $\varepsilon \to 0$, where $|\alpha| < 1$.

$$\hat{Q}_z \equiv \int_a^b \left(1 - \frac{\nu(r-R)}{R}\right)\bar{\tau}_{rz}(r)\frac{r}{R}\,\mathrm{d}r = 0, \tag{52}$$

$$\hat{M}_z \equiv \int_a^b (r-R)\left(1 - \frac{\nu(r-R)}{2R}\right)\bar{\tau}_{zz}(r)\frac{r}{R}\,\mathrm{d}r = 0. \tag{53}$$

The correctness of the conditions (52) and (53) is verified explicitly in Appendix A. These conditions should be compared with the corresponding conditions for flat plates obtained in Gregory and Wan (1985a).

4.2. *Case B—Mixed data*

Let τ_{zz} and u_r be prescribed on $z = 0$ as in (2b). In this case, the reciprocity argument implies that if this end-data generates a rapidly decaying state then it must satisfy necessary conditions of the form

$$\int_a^b [U_z\bar{\tau}_{zz} - T_{rz}\bar{u}_r]r\,\mathrm{d}r = 0 \tag{54}$$

on $z = 0$, where the secondary state $\{\mathbf{T}, \mathbf{U}\}$ satisfies the same conditions as in Case A, except that (36) is replaced by

$$T_{zz} = U_r = 0 \quad \text{on} \quad \{z = 0, a < r < b\}. \tag{55}$$

An *exact* choice for $\mathbf{T}$ is

$$\mathbf{T} = -\frac{1}{2\omega}[\boldsymbol{\tau}^R + \boldsymbol{\tau}^I]_{\lambda\to-\lambda} + \frac{1}{2\omega}[\boldsymbol{\tau}^R + \boldsymbol{\tau}^I] \tag{56}$$

and corresponding values of U_z, T_{rz} on $z = 0$ are given by (see Appendix B)

$$R^{-1}U_z = 4\xi\varepsilon^{1/2} - \tfrac{2}{15}(10 + 29\nu + 15\nu\xi^2)\varepsilon^{3/2} + O(\varepsilon^{5/2}), \tag{57}$$

$$E^{-1}T_{rz} = \tfrac{1}{5}(1-\xi^2)(33 + 5\xi^2)\varepsilon^{3/2} + O(\varepsilon^{5/2}) \tag{58}$$

as $\varepsilon \to 0$. On substituting (57), (58) into (54), the terms of (57) which are independent of ξ give no contribution [because of the restriction (3)], and the resulting condition is

$$\int_{-1}^{1} (1+\xi\varepsilon)[\xi(1-\tfrac{1}{2}\nu\xi\varepsilon)E^{-1}\bar{\tau}_{zz}(\xi) - \tfrac{1}{20}(1-\xi^2)(33+5\xi^2)\varepsilon R^{-1}\bar{u}_r(\xi)]\,\mathrm{d}\xi = O(\varepsilon^2) \tag{59}$$

as $\varepsilon \to 0$.

A second exact choice for $\mathbf{T}$ is

$$\mathbf{T} = -\frac{\omega}{2}[\boldsymbol{\tau}^R - \boldsymbol{\tau}^I]_{\lambda\to-\lambda} + \frac{\omega}{2}[\boldsymbol{\tau}^R - \boldsymbol{\tau}^I], \tag{60}$$

which leads to the condition

$$\int_{-1}^{1} (1+\xi\varepsilon)[(3(1-\xi^2) - (1+\nu)\xi(1-\xi^2)\varepsilon)R^{-1}\bar{u}_r(\xi) + (1+\nu)((2-\nu)\xi^3 + \tfrac{1}{5}(16-31\nu)\xi)\varepsilon E^{-1}\bar{\tau}_{zz}(\xi)]\,\mathrm{d}\xi = O(\varepsilon^2). \tag{61}$$

Equations (59), (61) can be used to simplify each other. On retaining only leading terms, these equations reduce to

$$\int_{-1}^{1} \xi\bar{\tau}_{zz}(\xi)\,\mathrm{d}\xi = O(\varepsilon), \qquad \int_{-1}^{1} (1-\xi^2)\bar{u}_r(\xi)\,\mathrm{d}\xi = O(\varepsilon), \tag{62, 63}$$

and these equations enable certain of the $O(\varepsilon)$ terms in (59), (61) to be removed. The simplified equations are

$$\int_{-1}^{1} (1+\xi\varepsilon)[\xi(1-\tfrac{1}{2}\nu\xi\varepsilon)E^{-1}\bar{\tau}_{zz}(\xi)-\tfrac{1}{4}\xi^2(1-\xi^2)\varepsilon R^{-1}\bar{u}_r(\xi)]\,\mathrm{d}\xi = O(\varepsilon^2), \tag{64}$$

$$\int_{-1}^{1} (1+\xi\varepsilon)\left[\left\{\frac{3}{1+\nu}(1-\xi^2)-\xi(1-\xi^2)\varepsilon\right\}R^{-1}\bar{u}_r(\xi)+(2-\nu)\xi^3\varepsilon E^{-1}\bar{\tau}_{zz}(\xi)\right]\mathrm{d}\xi = O(\varepsilon^2). \tag{65}$$

The two integral conditions (64), (65) are the required necessary conditions that the data $\bar{\tau}_{zz}, \bar{u}_r$ should generate a rapidly decaying state.

4.3. *Case C—Mixed data*

Let τ_{zr} and u_z be prescribed on $z = 0$ as in (2c). For these end-data to generate a rapidly decaying state, they must satisfy necessary conditions of the form

$$\int_a^b [U_r\bar{\tau}_{rz}-T_{zz}\bar{u}_z]r\,\mathrm{d}r = 0 \tag{66}$$

on $z = 0$, where the secondary state $\{\mathbf{T}, \mathbf{U}\}$ satisfies the same conditions as in Case A, except that (36) is replaced by

$$T_{rz} = U_z = 0 \quad \text{on} \quad \{z = 0, a \leqslant r \leqslant b\}. \tag{67}$$

Two *exact* choices for $\mathbf{T}$ are

$$\mathbf{T} = \tfrac{1}{2}[\boldsymbol{\tau}^R]_{\lambda\to-\lambda}+\tfrac{1}{2}[\boldsymbol{\tau}^R], \tag{68}$$

and

$$\mathbf{T} = \frac{1-\nu^2}{2\omega^2}[\boldsymbol{\tau}']_{\lambda\to-\lambda}+\frac{1-\nu^2}{2\omega^2}[\boldsymbol{\tau}']. \tag{69}$$

Since in the present case we have $u_z(r, 0)$ (and not τ_{zz}) prescribed, we need an additional necessary condition on $\bar{\tau}_{rz}, \bar{u}_z$ to ensure that the induced displacement field $u_z(r, z)$ (as well as the stress field) is rapidly decaying as $z \to \infty$. This is easily obtained however by taking $\{\mathbf{T}, \mathbf{U}\}$ to be a state of simple tension parallel to the axis of z. The resulting condition is

$$\int_{-1}^{1} (1+\xi\varepsilon)[R^{-1}\bar{u}_z(\xi)+\nu(1+\xi\varepsilon)E^{-1}\bar{\tau}_{rz}(\xi)]\,\mathrm{d}\xi = 0. \tag{70}$$

The conditions arising from (68), (69) may be used to simplify each other, as in Case B. The simplified formulae are

$$\int_{-1}^{1}(1+\xi\varepsilon)[(1-\nu\xi\varepsilon)E^{-1}\bar{\tau}_{rz}(\xi)+\xi^3\varepsilon R^{-1}\bar{u}_z(\xi)]\,\mathrm{d}\xi = O(\varepsilon^2), \tag{71}$$

$$\int_{-1}^{1}(1+\xi\varepsilon)[\xi(1-\tfrac{1}{2}\nu\xi\varepsilon)R^{-1}\bar{u}_z(\xi)+\tfrac{1}{2}\nu(1+\nu)\xi^2\varepsilon E^{-1}\bar{\tau}_{rz}(\xi)]\,\mathrm{d}\xi = O(\varepsilon^2). \tag{72}$$

Equations (70)–(72) are the required necessary conditions that the data $\bar{\tau}_{rz}, \bar{u}_z$ should generate a rapidly decaying state.

Case D—Pure displacement data

In this case it does not seem feasible to obtain explicit boundary conditions [similar to (47)–(49), (64)–(65) and (70)–(72)] for the refined shell theory, though the method of reciprocal relation still applies. However we may still deduce the correct boundary conditions for the (leading term) *thin* shell theory in terms of known results by using a different argument to obtain the appropriate canonical secondary states $\{\mathbf{T}, \mathbf{U}\}$. Despite the long history of thin shell theory, *these conditions are new to the literature.*

From Appendix A, we can verify that the leading terms (as $\varepsilon \to 0$) in the PF-eigenfunctions are identical to those of the plane strain semi-infinite strip $-1 \leqslant \xi \leqslant 1$, $\zeta \geqslant 0$, where $\zeta = z/\varepsilon$. The displacement fields (on $\zeta = 0$) of these strip eigenfunctions are known to satisfy conditions of the form [see Gregory and Wan (1984)]

$$\int_{-1}^{1}[\tau^X_{\xi\zeta}(\xi,0)u_\xi(\xi,0)+\tau^X_{\zeta\zeta}(\xi,0)u_\zeta(\xi,0)]\,\mathrm{d}\xi = 0, \tag{73}$$

where $\boldsymbol{\tau}^X(\xi,\zeta)$ ($X = T$, B or F) are the stress fields of three canonical problems for the strip, whose end $\zeta = 0$ is clamped, and which is subject to unit tension (T), bending (B) or flexure (F) at $\zeta = +\infty$.

A rapidly decaying state in the shell is just a linear sum of its PF-eigenfunctions. It follows that, if the end data $\bar{u}_r, \bar{u}_z$ for the shell generates a rapidly decaying state, we have

$$\int_{-1}^{1}[\tau^X_{\xi\zeta}(\xi,0)\bar{u}_r(\xi)+\tau^X_{\zeta\zeta}(\xi,0)\bar{u}_z(\xi)]\,\mathrm{d}\xi = O(\varepsilon) \tag{74}$$

as $\varepsilon \to 0$. The equations (74) (with $X = T$, B or F) are the required necessary conditions for rapid decay, correct to order $O(1)$.

It should be noted that $\tau^X_{\xi\zeta}(\xi,0)$, $\tau^X_{\zeta\zeta}(\xi,0)$ are *not* polynomials in ξ [see Gregory and Gladwell (1982)]; in particular then, the plausible conditions

$$\int_{-1}^{1}\bar{u}_r(\xi)\,\mathrm{d}\xi = \int_{-1}^{1}\bar{u}_z(\xi)\,\mathrm{d}\xi = \int_{-1}^{1}\xi\bar{u}_z(\xi)\,\mathrm{d}\xi = 0 \tag{75}$$

are generally not the conditions for rapid decay, *even in thin shell theory*. However the conditions (74) and (75) do happen to coincide for the special case in which

$$\bar{u}_r = w_0, \quad \bar{u}_z = u_0+\phi_0\xi, \tag{76}$$

where w_0, u_0, ϕ_0 are constants. In this case, both (74) and (75) predict that the data (76) would induce a rapidly decaying state only if $w_0 = u_0 = \phi_0 = 0$.

5. EXAMPLES

Example 1. Our first example is a numerical comparison of the exact theory, refined shell theory, and thin shell theory for a particular set of end-data of Case A.

For the case in which $\varepsilon = 0.1$ and $\nu = 0.3$ we solved eqn (11) numerically to find the dimensionless shell eigenvalue $R\lambda$, and the dimensionless shell eigenfunction $\{E^{-1}\boldsymbol{\tau}^{\lambda}, R^{-1}\mathbf{u}^{\lambda}\}$, conveniently normalized. The real parts of $\tau^{\lambda}_{rz}(r, 0)$, $\tau^{\lambda}_{zz}(r, 0)$ were then taken to be the data $\bar{\tau}_{rz}, \bar{\tau}_{zz}$; this data is depicted in Fig. 1. By construction, the exact solution corresponding to this data is known.

The refined shell theory solution corresponding to $\bar{\tau}_{rz}, \bar{\tau}_{zz}$ has the form

$$\{A\boldsymbol{\tau}^R + B\boldsymbol{\tau}^I, A\mathbf{u}^R + B\mathbf{u}^I\} \tag{77}$$

(see (37), (38) and Appendix B), where the real coefficients A, B are to be chosen so that the *difference* between the tractions of (77) and the data $\bar{\tau}_{rz}, \bar{\tau}_{zz}$ satisfy (47), (49) with the right-hand sides set equal to zero. This implies that

$$\frac{\varepsilon^{1/2}}{\omega}(-2 + \tfrac{68}{15}\omega^2\varepsilon)A + \frac{\varepsilon^{1/2}}{\omega}(2 + \tfrac{68}{15}\omega^2\varepsilon)B = \int_{-1}^{1}(1+\xi\varepsilon)(1-\nu\xi\varepsilon)E^{-1}\bar{\tau}_{rz}(\xi)\,\mathrm{d}\xi, \tag{78}$$

$$(-\tfrac{56}{15}\varepsilon)A + \left(-\frac{8\omega^2}{3(1-\nu^2)}\right)B = \int_{-1}^{1}(1+\xi\varepsilon)\xi(1-\tfrac{1}{2}\nu\xi\varepsilon)E^{-1}\bar{\tau}_{zz}(\xi)\,\mathrm{d}\xi. \tag{79}$$

The integrals in (78), (79) were computed numerically and this 2×2 system was then solved for A, B. These values were then inserted into (77) to determine the refined shell solution. The corresponding thin shell solution was also evaluated by retaining only leading terms (as $\varepsilon \to 0$) in (77)–(79). Since the exact solution has no "PF-part", it is meaningful to compare the exact and approximate solutions at $z = 0$. Figure 1 compares the actual data $\bar{\tau}_{rz}, \bar{\tau}_{zz}$ with the representation of that data by the refined and thin shell solutions. The two approximations both represent the end data well, being indistinguishable from $\bar{\tau}_{rz}$, but with the refined theory representing $\bar{\tau}_{zz}$ more accurately.

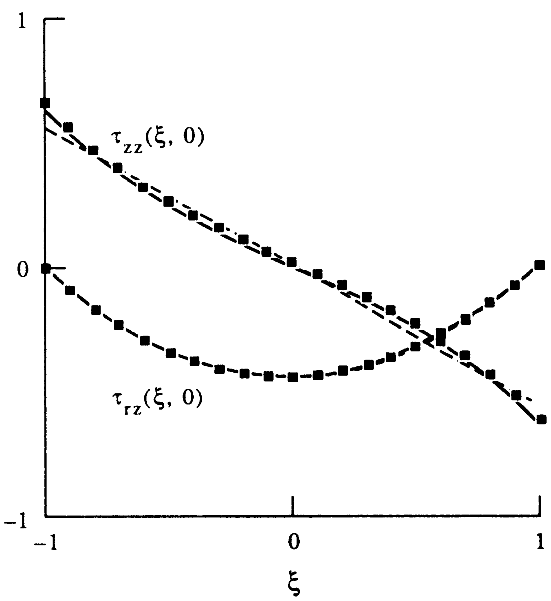

Fig. 1. The data on $z = 0$: the dots are points of the actual data $\bar{\tau}_{rz}, \bar{\tau}_{zz}$, while the curves are representations of this data in the refined shell (solid) and thin shell (dashed) solutions; $\varepsilon = 0.1$, $\nu = 0.3$.

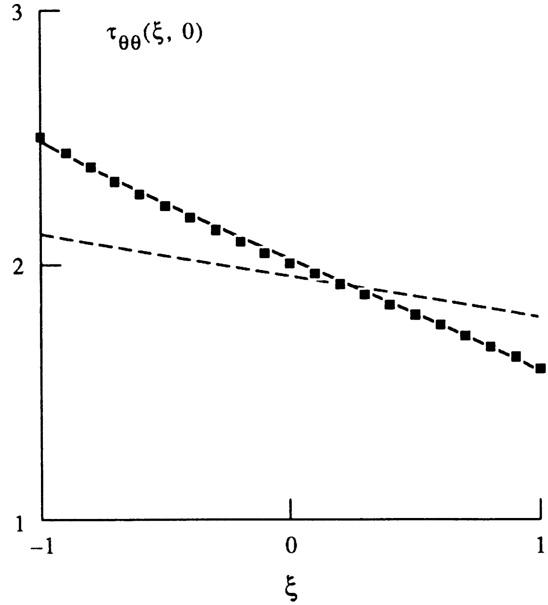

Fig. 2. The exact value of $\tau_{\theta\theta}(\xi, 0)$ (dots) compared with values predicted by refined shell (solid) and thin shell (dashed) theories; $\varepsilon = 0.1$, $\nu = 0.3$.

The situation is quite different however when we compare the *predictions* (i.e. the solutions) of the two approximate theories. Figure 2 shows the exact values of the important induced circumferential stress $\tau_{\theta\theta}(\xi, 0)$, together with the predictions of the refined and thin shell theories ($\varepsilon = 0.1$). Thin shell theory is significantly inaccurate, underestimating the maximum value of $\tau_{\theta\theta}$ (which occurs on the inner surface of the shell) by about 15%; the corresponding error in the refined shell theory is about 0.7%. With $\varepsilon = 0.2$, the errors in $\tau_{\theta\theta}$ predicted by the thin and refined shell theories were about 26% and 2.5% respectively.

We conclude that the accuracy of the *predictions* of thin shell theory cannot be inferred from its ability to represent the prescribed data and also that the refined theory appears to be sufficiently accurate for practical purposes when ε is as large as 0.2 (h/R as large as 0.4) i.e. for moderately thick shells.

Example 2. *Our second example relates to the particular Case A data*

$$\bar{\tau}_{rz}(r) = -\frac{P}{aR}\,\delta\!\left(\frac{r-a}{R}\right), \quad \bar{\tau}_{zz}(r) = 0, \tag{80}$$

where $\delta(\cdot)$ is the Dirac δ-function. (*This combination corresponds to a radially directed line loading P* (*per unit angle*) *applied around the inner rim $r = a$, $z = 0$ of the end of the shell.*)

In terms of ξ, (80) becomes

$$\bar{\tau}_{rz}(\xi) = -\frac{P}{R^2\varepsilon(1-\varepsilon)}\,\delta(\xi+1), \quad \bar{\tau}_{zz}(\xi) = 0. \tag{81}$$

The refined shell solution corresponding to (81) has the form (77), where the real coefficients A, B satisfy (78), (79) with $\bar{\tau}_{rz}$, $\bar{\tau}_{zz}$ given by (81). This gives

$$A = \frac{E^{-1}P\omega}{2R^2\varepsilon^{3/2}}\left[1+(\nu+\tfrac{2}{5}\omega^2)\varepsilon\right], \qquad B = -\frac{E^{-1}P\omega}{2R^2\varepsilon^{3/2}}\left(\tfrac{28}{15}\,\omega^2\varepsilon\right). \tag{82, 83}$$

The refined shell theory solution is obtained by substituting (82), (83) into (77). In particular we find that†

$$\tau_{\theta\theta}(\xi,0) = \frac{P\omega}{R^2\varepsilon^{3/2}}[1+(v+\tfrac{2}{5}\omega^2)\varepsilon-\tfrac{1}{10}\xi(5v\xi^2+9v+10)\varepsilon], \tag{84}$$

so that on the inner surface of the shell and with $v = 1/2$

$$\tau_{\theta\theta}(-1,0) = \frac{P\sqrt{3}}{2R^2\varepsilon^{3/2}}[1+\tfrac{5}{2}\varepsilon]. \tag{85}$$

Thus, in this case with $\varepsilon = 0.1$, thin shell theory will be about 25% in error. (Of course, thin shell theory was never intended to be applied to this type of problem.)

Example 3. Here we have essentially the same radial line loading as in Example 2, but now applied around the inner surface of an infinite *cylindrical shell.*

By symmetry, the boundary conditions at $z = 0$ to be applied to the right-hand half are

$$\bar{u}_z = 0, \quad \bar{\tau}_{rz} = -\frac{P}{2aR\varepsilon}\delta(\xi+1), \tag{86}$$

which is Case C mixed data. The refined shell theory solution corresponding to (86) has a displacement field of the form

$$A(\mathbf{u}^R+\mathbf{u}^I)+B(\mathbf{u}^R-\mathbf{u}^I)+C(R\mathbf{k}), \tag{87}$$

where A, B, C are real constants, and $\mathbf{k}$ is the unit vector pointing in the z-direction. The coefficients A, B, C are determined from the conditions for rapid decay (70)–(72), which must be satisfied by the *difference* between the data (86) and the refined shell solution.

From (70), we obtain

$$2C = -\frac{E^{-1}P}{2aR\varepsilon}\int_{-1}^{1} v(1+\xi\varepsilon)^2\delta(\xi+1)\,d\xi$$

so that

$$C = -\frac{E^{-1}Pv(1-\varepsilon)}{4R^2\varepsilon}, \tag{88}$$

with no approximations. On neglecting terms of order $O(\varepsilon^2)$, (71) and (72) give

$$(\tfrac{32}{3}\omega^2\varepsilon)A-4B = -K(1+v\varepsilon),$$

$$(\tfrac{8}{3}\omega^2)A+[\tfrac{2}{15}(2-3v)(11+14v)\varepsilon]B = -K\frac{v}{6}(1+4v)\varepsilon, \tag{89}$$

where $K = E^{-1}P\omega/(2R^2\varepsilon^{3/2})$. Thus, we have

$$A = -\frac{11}{40}\frac{K}{\omega^2}(1-v^2)\varepsilon, \quad B = \tfrac{1}{4}K(1+v\varepsilon). \tag{90}$$

† We are aware that at $z = 0$, the PF-part of the solution cannot be neglected. However $\tau_{\theta\theta}(\xi,0)$ is still a convenient measure of the difference between the refined and thin shell theories.

The expressions (88), (90) determine the refined shell solution corresponding to the data (86). In particular, when $\nu = \frac{1}{2}$, we find (see footnote on p. 1975)

$$\tau_{\theta\theta}(\xi, 0) = \frac{P\sqrt{3}}{8R^2\varepsilon^{3/2}}[(1+\xi)+\tfrac{1}{20}(33-25\xi-15\xi^2-5\xi^3)\varepsilon]. \tag{91}$$

The corresponding stress resultant N_θ is given by

$$N_\theta = \int_a^b \tau_{\theta\theta}(r)\,\mathrm{d}r = R\varepsilon\int_{-1}^{1}\tau_{\theta\theta}(\xi)\,\mathrm{d}\xi = \frac{P\sqrt{3}}{4R\varepsilon^{1/2}}[1+\tfrac{7}{5}\varepsilon]. \tag{92}$$

Example 4. In this example we obtain the thin *shell solution for a shell with pure displacement end data (Case D).*

In classical thin shell theory there is really no known way of treating this case except when the prescribed displacements $\bar{u}_r(\xi), \bar{u}_z(\xi)$ have exactly the same ξ-variation as some linear combination of the thin shell eigenfunctions $\mathbf{u}^R, \mathbf{u}^I$; in particular they must be linear functions of ξ. Consider the particular data†

$$\bar{u}_r = 0, \quad \bar{u}_z = \beta h\xi^3, \tag{93}$$

where h is the shell thickness, and β a small dimensionless constant. The thin shell solution corresponding to (93) has the form

$$\mathbf{u} = A\mathbf{u}^R + B\mathbf{u}^I, \tag{94}$$

where $\mathbf{u}^R, \mathbf{u}^I$ now mean just the leading terms of the expressions in Appendix B. (In general, (94) should also include an unknown rigid body displacement in the z-direction, but the anti-symmetry of $\bar{u}_z$ implies that this is zero.) The coefficients A, B are determined from the conditions for rapid decay (74) with $X = \mathrm{B}$ (Bending) and $X = \mathrm{F}$ (Flexure). This implies that

$$-(2\varepsilon^{1/2}\omega)A-(2\varepsilon^{1/2}\omega)B = \beta\varepsilon\int_{-1}^{1}\xi^3\tau^{\mathrm{B}}_{\zeta\zeta}(\xi,0)\,\mathrm{d}\xi, \tag{95}$$

$$2A = \varepsilon\beta\int_{-1}^{1}\xi^3\tau^{\mathrm{F}}_{\zeta\zeta}(\xi,0)\,\mathrm{d}\xi, \tag{96}$$

where $\boldsymbol{\tau}^{\mathrm{B}}(\xi,\zeta)$ and $\boldsymbol{\tau}^{\mathrm{F}}(\xi,\zeta)$ are the stress fields for the plane strain cantilevered semi-infinite strip $\zeta \geqslant 0$, $|\xi| \leqslant 1$ under unit bending (B) and flexure (F) at $\zeta = +\infty$, respectively [see Gregory and Wan (1984)]. It is convenient to use the notation of that paper, namely

$$n_j^X \equiv \int_{-1}^{1}\xi^j\tau^X_{\zeta\zeta}(\xi,0)\,\mathrm{d}\xi, \qquad t_j^X \equiv \int_{-1}^{1}\xi^j\tau^X_{\xi\zeta}(\xi,0)\,\mathrm{d}\xi, \tag{97, 98}$$

where $X = \mathrm{B}$ or F. The integrals in (95), (96) are therefore $n_3^{\mathrm{B}}, n_3^{\mathrm{F}}$ respectively; their values depend only on ν and are tabulated for $\nu = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ in Gregory and Wan (1984). The conditions (95), (96) then imply that

$$A = \tfrac{1}{2}\varepsilon\beta n_3^{\mathrm{F}}, \qquad B = -\tfrac{1}{2}\varepsilon\beta\left(\frac{n_3^{\mathrm{B}}}{\varepsilon^{1/2}\omega}+n_3^{\mathrm{F}}\right), \tag{99, 100}$$

† If $\beta h\xi^3$ were replaced by $\beta h\xi$, this would correspond to a rotation of the radial line elements through a small angle 2β. The data (93) gives the same displacements at $\xi = \pm 1$, but has a curved (cubic) profile.

and this determines the thin shell solution corresponding to the data (93). In particular, we have

$$\tau_{\theta\theta}(\xi,0)=\varepsilon\beta E\left[n_3^{\mathrm{F}}+\frac{2\nu\omega^2}{1-\nu^2}\left(\frac{n_3^{\mathrm{B}}}{\varepsilon^{1/2}\omega}+n_3^{\mathrm{F}}\right)\xi\right]. \tag{101}$$

When $\bar{u}_z=\beta h\xi$, the corresponding value is

$$\tau_{\theta\theta}(\xi,0)=\varepsilon\beta E\left[\frac{2\nu\omega^2}{1-\nu^2}\left(\frac{n_1^{\mathrm{B}}}{\varepsilon^{1/2}\omega}\right)\xi\right]. \tag{102}$$

Since $n_1^{\mathrm{B}}=-1$, $n_3^{\mathrm{B}}\approx-0.63$, $n_3^{\mathrm{F}}\approx-0.08$, we see that (101), (102) have quite different values. It follows that the displacement profile at $z=0$ has a first order effect on the induced stresses. Note also that our solution for the end data (93) gives the same leading term bending couple and transverse shear resultant at $z=0$ as the Nair–Reissner (1978) solution which assumes transverse inextensibility.

6. THE REFINED THEORY WITH KNOWN STRESS RESULTANT AND COUPLE ONLY

As Koiter (1970) has pointed out, the pointwise distribution of the boundary data on the edge of a shell is usually not known exactly in practical problems. (Important exceptions include a free edge, a clamped edge and certain special loadings such as those in Section 5, Examples 2, 3.) Can we establish bounds on the magnitude of the corrections given by the refined theory if only *resultant* forces and moments are known at $z=0$? The answer is positive if some constraint is imposed on the end data through physical consideration. The following two cases illustrate this point:

(i) Suppose we have Case A (pure stress) data with $\bar{\tau}_{zz}\equiv 0$ and some $\bar{\tau}_{rz}(\xi)$ such that the corresponding value of Q_z is positive. In the refined theory, the influence of this data is dependent [see (52)] upon $\hat{Q}_z$, given by

$$\hat{Q}_z=\tfrac{1}{2}h\int_{-1}^{1}(1+\xi\varepsilon)(1-\nu\xi\varepsilon)\bar{\tau}_{rz}(\xi)\,\mathrm{d}\xi, \tag{103}$$

whereas we know only the value of Q_z, given by

$$Q_z=\tfrac{1}{2}h\int_{-1}^{1}(1+\xi\varepsilon)\bar{\tau}_{rz}(\xi)\,\mathrm{d}\xi. \tag{104}$$

In general $\hat{Q}_z$ is not closely related to Q_z. However, for the case in which

$$\bar{\tau}_{rz}(\xi)\geqslant 0\quad(-1\leqslant\xi\leqslant 1), \tag{105}$$

(which might be expected on physical grounds) the Mean Value Theorem for integrals gives

$$\hat{Q}_z=(1-\nu\eta\varepsilon)\tfrac{1}{2}h\int_{-1}^{1}(1+\xi\varepsilon)\bar{\tau}_{rz}(\xi)\,\mathrm{d}\xi \tag{106}$$

for some η, $-1\leqslant\eta\leqslant 1$. Thus, in this case

$$(1-|\nu|\varepsilon)Q_z\leqslant\hat{Q}_z\leqslant Q_z(1+|\nu|\varepsilon), \tag{107}$$

which delimits the effect on $\hat{Q}_z$ of the pointwise distribution of $\bar{\tau}_{rz}(\xi)$.

(ii) Suppose now that $\bar{\tau}_{rz}(\xi)\equiv 0$ and that we have some $\bar{\tau}_{zz}(\xi)$ such that the corresponding value of M_z is positive, and (3) holds; we wish to establish some connection between $\hat{M}_z$ and M_z. From (53), we have

$$\frac{4}{h^2}\hat{M}_z = \int_{-1}^{1}(1+\xi\varepsilon)\xi(1-\tfrac{1}{2}\nu\xi\varepsilon)\bar{\tau}_{zz}(\xi)\,\mathrm{d}\xi$$

$$= \frac{4}{h^2}M_z - \tfrac{1}{2}\nu\varepsilon\int_{-1}^{1}\xi^2(1+\xi\varepsilon)\bar{\tau}_{zz}(\xi)\,\mathrm{d}\xi, \tag{108}$$

and on integrating by parts

$$\int_{-1}^{1}\xi^2(1+\xi\varepsilon)\bar{\tau}_{zz}(\xi)\,\mathrm{d}\xi = [\xi^2 F(\xi)]_{-1}^{1} - \int_{-1}^{1}2\xi F(\xi)\,\mathrm{d}\xi, \tag{109}$$

where

$$F(\xi) = \int_{-1}^{\xi}(1+\xi'\varepsilon)\bar{\tau}_{zz}(\xi')\,\mathrm{d}\xi'. \tag{110}$$

For the case in which $\bar{\tau}_{zz}(\xi)$ is an increasing function of ξ in $[-1,1]$, which might be expected on physical grounds, it follows from (3) that $F(\xi)\leqslant 0$ and $F(-1)=F(1)=0$. Hence, in this case,

$$\frac{4}{h^2}\hat{M}_z = \frac{4}{h^2}M_z + \nu\varepsilon\int_{-1}^{1}\xi F(\xi)\,\mathrm{d}\xi$$

$$= \frac{4}{h^2}M_z + \nu\eta\varepsilon\int_{-1}^{1}F(\xi)\,\mathrm{d}\xi \tag{111}$$

for some η $(-1\leqslant\eta\leqslant 1)$, on using the Mean Value Theorem. But, on integrating by parts,

$$\int_{-1}^{1}F(\xi)\,\mathrm{d}\xi = [\xi F(\xi)]_{-1}^{1} - \int_{-1}^{1}\xi(1+\xi\varepsilon)\bar{\tau}_{zz}(\xi)\,\mathrm{d}\xi$$

$$= -\frac{4}{h^2}M_z \tag{112}$$

and so

$$(1-|\nu|\varepsilon)M_z \leqslant \hat{M}_z \leqslant (1+|\nu|\varepsilon)M_z. \tag{113}$$

This delimits the effect on $\hat{M}_z$ of the pointwise distribution of $\bar{\tau}_{zz}(\xi)$.

We remark that the two results above deal only with corrections arising from the *boundary conditions*; further corrections of the same order arise from the refined shell eigenfunction itself.

7. CONCLUSION

In Sections 3–6 we have developed a complete refined theory for the semi-infinite (complete) cylindrical shell in axially symmetric deformation. This refined theory is a bridge between thin shell theory and the three-dimensional elasticity theory away from the end. The refined eigenfunctions in Appendices A, B can be used as an approximate elasticity theory solution; or the shell eigenfunction of Appendix B, together with the boundary conditions of Section 4, can be used as a refined *shell* theory for the shell interior. (An exception is the displacement end-data case; some additional calculation, beyond that done in Section 4, is needed for a refined theory.) In either case the refined theory is accurate for moderately thick shells for which the use of thin shell theory is inappropriate. It should be possible to develop a similar refined theory for nonaxially symmetric deformations, and also for other shells of revolution.

In addition to completing the formulation of a refined shell theory, the method of obtaining the appropriate boundary conditions for the interior solution (without any reference to the boundary layer solution) is also of fundamental importance to the formulation of any shell theory or the actual interior solution. In particular, the method allows us to obtain for the first time the generally appropriate *displacement* (and mixed) boundary conditions for a leading term asymptotic theory for thin shells.

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APPENDIX A: THE PF-EIGENFUNCTIONS

We showed in Section 3 that $\{\lambda_n\}$, the PF-eigenvalues lying in the first quadrant, are given by

$$\lambda_n = \frac{w_n}{R\varepsilon}(1+O(\varepsilon^2)) \tag{A1}$$

as $\varepsilon \to 0$, where $\{w_n\}$ $(n \geqslant 1)$ are the ordered roots of $\sin^2 2w - 4w^2 = 0$ lying in the first quadrant. Thus, in a theory correct to $O(\varepsilon)$, the correction term in (27) is negligible so that λ_n can be approximated simply by $w_n/(R\varepsilon)$.

Let the elastostatic state corresponding to λ_n be denoted by $\{\boldsymbol{\tau}^{(n)}(r)\,\mathrm{e}^{-\lambda_n z}, \mathbf{u}^{(n)}(r)\,\mathrm{e}^{-\lambda_n z}\}$ and define the dimensionless radial co-ordinate ξ by

$$r = R(1+\xi\varepsilon), \tag{A2}$$

so that $a \leqslant r \leqslant b$ corresponds to $-1 \leqslant \xi \leqslant 1$. Then as $\varepsilon \to 0$ the components of $\mathbf{u}^{(n)}, \boldsymbol{\tau}^{(n)}$ (when conveniently normalized) are

$$\begin{aligned} R^{-1}u_r^{(n)} = {} & w_n(1+v)[-2(1-v)(-1)^n \cos w_n(1+\xi) - 2(1-v)\cos w_n(1-\xi) \\ & + w_n(1-\xi)(-1)^n \sin w_n(1+\xi) + w_n(1+\xi)\sin w_n(1-\xi)]\varepsilon + O(\varepsilon^2), \end{aligned} \tag{A3}$$

$$\begin{aligned} R^{-1}u_z^{(n)} = {} & -w_n(1+v)[-w_n(1-\xi)(-1)^n \cos w_n(1+\xi) + w_n(1+\xi)\cos w_n(1-\xi) \\ & + (1-2v)(-1)^n \sin w_n(1+\xi) - (1-2v)\sin w_n(1-\xi)]\varepsilon + O(\varepsilon^2), \end{aligned} \tag{A4}$$

$$\begin{aligned} E^{-1}\tau_{rr}^{(n)} = {} & w_n^2[w_n(1-\xi)(-1)^n \cos w_n(1+\xi) \\ & - w_n(1+\xi)\cos w_n(1-\xi) + (-1)^n \sin w_n(1+\xi) - \sin w_n(1-\xi)] \\ & + \tfrac{1}{2}w_n^2[-w_n\xi(1-\xi)(-1)^n \cos w_n(1+\xi) \\ & + w_n\xi(1+\xi)\cos w_n(1-\xi) + \{-4v(1-\xi)+(1-2\xi)\}(-1)^n \sin w_n(1+\xi) \\ & - \sin w_n(1-\xi)]\varepsilon + O(\varepsilon^2), \end{aligned} \tag{A5}$$

$$\begin{aligned} E^{-1}\tau_{\theta\theta}^{(n)} = {} & 2vw_n^2[(-1)^n \sin w_n(1+\xi) - \sin w_n(1-\xi)] \\ & + w_n[(4v^2-v-2)(-1)^n \cos w_n(1+\xi) - (2-v)\cos w_n(1-\xi) \\ & + w_n(-v\xi-\xi+1)(-1)^n \sin w_n(1+\xi) + w_n(v\xi+\xi+1)\sin w_n(1-\xi)]\varepsilon + O(\varepsilon^2), \end{aligned} \tag{A6}$$

$$\begin{aligned} E^{-1}\tau_{zz}^{(n)} = {} & w_n^2[-w_n(1-\xi)(-1)^n \cos w_n(1+\xi) \\ & + w_n(1+\xi)\cos w_n(1-\xi) + (-1)^n \sin w_n(1+\xi) - \sin w_n(1-\xi)] \\ & - \tfrac{1}{2}w_n[\{2(1-2v) - w_n^2\xi(1-\xi)\}(-1)^n \cos w_n(1+\xi) \\ & + \{w_n^2\xi(1+\xi) - 2(1-2v)\}\cos w_n(1-\xi) \\ & + w_n\{-4v(1-\xi)+(3-2\xi)\}(-1)^n \sin w_n(1+\xi) + w_n \sin w_n(1-\xi)]\varepsilon + O(\varepsilon^2), \end{aligned} \tag{A7}$$

$$\begin{aligned} E^{-1}\tau_{rz}^{(n)} = {} & -w_n^3[(1-\xi)(-1)^n \sin w_n(1+\xi) + (1+\xi)\sin w_n(1-\xi)] \\ & + \tfrac{1}{2}w_n[2w_n(1-2v)(1-\xi)(-1)^n \cos w_n(1+\xi) \\ & + w_n^2\xi(1-\xi)(-1)^n \sin w_n(1+\xi) \\ & + \{w_n^2\xi(1+\xi) - 2(1-2v)\}\sin w_n(1-\xi)]\varepsilon + O(\varepsilon^2). \end{aligned} \tag{A8}$$

The PF-elastostatic states $\{\boldsymbol{\tau}^{(n)}(r)\,\mathrm{e}^{-\lambda_n z}, \mathbf{u}^{(n)}(r)\,\mathrm{e}^{-\lambda_n z}\}$ are specific examples of "rapidly decaying states", as defined in Section 4. As such, they should satisfy the decaying state conditions (47), (49) rather than the (plausible) Saint-Venant conditions (50), (51). This may be verified by direct integration using the expressions (A7), (A8) above. It is found for instance that

$$\int_{-1}^{1} (1+\xi\varepsilon)\tau_{rz}^{(n)}(\xi)\,\mathrm{d}\xi = [4v\{1-(-1)^n\}\sin^2 w_n]\varepsilon + O(\varepsilon^2), \tag{A9}$$

while

$$\int_{-1}^{1} (1+\xi\varepsilon)\tau_{rz}^{(n)}(\xi)(1-v\xi\varepsilon)\,\mathrm{d}\xi = O(\varepsilon^2), \tag{A10}$$

as predicted by (47). The fact that the PF-eigenfunctions are found to satisfy (47) (and all the other decay conditions in Section 4) is a valuable check not only on the expressions in this Appendix, but also on the decay conditions themselves. In contrast, eqn (A9) shows conclusively that Saint-Venant's principle cannot be applied as a condition for rapid decay in the refined theory.

APPENDIX B: THE SHELL EIGENFUNCTION

We showed in Section 3 that the shell eigenvalue lying in the first quadrant is given by

$$\lambda = \frac{\omega(1+\mathrm{i})}{R\varepsilon^{1/2}}[1-\tfrac{2}{5}\mathrm{i}\omega^2\varepsilon+O(\varepsilon^2)] \tag{B1}$$

as $\varepsilon \to 0$, where the constant $\omega = \{3(1-v^2)/4\}^{1/4}$. Let the corresponding shell elastostatic state be denoted by $\{\boldsymbol{\tau}^{\lambda}(r)\,\mathrm{e}^{-\lambda z}, \mathbf{u}^{\lambda}(r)\,\mathrm{e}^{-\lambda z}\}$, and define the dimensionless radial co-ordinate ξ by

$$r = R(1+\xi\varepsilon). \tag{B2}$$

Then the real and imaginary parts of the components of $\mathbf{u}^{\lambda}, \boldsymbol{\tau}^{\lambda}$ are given (as $\varepsilon \to 0$) by

$$R^{-1}\Re(u_r^{\lambda}) = 2-2v\xi\varepsilon+O(\varepsilon^2), \tag{B3}$$

$$R^{-1}\Im(u_r^{\lambda}) = \frac{2\omega^2}{3(1-v)}(3v\xi^2+5v-8)\varepsilon+O(\varepsilon^2), \tag{B4}$$

$$R^{-1}\Re(u_z^{\lambda}) = \omega^{-1}(v+2\omega^2\xi)\varepsilon^{1/2}+\frac{\omega^{-1}}{30}[15(1+v)(2-v)\xi^3-30v\omega^2\xi^2 + (1+v)(48-93v)\xi-2\omega^2(10+29v)]\varepsilon^{3/2}+O(\varepsilon^{5/2}), \tag{B5}$$

$$R^{-1}\Im(u_z^{\lambda}) = \omega^{-1}(2\omega^2\xi-v)\varepsilon^{1/2}-\frac{\omega^{-1}}{30}[15(1+v)(2-v)\xi^3+30v\omega^2\xi^2 + (1+v)(48-93v)\xi+2\omega^2(10+29v)]\varepsilon^{3/2}+O(\varepsilon^{5/2}), \tag{B6}$$

$$E^{-1}\Re(\tau_{rr}^{\lambda}) = -\xi(1-\xi^2)\varepsilon+O(\varepsilon^2), \tag{B7}$$

$$E^{-1}\Im(\tau_{rr}^{\lambda}) = \frac{2v\omega^2}{1-v^2}(1-\xi^2)\varepsilon+O(\varepsilon^2), \tag{B8}$$

$$E^{-1}\Re(\tau_{\theta\theta}^{\lambda}) = 2-\tfrac{1}{5}\xi(5v\xi^2+37v+10)\varepsilon+O(\varepsilon^2), \tag{B9}$$

$$E^{-1}\Im(\tau_{\theta\theta}^{\lambda}) = -\frac{4v\omega^2}{1-v^2}\xi+\frac{2\omega^2}{3(1-v)}(3v\xi^2+7v-8)\varepsilon+O(\varepsilon^2), \tag{B10}$$

$$E^{-1}\Re(\tau_{zz}^{\lambda}) = -\tfrac{2}{5}\xi(5\xi^2+11)\varepsilon+O(\varepsilon^2), \tag{B11}$$

$$E^{-1}\Im(\tau_{zz}^{\lambda}) = -\frac{4\omega^2}{1-v^2}\xi+\frac{2\omega^2}{3(1-v^2)}(3v\xi^2+2-v)\varepsilon+O(\varepsilon^2), \tag{B12}$$

$$E^{-1}\Re(\tau_{rz}^{\lambda}) = -\tfrac{3}{2}\omega^{-1}(1-\xi^2)\varepsilon^{1/2}+\frac{\omega^{-1}}{10}(1-\xi^2)[5\omega^2\xi^2+5(1+v)\xi+33\omega^2]\varepsilon^{3/2}+O(\varepsilon^{5/2}), \tag{B13}$$

$$E^{-1}\Im(\tau_{rz}^{\lambda}) = \tfrac{3}{2}\omega^{-1}(1-\xi^2)\varepsilon^{1/2}+\frac{\omega^{-1}}{10}(1-\xi^2)[5\omega^2\xi^2-5(1+v)\xi+33\omega^2]\varepsilon^{3/2}+O(\varepsilon^{5/2}). \tag{B14}$$